BANACH LATTICES WITH UPPER p-ESTIMATES: RENORMING AND FACTORIZATION

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ABSTRACT. The notions of p-convexity and concavity are fundamental tools for studying Banach lattices, as they partition the class of Banach lattices into a scale of spaces with L_p-like properties. Upper and lower p-estimates provide a refinement of this scale, modeled by the Lorentz spaces $L_{p,\infty}$ and $L_{p,1}$, respectively. In this article, we provide a comprehensive treatment of Banach lattices with upper p-estimates. In particular, we show that many well-known theorems about p-convex Banach lattices have analogues in the upper p-estimate setting, including the ability to represent all such spaces inside of infinity sums of model spaces, to canonically factor the convex operators and identify their associated operator ideals, as well as to give a precise description of the free objects and push-outs. Proving these results is far from straightforward and will require the development of a variety of new tools that avoid convexification and concavification procedures. In fact, we will identify many fundamental differences between the theories of p-convexity and upper p-estimates, particularly with regards to isometric problems and renormings.

CONTENTS

1. Introduction

Convexity properties have long played a fundamental role in the theory of Banach lattices [17]. Recall that a Banach lattice X is *p-convex* if there exists a constant $K \geqslant 1$ such that for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$ we have

$$(1.1) \qquad \left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \right\| \leqslant K \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}}.$$

Evidently, every Banach lattice is 1-convex, and $L_p(\mu)$ is p-convex. By reversing the inequality in (1.1) one obtains the notion of a *p-concave* Banach lattice, and it is well-known that a Banach lattice X is both p-convex and p-concave if and only if it is lattice isomorphic to $L_p(\mu)$ for some measure μ.

The above convexity and concavity properties partition the class of Banach lattices into a scale of spaces with properties similar to those of L_p. In fact, by Maurey's Factorization Theorem, it follows that a Banach lattice is p-convex if and only if it is lattice isomorphic to a closed sublattice of an ℓ_∞-sum of L_p-spaces. This theorem unlocks a host of concrete tools that can be used to study a wide class of function spaces and is in some sense analogous to the well-known representation of Banach spaces as closed subspaces of ℓ_∞-sums of copies of the real line.

It is natural to replace one of the p-sums by a q-sum in (1.1). However, it is well-known that this only leads to one additional non-trivial property; namely, (p, ∞)-convexity. Here, we recall that a Banach lattice X is *(p, ∞)-convex* (or has an *upper p-estimate*) if there exists a constant $K \geqslant 1$ such that for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$ we have

$$(1.2) \qquad \left\| \bigvee_{i=1}^{n} |x_i| \right\| \leqslant K \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}}.$$

It is well-known that X is (p, ∞)-convex if and only if the inequality (1.1) holds for all pairwise disjoint $x_1, \ldots, x_n \in X$. Although (p, ∞)-convexity is also a classical notion [17], the corresponding theory is far more difficult and underdeveloped than that of p-convexity – the primary objective of this article is to explain and rectify this issue.

As we shall see, the class of (p, ∞)-convex Banach lattices is modeled by weak-L_p. In fact, using Pisier's Factorization Theorem we will be able to represent every (p, ∞)-convex Banach lattice as a closed sublattice of an ℓ_∞-sum of weak-L_p-spaces. Then, by studying the properties of this representing space, we will give quick and transparent proofs of many of the fundamental structural properties of the full class of (p, ∞)-convex Banach lattices, including the fact that (p, ∞)-convexity implies q-convexity for $q < p$ with sharp constant growing like $(p - q)^{-\frac{1}{p}}$ as $q \uparrow p$. This is the main content of Section 2.

In the remainder of the article, we will analyze the fine structure of (p, ∞)-convex and $(p, 1)$-concave Banach lattices and their operators. This will present major challenges compared to the analogous study of p-convexity and concavity. This is perhaps most strikingly illustrated by contrasting Proposition 2.7 and Example 3.4: Every p-convex Banach lattice with constant one embeds lattice *isometrically* into an ℓ_∞-sum of L_p-spaces and every (p, ∞)-convex Banach lattice with

constant one embeds lattice *isomorphically* into an ℓ_∞-sum of weak-L_p-spaces, but there are (p, ∞)-convex Banach lattices with constant one that do not embed lattice isometrically into any ℓ_∞-sum of weak-L_p-spaces. This fact will make the isometric theory of (p, ∞)-convex Banach lattices particularly subtle, and will have immediate implications for the study of free Banach lattices with upper p-estimates as well as all other universal constructions.

Another difficulty in the study of upper p-estimates is that one no longer has access to convexification and concavification arguments. This is a standard and often indispensable tool in the theory of p-convex Banach lattices. For example, one may use it in combination with the fact that every separable Banach lattice embeds lattice isometrically into $C(\Delta, L_1)$ to immediately deduce that every separable p-convex Banach lattice embeds lattice isometrically into $C(\Delta, L_p)$, i.e., $C(\Delta, L_p)$ is injectively universal for separable p-convex Banach lattices. Identifying an analogous injectively universal object in the upper p-estimate case is much more difficult, not only because the above proof technique breaks down, but also because $L_{p,\infty}(\mu)$ is non-separable, depends in a more complicated way on the choice of measure μ, and can be equipped with more than one norm. However, in Section 6 we will give strong evidence that $C(\Delta, L_{p,\infty}^\circ[0, \infty))$ might be the correct universal space.

With the above difficulties in mind, the primary objective of this article will be to develop a host of new techniques to understand (p, ∞)-convex spaces and operators. In particular, we will develop a series of factorization results in Section 4 and a complete theory of the relevant operator ideals in Section 7. In many cases, the correct route to obtain a (p, ∞)-convex analogue of a result on p-convex Banach lattices is to replace L_p with the far more difficult to work with $L_{p,\infty}$ (contrast Pisier and Maurey's factorization theorems) and then overcome the additional difficulties. Interestingly, however, in other cases it will be more appropriate to replace certain ℓ_p^n-norms with ℓ_∞^n-norms (compare (1.1) with (1.2), or the general strategy in Section 7) or to instead replace multiple ℓ_p^n norms with $\ell_{p,\infty}^n$ norms (compare the norm of $\mathrm{FBL}^{(p)}$ with $\mathrm{FBL}^{(p,\infty)}$). In all cases, the proof that the desired (p, ∞)-convex result holds will be significantly more complicated than the p-convex one.

1.1. **Content of the paper.** We begin the paper by reviewing in Section 2 some basic facts about p-convexity and upper p-estimates. In particular, we recall some of the properties of $L_{p,\infty}$, as well as its role as a model space in the factorization of (p, ∞)-convex operators (Theorem 2.4) and the representation of Banach lattices with upper p-estimates (Proposition 2.7). We also provide a simple proof of the well-known fact that every Banach lattice with upper p-estimates is q-convex for every $1 \leqslant q < p$ with a constant that explodes at a precise rate as q approaches p, and give an example that shows that the converse is not true.

Next, in Section 3 we delve into the connection between the class of Banach lattices with upper p-estimates (with constant 1) and that of sublattices of ℓ_∞-sums of $L_{p,\infty}$-spaces. Proposition 2.7 implies that these classes coincide isomorphically, with a universal constant γ_p depending only on p. However, in Section 3.1 we establish that they cannot coincide isometrically, as we provide an example of a (three-dimensional) Banach lattice that satisfies an upper p-estimate with constant 1, but does not embed isometrically into any ℓ_∞-sum of $L_{p,\infty}$-spaces. Moreover, we show in Section 3.2 that the ambiguity in the choice of the renorming of $L_{p,\infty}$ (see (2.5)) is not the cause of this discrepancy. Implications for free Banach lattices are also discussed.

Section 4 is devoted to extending a series of results on abstract factorizations of convex and concave operators through Banach lattices with certain convexity and concavity properties due to

Reisner [25], Meyer-Nieberg [20], and Raynaud and the last author [24]. We take the ideas from [20, 24] to develop a general theory of convex and concave Banach lattices and operators, which is developed through Sections 4.1 to 4.4. This allows us to extend the results from [24] on minimal and maximal factorizations for p-convex and q-concave operators, respectively, to the setting of (p, ∞)-convex and $(q, 1)$-concave operators (see Theorem 4.1), refining [20, Theorem 2.8.7]. Finally, in Section 4.5 we apply this abstract theory to the study of factorizations of operators between Banach lattices which are simultaneously (p, p_2)-convex and (q, q_2)-concave, for $p_2 \in \{p, \infty\}$ and $q_2 \in \{1, q\}$. The corresponding result when $p_2 = p$ and $q_2 = q$ was established in [24]. Therefore, here we deal with the three remaining cases.

In Section 5, we revisit the construction of push-outs in subcategories of Banach lattices established in [8] with the aim of improving the isomorphism constant of Theorem 4.7 from the aforementioned reference. We show using a p-convexification and p-concavification argument that in the p-convex case this constant can be reduced to 1, that is, [3, Theorem 4.4] holds also for push-outs in the subcategory of p-convex Banach lattices. We also apply the abstract factorization techniques from the previous section to improve the proof for the upper p-estimates case.

Next, in Section 6, we continue the study of universal constructions in subcategories of Banach lattices. More specifically, we show that $C(\Delta, L_p[0, 1])$ is universal for the class of separable p-convex Banach lattices, and provide some partial results that point towards $C(\Delta, L_{p, \infty}^{\circ}[0, \infty))$ as the natural candidate to satisfy an analogous property in the setting of upper p-estimates.

Finally, we devote Section 7 to extending some results from Reisner [25] concerning the ideal of operators that factor through a Banach lattice with p-convex and q-concave factors to the other three relevant cases: We show that the ideal $M_{(p, p_2; q, q_2)}$ of operators that factor through a (p, p_2)-convex and a (q, q_2)-concave operator, with $p_2 \in \{p, \infty\}$ and $q_2 \in \{1, q\}$, is perfect, and can be identified with the ideal of operators that factor through a multiplication operator from L_{p, p_2} to L_{q, q_2}.

2. An overview of p-convexity and upper p-estimates

Throughout the paper, Banach lattices will be denoted by $X, Y, Z...$ and Banach spaces by $E, F, G...$ The closed unit ball of a Banach space E is denoted by B_E. Given $1 \leqslant p \leqslant \infty$, we will write $p^* = \frac{p}{p-1}$. Expressions of the form $\left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$ are defined via functional calculus (cf. [17, Section 1.d]) and must be understood as the finite supremum $\bigvee_{i=1}^{n} |x_i|$ for $p = \infty$. We refer the reader to [17, 20] for standard notation and terminology concerning Banach lattices.

As explained in the introduction, this paper is devoted to the study of upper p-estimates, and more specifically its differences and similarities with p-convexity. We begin by recalling the classical notions of convexity and concavity in full generality (see [6, Chapter 16]):

Definition 2.1. *Let X be a Banach lattice, E a Banach space and $1 \leqslant p, q \leqslant \infty$.*

(1) An operator $T : E \to X$ is called (p, q)-convex if there exists a constant $K \geqslant 1$ such that

$$(2.1) \qquad \left\| \left(\sum_{i=1}^{n} |Tx_i|^q \right)^{\frac{1}{q}} \right\| \leqslant K \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}}$$

for every $x_1, \ldots, x_n \in E$ *and* $n \in \mathbb{N}$. *The best constant* K *satisfying* (2.1) *is called the* (p, q)-*convexity constant of* T *and is denoted* $K^{(p,q)}(T)$.

(2) An operator $T : X \to E$ *is called* (p, q)-*concave if there exists a constant* $K \geqslant 1$ *such that*

$$(2.2) \qquad \left(\sum_{i=1}^{n} \|Tx_i\|^p \right)^{\frac{1}{p}} \leqslant K \left\| \left(\sum_{i=1}^{n} |x_i|^q \right)^{\frac{1}{q}} \right\|$$

for every $x_1, \ldots, x_n \in X$ *and* $n \in \mathbb{N}$. *The best constant* K *satisfying* (2.2) *is called the* (p, q)-*concavity constant of* T *and is denoted* $K_{(p,q)}(T)$.

(3) A Banach lattice X *is said to be* (p, q)-*convex (respectively,* (p, q)-*concave) if the identity operator* $id_X : X \to X$ *is* (p, q)-*convex (respectively,* (p, q)-*concave). We write* $K^{(p,q)}(X) = K^{(p,q)}(id_X)$ *and* $K_{(p,q)}(X) = K_{(p,q)}(id_X)$ *for the* (p, q)-*convexity and* (p, q)-*concavity constants of* X, *respectively.*

If $p = q$, *we will simply say that* T *or* X *is* p-*convex (respectively,* p-*concave) and write* $K^{(p)}(T)$ *or* $K^{(p)}(X)$ *(respectively,* $K_{(p)}(T)$ *or* $K_{(p)}(X)$*) instead.*

If we replace arbitrary collections of vectors by disjoint collections, we arrive at the notions of upper and lower estimates:

Definition 2.2. *Let* X *be a Banach lattice and* $1 \leqslant p \leqslant \infty$.

(1) X *is said to satisfy an* upper p-estimate *if there exists a constant* $K \geqslant 1$ *such that*

$$(2.3) \qquad \left\| \sum_{i=1}^{n} x_i \right\| \leqslant K \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}}$$

for every (positive) pairwise disjoint $x_1, \ldots, x_n \in X$ *and* $n \in \mathbb{N}$. *The best constant* K *satisfying* (2.3) *is called the* upper p-estimates constant *of* X *and is denoted* $K^{(\uparrow p)}(X)$.

(2) X *is said to satisfy a* lower p-estimate *if there exists a constant* $K \geqslant 1$ *such that*

$$(2.4) \qquad \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}} \leqslant K \left\| \sum_{i=1}^{n} x_i \right\|$$

for every (positive) pairwise disjoint $x_1, \ldots, x_n \in X$ *and* $n \in \mathbb{N}$. *The best constant* K *satisfying* (2.4) *is called the* lower p-estimates constant *of* X *and is denoted* $K_{(\downarrow p)}(X)$.

The first thing that should be mentioned is that convexity and concavity are dual notions: An operator T is (p, q)-convex (respectively, (p, q)-concave) if and only if T^* is (p^*, q^*)-concave (respectively, (p^*, q^*)-convex), with the same constant, and a similar relation holds for upper and lower p-estimates (see, for instance, [17, Sections 1.d and 1.f] or [6, Chapter 16]). Next, note that when $q < p$, the only (p, q)-convex operator is the null operator, and the same happens for (p, q)-concavity when $q > p$. Additionally, an operator T is (p, q)-convex for some $q > p$ (respectively, (p, q)-concave for some $q < p$) if and only if it is (p, ∞)-convex (respectively, $(p, 1)$-concave), with equivalent constants; and a Banach lattice is (p, ∞)-convex (respectively, $(p, 1)$-concave) if and only if it satisfies an upper p-estimate (respectively, lower p-estimate), with the same constant. Finally, it is straightforward to check that for every $1 \leqslant q \leqslant \infty$, every operator $T : E \to X$ is $(1, q)$-convex and every operator $T : X \to E$ is (∞, q)-concave, both with constant 1. Therefore, the only meaningful cases, up to constants, are p-convexity and p-concavity for $1 \leqslant p \leqslant \infty$, and (p, ∞)-convexity/upper

p-estimates and $(p, 1)$-concavity/lower p-estimates for $1 < p < \infty$. Throughout the paper, either of the synonymous terms (p, ∞)-convexity and upper p-estimates will be employed when deemed more appropriate.

As expected, the notions of p-convexity and (p, ∞)-convexity are strongly related, and therefore so are p-concavity and $(p, 1)$-concavity. First, it is clear that every p-convex operator is (p, ∞)-convex with the same constant. Moreover, it can be shown that every p-convex operator is q-convex for every $q < p$, and the q-convexity constant is non-decreasing [17, Proposition 1.d.5]. On the other hand, every (p, ∞)-convex operator is q-convex for every $q < p$, with constant $K^{(q)}(T) \leqslant C_{p,q} K^{(p,\infty)}(T)$, where $C_{p,q}$ is a constant that explodes like $(p - q)^{-\frac{1}{p}}$ when $q \to p^-$. Below, we elaborate on the proof of this fact and provide a new argument (see Corollary 2.8). Moreover, we show that the converse is not true via Example 2.10.

The prototypical examples of p-convex and p-concave Banach lattices are L_p-spaces. Actually, these spaces play a crucial role in many representation and factorization results for p-convex and p-concave Banach lattices and operators, as we will recall later. Some of these results have an analogue in the setting of (p, ∞)-convexity and $(p, 1)$-concavity, with the Lorentz spaces $L_{p,\infty}$ and $L_{p,1}$ playing the role of model spaces. Let us recall some facts about these spaces.

Let (Ω, Σ, μ) be a measure space and f a measurable function on Ω. We define its decreasing rearrangement $f^* : [0, \infty) \to [0, \infty)$ by

$$f^*(t) := \inf\{\lambda > 0 : \mu\{|f| > \lambda\} \leqslant t\}.$$

Given $1 < p < \infty$, the space $L_{p,1}(\mu)$ is the set of all measurable functions $f : \Omega \to \mathbb{R}$ such that the norm

$$\|f\|_{L_{p,1}} := \int_0^\infty t^{\frac{1}{p}} f^*(t) \frac{dt}{t}$$

is finite. Similarly, the space $L_{p,\infty}(\mu)$ is the set of all measurable functions f such that the quasinorm

$$\|f\|_{L_{p,\infty}} := \sup_{t>0} t^{\frac{1}{p}} f^*(t) = \sup_{\lambda>0} \lambda \mu(\{|f| > \lambda\})^{\frac{1}{p}}$$

is finite. Note, in particular, that every $f \in L_{p,\infty}(\mu)$ has σ-finite support. It is easy to check that the quasinorm of $L_{p,\infty}(\mu)$ is equivalent to the dual norm of $L_{p^*,1}(\mu)$ with equivalence constant p^*, in other words, $L_{p,\infty}(\mu)$ is the dual of $L_{p^*,1}(\mu)$ (up to a lattice isomorphic renorming). Actually, $L_{p,\infty}(\mu)$ admits the following family of lattice renormings: For each $1 \leqslant r < p$, let

$$\|f\|_{L_{p,\infty}^{[r]}} := \sup_{0<\mu(A)<\infty} \mu(A)^{\frac{1}{p}-\frac{1}{r}} \left(\int_A |f|^r \, d\mu \right)^{\frac{1}{r}}.$$

All of these norms are equivalent to the quasinorm [11, Exercise 1.1.12]:

$$(2.5) \qquad \|f\|_{L_{p,\infty}} \leqslant \|f\|_{L_{p,\infty}^{[r]}} \leqslant \left(\frac{p}{p-r} \right)^{\frac{1}{r}} \|f\|_{L_{p,\infty}} \quad \text{for every} \quad f \in L_{p,\infty}(\mu).$$

Moreover, given $f \in L_{p,\infty}(\mu)$, the norms $\|f\|_{L_{p,\infty}^{[r]}}$ are increasing in $r \in [1, p)$. Observe that for each $1 \leqslant r < p$, $L_{p,\infty}^{[r]}(\mu) = (L_{p,\infty}(\mu), \| \cdot \|_{L_{p,\infty}^{[r]}})$ is a Banach lattice that satisfies an upper p-estimate with constant 1. Hence, $L_{p^*,1}(\mu)$ satisfies a lower p^*-estimate.

Unless stated otherwise, in this paper, we will assume that $L_{p,\infty}(\mu)$ is endowed with the norm $\|\cdot\|_{L_{p,\infty}^{[1]}}$, and we will simply denote this norm by $\|\cdot\|_{L_{p,\infty}}$ or $\|\cdot\|_{p,\infty}$. Similarly, when referring to the predual space $L_{p*,1}(\mu)$, we will write $L_{p*,1}^{[r]}(\mu)$ when it is endowed with the dual norm induced by $\|\cdot\|_{L_{p,\infty}^{[r]}}$, and we will simply write $L_{p*,1}(\mu)$ in the default case $r = 1$. As usual, when the measure space is the natural numbers with the counting measure, we will write $\ell_{p,\infty}$ and $\ell_{p*,1}$.

As mentioned earlier, the spaces L_p, $L_{p,\infty}$ and $L_{p,1}$ play a special role in the theory of concavity and convexity, as the following results due to Maurey [19] and Pisier [23] illustrate. It should be noted that both theorems can be stated for $0 < r < p$, but in this work, we will only consider the range $1 \leqslant r < p$.

Theorem 2.3 (Maurey's Factorization Theorem). *Let $1 \leqslant r < p < \infty$, E a Banach space, $T : E \to L_r(\mu)$ a linear operator and $C > 0$. The following are equivalent:*

(1) T is p-convex with $K^{(p)}(T) \leqslant C$.
(2) There exists a normalized $g \in L_1(\mu)_+$ such that $\{g = 0\} \subseteq \{Tx = 0\}$ for every $x \in E$, and $T = MR$, where $R : E \to L_p(g \cdot \mu)$ given by $Rx = g^{-\frac{1}{r}}Tx$ is bounded with $\|R\| \leqslant C$ and $M : L_p(g \cdot \mu) \to L_r(\mu)$ is the multiplication operator by $g^{\frac{1}{r}}$.

$$E \xrightarrow{\quad T \quad} L_r(\mu)$$
$$R \searrow \quad \nearrow M$$
$$L_p(g \cdot \mu)$$

Theorem 2.4 (Pisier's Factorization Theorem). *Let $1 \leqslant r < p < \infty$, E a Banach space and $T : E \to L_r(\mu)$ a linear operator. The following are equivalent:*

(1) There exists $C > 0$ such that T is (p, ∞)-convex with $K^{(p,\infty)}(T) \leqslant C$.
(2) There exists $C' > 0$ and a normalized $g \in L_1(\mu)_+$ such that for every $x \in E$ and measurable subset $B \subseteq \Omega$

$$\|Tx\chi_B\|_{L_r(\mu)} \leqslant C'\|x\| \left(\int_B g \, d\mu\right)^{\frac{1}{r} - \frac{1}{p}}.$$

(3) There exists $C'' > 0$ and a normalized $g \in L_1(\mu)_+$ such that $\{g = 0\} \subseteq \{Tx = 0\}$ for every $x \in E$, and $T = MR$, where $R : E \to L_{p,\infty}^{[r]}(g \cdot \mu)$ given by $Rx = g^{-\frac{1}{r}}Tx$ is bounded with $\|R\| \leqslant C''$ and $M : L_{p,\infty}^{[r]}(g \cdot \mu) \to L_r(\mu)$ is the multiplication operator by $g^{\frac{1}{r}}$.

$$E \xrightarrow{\quad T \quad} L_r(\mu)$$
$$R \searrow \quad \nearrow M$$
$$L_{p,\infty}^{[r]}(g \cdot \mu)$$

Moreover, for the best constants, we have $C \leqslant C'' \leqslant C' \leqslant (1 - \frac{r}{p})^{\frac{1}{p} - \frac{1}{r}}C$.

Note that in the (p, ∞)-convex setting, we obtain a constant $(1 - \frac{r}{p})^{\frac{1}{p} - \frac{1}{r}}$ that does not appear in the p-convex case. For $r = 1$, this constant will play a significant role in subsequent sections, so we

will denote it by

$$\gamma_p = \left(1 - \frac{1}{p}\right)^{\frac{1}{p} - 1} = (p^*)^{\frac{1}{p^*}}.$$

We include for later use the corresponding dual statements for concave operators.

Theorem 2.5. *Let $1 < q < s < \infty$, E a Banach space, $T : L_s(\mu) \to E$ a linear operator and $C > 0$. The following are equivalent:*

(1) *T is q-concave with $K_{(q)}(T) \leqslant C$.*

(2) *There exists a normalized $g \in L_1(\mu)_+$ such that $T = S\widetilde{M}$, where $\widetilde{M} : L_s(\mu) \to L_q(g \cdot \mu)$ is the multiplication operator by $g^{-\frac{1}{s}}$ and $S : L_q(g \cdot \mu) \to E$ is bounded with $\|S\| \leqslant C$.*

Theorem 2.6. *Let $1 < q < s < \infty$, E a Banach space and $T : L_s(\mu) \to E$ a linear operator. The following are equivalent:*

(1) *There exists $C > 0$ such that T is $(q, 1)$-concave with $K_{(q,1)}(T) \leqslant C$.*

(2) *There exists $C' > 0$ and a normalized $g \in L_1(\mu)_+$ such that for every $f \in L_s(\mu)$ and measurable subset $B \subseteq \Omega$*

$$\|T(f\chi_B)\| \leqslant C' \|f\|_{L_s(\mu)} \left(\int_B g \, d\mu\right)^{\frac{1}{q} - \frac{1}{s}}.$$

(3) *There exists $C'' > 0$ and a normalized $g \in L_1(\mu)_+$ such that $T = S\widetilde{M}$, where $\widetilde{M} : L_s(\mu) \to L_{q,1}^{[s^*]}(g \cdot \mu)$ is the multiplication operator by $g^{-\frac{1}{s}}$ and $S : L_{q,1}^{[s^*]}(g \cdot \mu) \to E$ is bounded with $\|S\| \leqslant C''$.*

Moreover, for the best constants, we have $C \leqslant C'' \leqslant C' \leqslant (1 - \frac{s^*}{q^*})^{\frac{1}{q^*} - \frac{1}{s^*}} C$.

For concave operators defined on L_∞, or more generally on $C(K)$-spaces, the analogues of the last two results are Pietsch's Factorization Theorem for p-summing operators (see, for instance, [27, Theorem 9.2]) and Pisier's generalization for $(p, 1)$-summing operators [23, Theorem 2.4], since every operator defined on a $C(K)$-space is (p, q)-concave if and only if it is (p, q)-summing. Maurey's and Pisier's Factorization Theorems can be used, for instance, to prove the following representation result [8, Proposition 3.3], which generalizes the well-known representation of any Banach lattice as a sublattice of an ℓ_∞-sum of L_1 spaces (see [18, Lemma 3.4]). Throughout the paper, by the ℓ_∞-*sum of spaces* $(X_\gamma)_{\gamma \in \Gamma}$ we mean the space

$$\left(\bigoplus_{\gamma \in \Gamma} X_\gamma\right)_\infty := \left\{ x = (x_\gamma)_{\gamma \in \Gamma} : x_\gamma \in X_\gamma \; \forall \gamma \in \Gamma, \|x\| := \sup_{\gamma \in \Gamma} \|x_\gamma\| < \infty \right\}.$$

Proposition 2.7. *Let X be a Banach lattice and $1 < p < \infty$:*

(1) If X is p-convex, there are a family Γ of probability measures and a lattice isomorphism

$$J : X \to \left(\bigoplus_{\mu \in \Gamma} L_p(\mu) \right)_\infty$$

such that $\|x\| \leqslant \|Jx\| \leqslant K^{(p)}(X)\|x\|$ for all $x \in X$.

(2) If X is (p, ∞)-convex, there are a family Γ of probability measures and a lattice isomorphism

$$J : X \to \left(\bigoplus_{\mu \in \Gamma} L_{p,\infty}(\mu) \right)_\infty$$

such that $\|x\| \leqslant \|Jx\| \leqslant \gamma_p K^{(p,\infty)}(X)\|x\|$ for all $x \in X$, where $\gamma_p = (p^*)^{\frac{1}{p^*}}$ is the constant from Theorem 2.4 associated to $r = 1$.

Clearly, every closed sublattice of a p-convex or (p, ∞)-convex Banach lattice inherits the corresponding property. Therefore, the above conditions characterize p-convexity and (p, ∞)-convexity. Similarly, one can characterize the p-concave and $(p, 1)$-concave Banach lattices as quotients of ℓ_1-sums of L_p and $L_{p,1}$-spaces, respectively. As a corollary of Proposition 2.7, we obtain an alternative proof of the fact that every (p, ∞)-convex Banach lattice (and, by [20, Theorem 2.8.7] every (p, ∞)-convex operator) is q-convex for every $q < p$, with a constant that explodes like $(p - q)^{-\frac{1}{p}}$ when $q \to p^-$. To our knowledge, there existed two proofs of this fact: The proof presented in [6, Corollary 16.7] exploits a connection between (r, s)-concave operators and (r, s)-summing operators on $C(K)$-spaces (see [6, Theorem 16.5]) to apply Pisier's generalization of Pietsch's factorization theorem for $(r, 1)$-summing operators ([23, Theorem 2.4], see also [27, Theorem 21.3]). On the other hand, the proof presented in [17, Theorem 1.f.7] uses a probabilistic argument. The argument we provide here also relies on a result of Pisier (namely Theorem 2.4, which corresponds to [23, Theorem 1.1] instead of [23, Theorem 2.4]), but has some advantages over the other proofs. First of all, it is more direct than the one presented in [6], as it does not require one to go through (p, q)-summing operators. On the other hand, this proof is conceptually simpler than that of [17], and does not require probabilistic techniques.

Corollary 2.8. *Let $1 < p < \infty$ and assume that X is a Banach lattice satisfying an upper p-estimate. Then it is q-convex for every $1 \leqslant q < p$ with constant $K^{(q)}(X) \leqslant \left(\frac{p}{p-q} \right)^{\frac{1}{q}} \gamma_p K^{(\uparrow p)}(X)$.*

Proof. First of all, by Proposition 2.7 we know that X lattice embeds with distortion $\gamma_p K^{(\uparrow p)}(X)$ into a Banach lattice of the form $\left(\bigoplus_{\mu \in \Gamma} L_{p,\infty}(\mu) \right)_\infty$, with μ a probability measure for every $\mu \in \Gamma$. Now, let us fix $1 \leqslant q < p$, and observe that by (2.5) each of the factors $L_{p,\infty}(\mu)$ is lattice isomorphic to $L_{p,\infty}^{[q]}(\mu)$, with distortion $\left(\frac{p}{p-q} \right)^{\frac{1}{q}}$. Now, it suffices to show that $L_{p,\infty}^{[q]}(\mu)$ is q-convex with constant 1: Given $f_1, \dots, f_n \in L_{p,\infty}^{[q]}(\mu)$ and a measurable set A with $\mu(A) > 0$, it follows that

$$\mu(A)^{\frac{1}{p}-\frac{1}{q}} \left(\int_A \sum_{i=1}^n |f_i|^q \, d\mu \right)^{\frac{1}{q}} = \left(\sum_{i=1}^n \left(\mu(A)^{\frac{1}{p}-\frac{1}{q}} \left(\int_A |f_i|^q \, d\mu \right)^{\frac{1}{q}} \right)^q \right)^{\frac{1}{q}} \leqslant \left(\sum_{i=1}^n \|f\|_{L_{p,\infty}^{[q]}(\mu)}^q \right)^{\frac{1}{q}} .$$

Taking the supremum over all the sets A, we obtain

$$\left\|\left(\sum_{i=1}^{n}|f_i|^q\right)^{\frac{1}{q}}\right\|_{L_{p,\infty}^{[q]}(\mu)} \leqslant \left(\sum_{i=1}^{n}\|f\|_{L_{p,\infty}^{[q]}(\mu)}^q\right)^{\frac{1}{q}}.$$

Since q-convexity is preserved under taking ℓ_∞-sums, sublattices and isomorphisms, we conclude that X is q-convex with constant $K^{(q)}(X) \leqslant \left(\frac{p}{p-q}\right)^{\frac{1}{q}}\gamma_p K^{(\uparrow p)}(X)$. $\qquad\square$

Remark 2.9. Observe that the previous argument shows that for every $1 \leqslant q < p$, $L_{p,\infty}$ endowed with the norm $\|\cdot\|_{L_{p,\infty}^{[q]}}$ is q-convex with constant 1, while it simultaneously satisfies an upper p-estimate with constant 1. In general, by the above results, obtaining information on the structure of $\left(\bigoplus_{\mu\in\Gamma} L_{p,\infty}(\mu)\right)_\infty$ will have implications for arbitrary Banach lattices with upper p-estimates. In particular, a direct proof of the fact that $L_{p,\infty}$ is (p,q)-convex for every $q > p$ suffices to show that every (p,∞)-convex operator is (p,q)-convex following the same scheme of proof as in Corollary 2.8. We note that this strategy is part of a more general line of reasoning; for example, understanding the ideals of universal spaces was used in [3] to deduce information about separably injective Banach lattices.

The following example illustrates that the reverse implication in Corollary 2.8 is not true:

Example 2.10. Let $1 < p < \infty$. The space $X = \ell_{p,\infty}(\ell_p)$, formed by the sequences of vectors $\overline{x} = (x_k)_{k\in\mathbb{N}}$ such that $x_k \in \ell_p$ for every $k \in \mathbb{N}$ and $\|\overline{x}\|_X := \|(\|x_k\|_{\ell_p})_k\|_{\ell_{p,\infty}}$ is finite, does not satisfy an upper p-estimate, but it is q-convex with constant $\left(\frac{p}{p-q}\right)^{\frac{1}{q}}$ for every $1 \leqslant q < p$.

Proof. We start by showing that X is q-convex for any $1 \leqslant q < p$. Recall that both ℓ_p and $\ell_{p,\infty}$ are q-convex with constants 1 and $\left(\frac{p}{p-q}\right)^{\frac{1}{q}}$, respectively. Given $\overline{x}^{(i)} = (x_k^{(i)})_{k\in\mathbb{N}} \in X$, $i = 1, \ldots, n$, we have

$$\left\|\left(\sum_{i=1}^{n}|\overline{x}^{(i)}|^q\right)^{\frac{1}{q}}\right\|_X = \left\|\left(\left(\sum_{i=1}^{n}|x_k^{(i)}|^q\right)^{\frac{1}{q}}\right)_k\right\|_X = \left\|\left(\left\|\left(\sum_{i=1}^{n}|x_k^{(i)}|^q\right)^{\frac{1}{q}}\right\|_{\ell_p}\right)_k\right\|_{\ell_{p,\infty}}$$

$$\leqslant \left\|\left(\left(\sum_{i=1}^{n}\|x_k^{(i)}\|_{\ell_p}^q\right)^{\frac{1}{q}}\right)_k\right\|_{\ell_{p,\infty}} = \left\|\left(\sum_{i=1}^{n}\left|\left(\|x_k^{(i)}\|_{\ell_p}\right)_k\right|^q\right)^{\frac{1}{q}}\right\|_{\ell_{p,\infty}}$$

$$\leqslant \left(\frac{p}{p-q}\right)^{\frac{1}{q}}\left(\sum_{i=1}^{n}\left\|\left(\|x_k^{(i)}\|_{\ell_p}\right)_k\right\|_{\ell_{p,\infty}}^q\right)^{\frac{1}{q}} = \left(\frac{p}{p-q}\right)^{\frac{1}{q}}\left(\sum_{i=1}^{n}\|\overline{x}^{(i)}\|_X^q\right)^{\frac{1}{q}},$$

where the first and second inequalities follow from the q-convexity of ℓ_p and $\ell_{p,\infty}$, respectively.

To show that X does not satisfy an upper p-estimate (i.e., it is not (p,∞)-convex), let us assume that the coordinates in ℓ_p and $\ell_{p,\infty}$ are indexed starting at 0. Let us fix $n \geqslant 1$ and denote by $(e_j)_{j=0}^{\infty}$ the canonical basis of ℓ_p. For $k = 0, \ldots, n-1$, we write $a^{(k)} = \sum_{j=0}^{n-1}\alpha_{(k+j)_n}e_j$ for the cyclic permutations of the coefficients $\alpha_k = (k+1)^{\frac{1}{p^*}} - k^{\frac{1}{p^*}}$ over the set $\{0, \ldots, n-1\}$. Here, by i_n we

mean $i \mod n$. We consider for $i = 0, \ldots, n-1$ the (finite) sequences $\overline{x}^{(i)} = (x_k^{(i)})_{k=0}^\infty \in X$ given by $x_k^{(i)} = \alpha_{(k+i)_n} e_i$ for $k = 0, \ldots, n-1$ and $x_k^{(i)} = 0$ for $k \geqslant n$. It is clear that for every i,

$$\|\overline{x}^{(i)}\|_X = \|(\alpha_{(k+i)_n})_{k=0}^{n-1}\|_{\ell_{p,\infty}} = \|(\alpha_k)_{k=0}^{n-1}\|_{\ell_{p,\infty}} = 1, \quad \text{so} \quad \left(\sum_{i=0}^{n-1} \|\overline{x}^{(i)}\|_X^p\right)^{\frac{1}{p}} = n^{\frac{1}{p}}.$$

On the other hand, for every $k = 0, \ldots, n-1$,

$$\bigvee_{i=0}^{n-1} |x_k^{(i)}| = \sum_{i=0}^{n-1} \alpha_{(k+i)_n} e_i = a^{(k)},$$

so

$$\left\|\bigvee_{i=0}^{n-1} |\overline{x}^{(i)}|\right\|_X = \left\|\left(\left\|\bigvee_{i=0}^{n-1} |x_k^{(i)}|\right\|_{\ell_p}\right)_{k=0}^{n-1}\right\|_{\ell_{p,\infty}} = \left\|\left(\|a^{(k)}\|_{\ell_p}\right)_{k=0}^{n-1}\right\|_{\ell_{p,\infty}} = A_n n^{\frac{1}{p}},$$

where $A_n = \left(\sum_{j=0}^{n-1} \alpha_j^p\right)^{\frac{1}{p}}$ is the norm of $a^{(k)}$ in ℓ_p for every k, and $n^{\frac{1}{p}}$ is the norm in $\ell_{p,\infty}$ of the constant 1 vector on the set $\{0, \ldots, n-1\}$. It is not difficult to check that A_n diverges as $n \to \infty$. More specifically, $\alpha_k \approx (k+1)^{-\frac{1}{p}}$ uniformly in k, so $A_n \approx \left(\sum_{k=1}^n k^{-1}\right)^{\frac{1}{p}}$, which diverges. To conclude the argument, observe that the quotients $\left\|\bigvee_{i=0}^{n-1} |\overline{x}^{(i)}|\right\|_X / \left(\sum_{i=0}^{n-1} \|\overline{x}^{(i)}\|_X^p\right)^{\frac{1}{p}} = A_n$ are not uniformly bounded, so X cannot be (p, ∞)-convex. $\qquad\square$

Remark 2.11. Observe that for the previous argument to hold, it suffices to consider $\ell_{p,\infty}(\ell_p^n)_{n\in\mathbb{N}}$, an $\ell_{p,\infty}$-sum of ℓ_p^n with $n \in \mathbb{N}$. However, it is essential that the dimensions of the inner terms in the sum increase.

3. BANACH LATTICES WITH UPPER p-ESTIMATES VS. SUBLATTICES OF ℓ_∞-SUMS OF $L_{p,\infty}$

Proposition 2.7 points out a fundamental difference between p-convexity and upper p-estimates: While the class of p-convex Banach lattices (with constant 1) and the class of closed sublattices of ℓ_∞-sums of L_p-spaces can be identified isometrically, the class of Banach lattices with upper p-estimates (with constant 1) and the class of closed sublattices of ℓ_∞-sums of $L_{p,\infty}$-spaces do not coincide isometrically, but only isomorphically. Indeed, in this section, we provide an example of a Banach lattice that satisfies an upper p-estimate with constant 1 but does not embed isometrically into any ℓ_∞-sum of $L_{p,\infty}$-spaces. Moreover, we show that $L_{p,\infty}$ endowed with the different renormings from (2.5) can always be isometrically embedded into an ℓ_∞-sum of $L_{p,\infty}^{[1]}$, concluding that the choice of norm for $L_{p,\infty}$ is not the cause of the obstruction.

3.1. **A finite dimensional counterexample.** A natural question that arises from Theorem 2.4 and Proposition 2.7 is whether the constant γ_p can be improved to 1. In this section, we will show that the answer is generally negative.

Let X, Y be Banach lattices and let $1 \leqslant C < \infty$. We say that X C-lattice embeds into Y if there exists a lattice isomorphic injection $T : X \to Y$ such that $C^{-1}\|x\| \leqslant \|Tx\| \leqslant \|x\|$ for all $x \in X$. In this case, we call T a C-lattice embedding. If $C = 1$, then T is a lattice isometric embedding. A Banach space with a normalized 1-unconditional basis can be regarded as a Banach lattice with respect to the coordinate-wise order.

Theorem 3.1. *Let X be a Banach lattice with the order given by a normalized 1-unconditional basis (e_i). Denote the biorthogonal functionals by (e_i^*). For any $C \geqslant 1$, the following statements are equivalent:*

(1) For each normalized finitely supported $a = \sum_{i=1}^n a_i e_i \in X_+$ and every $\varepsilon > 0$, there exist $b = \sum_{i=1}^n b_i e_i^* \in X_+^*$ and $d = (d_i)_{i=1}^n \in \mathbb{R}_+^n$ so that $\sum_{i=1}^n a_i b_i > 1 - \varepsilon$, $\sum_{i=1}^n d_i = 1$ and $\|\sum_{i \in I} b_i e_i^*\|^{p^*} \leqslant C^{p^*} \sum_{i \in I} d_i$ for any $I \subseteq \{1, \ldots, n\}$.

(2) There is a set Γ of (probability) measures such that X C-lattice embeds into the space $\left(\bigoplus_{\mu \in \Gamma} L_{p,\infty}(\mu)\right)_\infty$.

Proof. $(1) \Longrightarrow (2)$ Let $a \in X_+$ be normalized and finitely supported, and let $\varepsilon > 0$ be given. Obtain b and d from (1). Let $(U_i)_{i=1}^n$ be a measurable partition of $(0,1)$ such that $\lambda(U_i) = d_i$, $1 \leqslant i \leqslant n$, where λ denotes the Lebesgue measure. Define

$$S_{a,\varepsilon} : X \to L_{p,\infty}(0,1) \text{ by } S_{a,\varepsilon}\left(\sum_{i=1}^\infty c_i e_i\right) = \sum_{i=1}^n \frac{c_i b_i \chi_{U_i}}{d_i}.$$

We have

$$\|S_{a,\varepsilon} a\| \geqslant \lambda(0,1)^{-\frac{1}{p^*}} \int_0^1 |S_{a,\varepsilon} a| \, d\lambda = \sum_{i=1}^n a_i b_i > 1 - \varepsilon.$$

On the other hand, if $\|\sum_{i=1}^\infty c_i e_i\| \leqslant 1$ then there exists $I \subseteq \{1, \ldots, n\}$ so that

$$\|S_{a,\varepsilon} c\| = \lambda\left(\bigcup_{i \in I} U_i\right)^{-\frac{1}{p^*}} \int_{\bigcup_{i \in I} U_i} |S_{a,\varepsilon} c| \, d\lambda = \left(\sum_{i \in I} d_i\right)^{-\frac{1}{p^*}} \sum_{i \in I} |c_i| b_i$$

$$= \left(\sum_{i \in I} d_i\right)^{-\frac{1}{p^*}} \left\langle \sum_{i=1}^\infty |c_i| e_i, \sum_{i \in I} b_i e_i^* \right\rangle \leqslant \left(\sum_{i \in I} d_i\right)^{-\frac{1}{p^*}} \left\|\sum_{i \in I} b_i e_i^*\right\| \leqslant C.$$

Let Γ be the Cartesian product of the set of normalized finitely supported elements in X_+ and the interval $(0, \frac{1}{2})$. It is now clear that the map $T : X \to \left(\bigoplus_{(a,\varepsilon) \in \Gamma} L_{p,\infty}(0,1)\right)_\infty$ given by

$$Tc = (C^{-1} S_{a,\varepsilon} c)_{(a,\varepsilon) \in \Gamma}$$

is a C-lattice embedding.

$(2) \Longrightarrow (1)$ Let $T : X \to \left(\bigoplus_{\mu \in \Gamma} L_{p,\infty}(\mu)\right)_\infty$ be a C-lattice embedding. Suppose that $a = \sum_{i=1}^n a_i e_i$ is normalized and $\varepsilon > 0$. For each $\mu_0 \in \Gamma$, let P_{μ_0} be the projection of $\left(\bigoplus_{\mu \in \Gamma} L_{p,\infty}(\mu)\right)_\infty$ onto the μ_0-th component. There exists μ_0 such that $C \|P_{\mu_0} Ta\| > 1 - \varepsilon$. Let $f_i = P_{\mu_0} Te_i$. Choose a μ_0-measurable set U such that $0 < \mu_0(U) < \infty$ and

$$C \int_U P_{\mu_0} Ta \, d\mu_0 > (1 - \varepsilon) \mu_0(U)^{\frac{1}{p^*}}.$$

Set $U_i = U \cap \operatorname{supp} f_i$ and $m = \left(\sum_{i=1}^n \mu_0(U_i)\right)^{\frac{1}{p^*}}$. Define

$$b_i = \frac{C \int_{U_i} f_i \, d\mu_0}{m}, \quad d_i = \frac{\mu_0(U_i)}{m^{p^*}}, \quad 1 \leqslant i \leqslant n.$$

We have $d_i \geqslant 0$, $\sum_{i=1}^{n} d_i = 1$ and

$$\sum_{i=1}^{n} a_i b_i = \frac{C}{m} \int_{\bigcup_{i=1}^{n} U_i} \sum_{i=1}^{n} a_i f_i \, d\mu_0 = \frac{C}{m} \int_U P_{\mu_0} T a \, d\mu_0 > 1 - \varepsilon.$$

On the other hand, suppose that $I \subseteq \{1, \ldots, n\}$. If $\|\sum_{i=1}^{\infty} c_i e_i\| \leqslant 1$, then

$$\left\langle \sum_{i=1}^{\infty} c_i e_i, \sum_{i \in I} b_i e_i^* \right\rangle = \sum_{i \in I} c_i b_i = \frac{C}{m} \int_{\bigcup_{i \in I} U_i} \sum_{i \in I} c_i f_i \, d\mu_0$$

$$\leqslant \frac{C}{m} \left\| P_{\mu_0} T \left(\sum_{i=1}^{\infty} c_i e_i \right) \right\| \mu_0 \left(\bigcup_{i \in I} U_i \right)^{\frac{1}{p^*}} \leqslant C \left(\sum_{i \in I} d_i \right)^{\frac{1}{p^*}}.$$

This proves that $\|\sum_{i \in I} b_i e_i^*\|^{p^*} \leqslant C^{p^*} \sum_{i \in I} d_i$. $\square$

Remark 3.2. If X is a two-dimensional Banach lattice that satisfies an upper p-estimate with constant 1, then both conditions (1) and (2) of Theorem 3.1 hold with $C = 1$. Indeed, let (e_1, e_2) be a normalized 1-unconditional basis for X. Suppose that $a = a_1 e_1 + a_2 e_2$ is normalized and non-negative. Choose $b = b_1 e_1^* + b_2 e_2^* \in X_+^*$ such that $\langle a, b \rangle = 1 = \|b\|$. Define $d_i = b_i^{p^*}/(b_1^{p^*} + b_2^{p^*})$, $i = 1, 2$. Note that X^* satisfies a lower p^*-estimate with constant 1 and e_1^* and e_2^* are disjoint, so

$$b_1^{p^*} + b_2^{p^*} = \|b_1 e_1^*\|^{p^*} + \|b_2 e_2^*\|^{p^*} \leqslant \|b\|^{p^*} = 1.$$

It is easy to check that $\|\sum_{i \in I} b_i e_i^*\|^{p^*} \leqslant \sum_{i \in I} d_i$ for any $I \subseteq \{1, 2\}$ and $d_1 + d_2 = 1$.

Corollary 3.3. *Suppose that X is a Banach lattice with the order given by a normalized 1-unconditional basis (e_i) that C-lattice embeds into $\left(\bigoplus_{\mu \in \Gamma} L_{p,\infty}(\mu) \right)_{\infty}$. For any normalized $b = \sum_{i=1}^{n} b_i e_i^* \in X_+^*$ and any $I_j \subseteq \{1, \ldots, n\}$, $1 \leqslant j \leqslant m$, such that $\sum_{j=1}^{m} \chi_{I_j} = l \chi_{\{1,\ldots,n\}}$ for some $l \in \mathbb{N}$, we have*

$$\sum_{j=1}^{m} \left\| \sum_{i \in I_j} b_i e_i^* \right\|^{p^*} \leqslant C^{p^*} l.$$

Proof. Choose a normalized $a = \sum_{i=1}^{n} a_i e_i \in X_+$ so that $\sum_{i=1}^{n} a_i b_i = 1$. First, assume that $[(e_i)_{i=1}^{n}]$ is uniformly convex. Let $\varepsilon > 0$ be given. There exists $\delta > 0$ so that if $b' = \sum_{i=1}^{n} b_i' e_i^* \in X_+^*$, $\|b'\| = 1$ and $\langle a, b' \rangle > 1 - \delta$, then $\|b - b'\| < \varepsilon$. By Theorem 3.1, there exists $b' \in X_+^*$ and $d = (d_i)_{i=1}^{n} \in \mathbb{R}_+^n$ so that $\sum_{i=1}^{n} d_i = 1$, $\langle a, b' \rangle > 1 - \delta$ and $\|\sum_{i \in I} b_i' e_i^*\|^{p^*} \leqslant C^{p^*} \sum_{i \in I} d_i$ for any $I \subseteq \{1, \ldots, n\}$. It follows that for any $I \subseteq \{1, \ldots, n\}$,

$$\left\| \sum_{i \in I} b_i e_i^* \right\| \leqslant \left\| \sum_{i \in I} |b_i - b_i'| e_i^* \right\| + \left\| \sum_{i \in I} b_i' e_i^* \right\| \leqslant \varepsilon |I| + C \left(\sum_{i \in I} d_i \right)^{\frac{1}{p^*}}.$$

Now assume that $I_j \subseteq \{1, \ldots, n\}$, $1 \leqslant j \leqslant m$, are such that $\sum_{j=1}^{m} \chi_{I_j} = l \chi_{\{1,\ldots,n\}}$ for some $l \in \mathbb{N}$. Set $A_j = \varepsilon |I_j|$ and $B_j = C(\sum_{i \in I_j} d_i)^{\frac{1}{p^*}}$. Then

$$\left(\sum_{j=1}^{m} \left\| \sum_{i \in I_j} b_i e_i^* \right\|^{p^*} \right)^{\frac{1}{p^*}} \leqslant \left(\sum_{j=1}^{m} (A_j + B_j)^{p^*} \right)^{\frac{1}{p^*}} \leqslant \left(\sum_{j=1}^{m} A_j^{p^*} \right)^{\frac{1}{p^*}} + \left(\sum_{j=1}^{m} B_j^{p^*} \right)^{\frac{1}{p^*}}$$

$$\leqslant \sum_{j=1}^{m} A_j + \left(\sum_{j=1}^{m} B_j^{p^*}\right)^{\frac{1}{p^*}} = \varepsilon \sum_{j=1}^{m} |I_j| + C \left(\sum_{j=1}^{m}\sum_{i\in I_j} d_i\right)^{\frac{1}{p^*}} \leqslant \varepsilon n l + C l^{\frac{1}{p^*}}.$$

Taking $\varepsilon \downarrow 0$ gives the desired result.

We now return to a general X with a normalized 1-unconditional basis (e_i). Given a normalized $b = \sum_{i=1}^{n} b_i e_i^* \in X_+^*$ and $\varepsilon > 0$, there is a uniformly convex lattice norm $\|\|\cdot\|\|$ on $[(e_i)_{i=1}^n]$ such that $\|x\| \leqslant \|\|x\|\| \leqslant (1+\varepsilon)\|x\|$ for any $x \in [(e_i)_{i=1}^n]$ (take, for instance, $\|\|\cdot\|\| = \|\cdot\| + \delta\|\cdot\|_{\ell_2^n}$ for $\delta > 0$ small enough, which is strictly convex and hence uniformly convex, since $[(e_i)_{i=1}^n]$ is finite dimensional). Set $u_i = e_i/\|\|e_i\|\|$, $u_i^* = e_i^*/\|\|e_i^*\|\|$ and $b' = \sum_{i=1}^{n} b_i' u_i^* = b/\|\|b\|\|$. Note that $(u_i)_{i=1}^n$ is a 1-unconditional basis for $[(e_i)_{i=1}^n] = [(u_i)_{i=1}^n]$ and b' is $\|\|\cdot\|\|$-normalized. Moreover, if $T : [(e_i)_{i=1}^n] \to \left(\bigoplus_{\mu\in\Gamma} L_{p,\infty}(\mu)\right)_\infty$ is a C-lattice embedding, then

$$\tilde{T}x = Tx : [(u_i)_{i=1}^n] \to \left(\bigoplus_{\mu\in\Gamma} L_{p,\infty}(\mu)\right)_\infty$$

is a $(1+\varepsilon)C$-lattice embedding. Assume that $I_j \subseteq \{1,\ldots,n\}$, $1 \leqslant j \leqslant m$, are such that $\sum_{j=1}^{m} \chi_{I_j} = l\chi_{\{1,\ldots,n\}}$ for some $l \in \mathbb{N}$. By the first part of the proof,

$$\sum_{j=1}^{m} \left\|\left\|\sum_{i\in I_j} b_i' u_i^*\right\|\right\|^{p^*} \leqslant (1+\varepsilon)^{p^*} C^{p^*} l.$$

Since $\|\|b\|\| \leqslant 1+\varepsilon$, for any $I \subseteq \{1,\ldots,n\}$ we have

$$\left\|\left\|\sum_{i\in I} b_i' u_i^*\right\|\right\| = \frac{1}{\|\|b\|\|}\left\|\left\|\sum_{i\in I} b_i e_i^*\right\|\right\| \geqslant \frac{1}{1+\varepsilon}\left\|\left\|\sum_{i\in I} b_i e_i^*\right\|\right\|.$$

Thus,

$$\sum_{j=1}^{m} \left\|\left\|\sum_{i\in I} b_i e_i^*\right\|\right\|^{p^*} \leqslant (1+\varepsilon)^{2p^*} C^{p^*} l.$$

Taking $\varepsilon \downarrow 0$ gives the desired result. $\square$

The next example shows that γ_p cannot be made to be 1 in Proposition 2.7 or Theorem 2.4.

Example 3.4. Suppose that $1 < p < \infty$. Denote by C_p the infimum of the set of constants C so that every Banach lattice satisfying an upper p-estimate with constant 1 C-lattice embeds into $\left(\bigoplus_{\gamma\in\Gamma} L_{p,\infty}(\mu)\right)_\infty$ for some set of measures Γ. Then $C_p^{p^*} \geqslant \frac{3\cdot 2^{p^*}}{2(1+2^{p^*})} > 1$.

Proof. Let $X = \mathbb{R}^3$ so that the norm on X^* is

$$\|(b_1, b_2, b_3)\|_{X^*} = \max\left\{\left(|b_i|^{p^*} + (|b_j| + |b_k|)^{p^*}\right)^{\frac{1}{p^*}} : \{i, j, k\} = \{1, 2, 3\}\right\}.$$

It is clear that the coordinate unit vectors form a normalized 1-unconditional basis for X. Suppose that $b = (b_1, b_2, b_3) \in X^*$ and I, J are disjoint non-empty subsets of $\{1, 2, 3\}$. Without loss of generality, we may assume that $I = \{i\}$ and $J = \{j, k\}$, where i, j, k are distinct. Then

$$\|b\chi_I\|_{X^*}^{p^*} + \|b\chi_J\|_{X^*}^{p^*} = |b_i|^{p^*} + (|b_j| + |b_k|)^{p^*} \leqslant \|b\|_{X^*}^{p^*}.$$

Thus, X^* satisfies a lower p^*-estimate with constant 1. Hence, X satisfies an upper p-estimate with constant 1. Suppose that X C-lattice embeds into some $\left(\bigoplus_{\gamma \in \Gamma} L_{p,\infty}(\mu)\right)_\infty$. The vector $(1 + 2^{p^*})^{-\frac{1}{p^*}}(1,1,1)$ is normalized in X_+^*. By Corollary 3.3,

$$\frac{1}{1 + 2^{p^*}}\left(\|(1,1,0)\|^{p^*} + \|(1,0,1)\|^{p^*} + \|(0,1,1)\|^{p^*}\right) \leqslant 2C^{p^*}.$$

Thus,

$$\frac{3 \cdot 2^{p^*}}{2(1 + 2^{p^*})} \leqslant C^{p^*}.$$

Taking infimum over the eligible C's gives the desired result. Direct verification shows that $\frac{3 \cdot 2^{p^*}}{2(1+2^{p^*})} > 1$ if $1 < p < \infty$. $\qquad\square$

Question 3.5. Is $\gamma_p = (p^*)^{\frac{1}{p^*}}$ the optimal value of the constant C_p defined in Example 3.4?

As a consequence of the previous discussion, we find that the free Banach lattices associated to the classes $\mathcal{C}_{p,\infty}$ of Banach lattices satisfying an upper p-estimate with constant 1, and $\mathcal{X}_{p,\infty}$ of $L_{p,\infty}(\mu)$ spaces, are not lattice isometric in general, even though they are lattice isomorphic (see [8]). Recall that, given a class $\mathcal{C}$ of Banach lattices and a Banach space E, there exists an (essentially unique) Banach lattice $\mathrm{FBL}^{\mathcal{C}}[E]$ together with a linear isometric embedding $\phi : E \to \mathrm{FBL}^{\mathcal{C}}[E]$ with the property that for every Banach lattice X in the class $\mathcal{C}$ and every bounded operator $T : E \to X$, there exists a unique extension $\widehat{T} : \mathrm{FBL}^{\mathcal{C}}[E] \to X$ as a lattice homomorphism such that $\widehat{T}\phi = T$ and $\|\widehat{T}\| = \|T\|$. This space is called the *free Banach lattice generated by E associated to the class $\mathcal{C}$*. In [8, Theorem 3.6] it was established that for every Banach space E, the norm of $\mathrm{FBL}^{\mathcal{C}_{p,\infty}}[E]$ is γ_p-equivalent to the norm of $\mathrm{FBL}^{\mathcal{X}_{p,\infty}}[E]$. However, we will see that this isomorphism constant cannot be made 1.

In order to show this claim, we need to recall the following property. A Banach lattice X is said to have the *isometric lattice-lifting property* if there exists a lattice isometric embedding $\alpha : X \to \mathrm{FBL}[X]$ such that $\widehat{id_X}\alpha = id_X$. This property was introduced in the article [2], inspired by previous work of Godefroy and Kalton [9] on Lipschitz-free spaces. Notably, Banach lattices ordered by a 1-unconditional basis have this property (see [2, Theorem 4.1]). In [21, Theorem 8.3] an alternative proof of this fact was given, which also worked when $\mathrm{FBL}[E]$ was replaced by $\mathrm{FBL}^{(p)}[E]$. We now show how to generalize these results to any class $\mathcal{C}$.

Proposition 3.6. *Let $\mathcal{C}$ be a class of Banach lattices, X a Banach space with a normalized 1-unconditional basis $(e_i)_i$, viewed as a Banach lattice with the order given by the basis, and $C \geqslant 1$. The following are equivalent.*

(1) X C-lattice embeds into some $\left(\bigoplus_{\gamma \in \Gamma} X_\gamma\right)_\infty$ with $X_\gamma \in \mathcal{C}$ for every $\gamma \in \Gamma$.

(2) For every operator $T : X \to X$, there exists a unique extension $\widehat{T} : \mathrm{FBL}^{\mathcal{C}}[X] \to X$ satisfying $\|\widehat{T}\| \leqslant C\|T\|$.

(3) There exists a unique extension $\widehat{id_X} : \mathrm{FBL}^{\mathcal{C}}[X] \to X$ of the identity $id_X : X \to X$ satisfying $\|\widehat{id_X}\| \leqslant C$.

(4) X C-lattice embeds into $\mathrm{FBL}^{\mathcal{C}}[X]$.

Moreover, the C-lattice embedding $\alpha : X \to \mathrm{FBL}^{\mathcal{C}}[X]$ in (4) can be chosen so that $\widehat{id_X}\alpha = id_X$.

Proof. (1) $\Rightarrow$ (2) is an adaptation of [8, Proposition 2.5] and (2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (4) Recall that by either [2, Theorem 4.1] or [21, Theorem 8.3] with $p = 1$ there is a lattice isometric embedding $\alpha_0 : X \to \mathrm{FBL}[X]$, the free Banach lattice (which can be seen as the free Banach lattice associated to the class of all Banach lattices), such that $\widehat{id_X}\alpha_0 = id_X$ (here, $\widehat{id_X} : \mathrm{FBL}[X] \to X$). Moreover, since $\mathcal{C}$ is a class of Banach lattices, the formal identity $j : \mathrm{FBL}[X] \to \mathrm{FBL}^{\mathcal{C}}[X]$ is a continuous norm one injection such that $\widehat{id_X}j\alpha_0 = id_X$ (now, we are abusing the notation by also denoting $\widehat{id_X} : \mathrm{FBL}^{\mathcal{C}}[X] \to X$ the extension of id_X to $\mathrm{FBL}^{\mathcal{C}}[X]$ given in (3)). We claim that $\alpha := j\alpha_0 : X \to \mathrm{FBL}^{\mathcal{C}}[X]$ defines a C-lattice embedding. To prove the claim, simply note that for every $x \in X$

$$\frac{1}{C}\|x\| \leqslant \frac{1}{\|\widehat{id_X}\|}\|\widehat{id_X}\alpha x\| \leqslant \|\alpha x\|_{\mathrm{FBL}^{\mathcal{C}}[X]} \leqslant \|\alpha_0 x\|_{\mathrm{FBL}[X]} \leqslant \|x\|.$$

This also proves the additional claim.

(4) $\Rightarrow$ (1) follows from [8, Proposition 2.7], as $\mathrm{FBL}^{\mathcal{C}}[X]$ embeds lattice isometrically into an ℓ_∞-sum of spaces in $\mathcal{C}$. $\square$

We can apply Proposition 3.6 to the Banach lattice X constructed in Example 3.4 to conclude that, since X does not 1-lattice embed into $\left(\bigoplus_{\mu\in\Gamma} L_{p,\infty}(\mu)\right)_\infty$ for any choice of measures Γ, the extension $\widehat{id_X}$ of the identity on X to $\mathrm{FBL}^{\mathcal{X}_{p,\infty}}[X]$ cannot be contractive. However, if we instead consider $\mathrm{FBL}^{\mathcal{C}_{p,\infty}}[X]$ (which by [8, Theorem 3.6] coincides as a set with $\mathrm{FBL}^{\mathcal{X}_{p,\infty}}[X]$, with an equivalent renorming), we observe that in this setting the norm of the extension $\widehat{id_X}$ becomes $\|id_X\| = 1$, since X belongs to the class $\mathcal{C}_{p,\infty}$. Therefore, the norms $\|\cdot\|_{\mathrm{FBL}^{\mathcal{X}_{p,\infty}}[X]}$ and $\|\cdot\|_{\mathrm{FBL}^{\mathcal{C}_{p,\infty}}[X]}$ cannot agree.

3.2. **The renormings of weak-L_p.** In this paper (and also in [8]), we have chosen to equip $L_{p,\infty}(\mu)$ with the norm

$$\|f\|_{L_{p,\infty}^{[1]}} := \sup_{0<\mu(A)<\infty} \mu(A)^{\frac{1}{p}-1} \int_A |f|\, d\mu.$$

However, for each $1 \leqslant r < p$, we could have equally chosen to equip $L_{p,\infty}(\mu)$ with the norm

$$\|f\|_{L_{p,\infty}^{[r]}} := \sup_{0<\mu(A)<\infty} \mu(A)^{\frac{1}{p}-\frac{1}{r}} \left(\int_A |f|^r\, d\mu\right)^{\frac{1}{r}},$$

as it also defines a lattice norm on $L_{p,\infty}(\mu)$ with upper p-estimate constant 1. For this reason, it is natural to ask whether the ambiguity in choosing a norm on $L_{p,\infty}(\mu)$ is the reason why γ_p cannot be chosen to be 1 in [8, Proposition 3.3 and Theorem 3.6].

The aim of this subsection is to show that $(L_{p,\infty}(\mu), \|\cdot\|_{L_{p,\infty}^{[r]}})$ embeds lattice isometrically into a Banach lattice of the form $\left(\bigoplus_{\mu\in\Gamma} L_{p,\infty}^{[1]}(\mu)\right)_\infty$. As a consequence, we deduce that the above renormings cannot eliminate the constant γ_p in [8, Proposition 3.3 and Theorem 3.6].

Recall that every Banach lattice with upper p-estimates is r-convex for every $1 \leqslant r < p$, with a constant that might explode like $(p-r)^{-\frac{1}{p}}$ as r approaches p. Moreover, every r-convex Banach lattice has an equivalent lattice norm whose r-convexity constant becomes 1 [17, Proposition 1.d.8]. As we saw in Remark 2.9, it turns out that in the case of $L_{p,\infty}$, we can use the $L_{p,\infty}^{[r]}$-renorming

to actually renorm $L_{p,\infty}$ so that the upper p-estimates constant and the r-convexity constant are simultaneously 1. Now that we have clarified the role of the renormings, we can proceed with the main goal of this section. We begin with some notation and technical lemmata.

Let $e = (1, \dots, 1) \in \mathbb{R}^n$ and let $(e_i)_{i=1}^n$ be the unit vector basis for $\mathbb{R}^n$. Let $\beta, b \in \mathbb{R}_+^n$, $\|\beta\|_1, \|b\|_1 \leqslant 1$, and let $0 < s < 1$. Define $\alpha : \mathbb{R}_+^n \to \mathbb{R}$ by

$$(3.1) \qquad \alpha(x) = (\beta \cdot x)^{1-s} (b \cdot x)^s,$$

where $\cdot$ is the dot product on $\mathbb{R}^n$. Clearly, α is positively homogeneous.

Lemma 3.7. *There exists $d = (d_i)_{i=1}^n \in \mathbb{R}_+^n$ with $\|d\|_1 \leqslant 1$ such that $\alpha(x) \leqslant d \cdot x$ for any $x \in \mathbb{R}_+^n$.*

Proof. Let $d = (1-s)\beta + sb$, so that $d \in \mathbb{R}_+^n$ and $\|d\|_1 \leqslant 1$. Given $x \in \mathbb{R}_+^n$, by Young's inequality for products, we have that

$$\alpha(x) = (\beta \cdot x)^{1-s} (b \cdot x)^s \leqslant (1-s)(\beta \cdot x) + s(b \cdot x) = ((1-s)\beta + sb) \cdot x = d \cdot x.$$

$\square$

Lemma 3.8. *Let (Ω, Σ, μ) be a measure space. Suppose that $(a_i)_{i=1}^n \in \mathbb{R}_+^n$ and $(U_i)_{i=1}^n$ is a disjoint sequence in Σ with $0 < \mu(U_i) < \infty$ so that $C := (\sum_{i=1}^n \mu(U_i))^{\frac{1}{p} - \frac{1}{r}} \| \sum_{i=1}^n a_i \chi_{U_i} \|_{L_r(\mu)} \leqslant 1$. There is a measure ν on Σ and a lattice homomorphism $S : L_{p,\infty}^{[r]}(\mu) \to L_{p,\infty}^{[1]}(\nu)$ so that $\|S\| \leqslant 1$ and $\|S(\sum_{i=1}^n a_i \chi_{U_i})\| \geqslant C^r$.*

Proof. Define $M = \sum_{i=1}^n \mu(U_i)$, $b_i = \frac{\mu(U_i)}{M}$, $b = (b_i)_{i=1}^n$ and $\beta_i = M^{\frac{r}{p}-1} \mu(U_i) a_i^r$, $\beta = (\beta_i)_{i=1}^n$. Clearly, $\beta, b \in \mathbb{R}_+^n$ and $\|b\|_1 = 1$. Moreover,

$$\|\beta\|_1 = M^{\frac{r}{p}-1} \left\| \sum_{i=1}^n a_i \chi_{U_i} \right\|_{L_r(\mu)}^r = C^r \leqslant 1.$$

Take $s = p^*(\frac{1}{r} - \frac{1}{p})$ and define α as in (3.1). Note that $\alpha(e) = \|\beta\|_1^{1-s} \|b\|_1^s = C^{r(1-s)} \leqslant 1$. Let d be given by Lemma 3.7. Define a measure ν on Σ by $\nu(A) = \sum_{i=1}^n \frac{d_i \mu(A \cap U_i)}{\mu(U_i)}$ and a linear operator

$$S : L_{p,\infty}^{[r]}(\mu) \to L_{p,\infty}^{[1]}(\nu) \quad \text{by} \quad Sf = M^{\frac{r}{p}} \sum_{i=1}^n \frac{a_i^{r-1} b_i}{d_i} f \chi_{U_i}.$$

Clearly, S is a lattice homomorphism. Note that $\nu(\bigcup_{i=1}^n U_i) = \|d\|_1 \leqslant 1$. Hence,

$$\left\| S \left(\sum_{i=1}^n a_i \chi_{U_i} \right) \right\| \geqslant M^{\frac{r}{p}} \left\| \sum_{i=1}^n \frac{a_i^r b_i}{d_i} \chi_{U_i} \right\|_{L_1(\nu)} = M^{\frac{r}{p}} \sum_{i=1}^n a_i^r b_i$$

$$= M^{\frac{r}{p}-1} \left\| \sum_{i=1}^n a_i \chi_{U_i} \right\|_{L_r(\mu)}^r = C^r.$$

On the other hand, let $f \in L_{p,\infty}^{[r]}(\mu)$ with $\|f\| \leqslant 1$. If $A \in \Sigma$ with $0 < \nu(A) < \infty$ then

$$\int_A |Sf| \, d\nu = M^{\frac{r}{p}} \sum_{i=1}^n \frac{a_i^{r-1} b_i}{d_i} \int_{A \cap U_i} |f| \, d\nu = M^{\frac{r}{p}-1} \sum_{i=1}^n a_i^{r-1} \int_{A \cap U_i} |f| \, d\mu$$

$$\leqslant M^{\frac{r}{p}-1} \left\| \sum_{i=1}^{n} a_i^{r-1} \chi_{A \cap U_i} \right\|_{L_{r*}(\mu)} \left\| f \chi_{\bigcup_{i=1}^{n}(A \cap U_i)} \right\|_{L_r(\mu)}$$

$$\leqslant \left(\sum_{i=1}^{n} \beta_i \frac{\mu(A \cap U_i)}{\mu(U_i)} \right)^{\frac{1}{r*}} \left(\sum_{i=1}^{n} b_i \frac{\mu(A \cap U_i)}{\mu(U_i)} \right)^{\frac{1}{r}-\frac{1}{p}}.$$

Let $x = \left(\frac{\mu(A \cap U_i)}{\mu(U_i)} \right)_{i=1}^{n} \in \mathbb{R}_+^n$. Since $1 - s = \frac{p*}{r*}$, it follows from Lemma 3.7 that

$$\int_A |Sf| \, d\nu \leqslant (\beta \cdot x)^{\frac{1-s}{p*}} (b \cdot x)^{\frac{s}{p*}} = \alpha(x)^{\frac{1}{p*}} \leqslant (x \cdot d)^{\frac{1}{p*}} = \nu(A)^{\frac{1}{p*}}.$$

Hence, $\|Sf\| \leqslant 1$. $\qquad\square$

We are now in disposition of proving the claim that we made at the beginning of the subsection.

Theorem 3.9. *Suppose that (Ω, Σ, μ) is a measure space and $1 < r < p$. There is a set Γ of measures on Σ so that $L_{p,\infty}^{[r]}(\mu)$ embeds lattice isometrically into $\left(\bigoplus_{\nu \in \Gamma} L_{p,\infty}^{[1]}(\nu) \right)_\infty$.*

Proof. Let Γ be the set of simple functions $f = \sum_{i=1}^{n} a_i \chi_{U_i}$, where $n \in \mathbb{N}$, $a_i > 0$, $0 < \mu(U_i) < \infty$ and $(U_i)_{i=1}^{n}$ is a disjoint sequence in Σ with

$$C_f = \left(\sum_{i=1}^{n} \mu(U_i) \right)^{\frac{1}{p}-\frac{1}{r}} \left\| \sum_{i=1}^{n} a_i \chi_{U_i} \right\|_r \leqslant 1.$$

For each $f \in \Gamma$, according to Lemma 3.8, there is a lattice homomorphism $S_f : L_{p,\infty}^{[r]}(\mu) \to L_{p,\infty}^{[1]}(\nu_f)$ such that $\|S_f\| \leqslant 1$ and $\|S_f f\| \geqslant C_f^r$. Define

$$T : L_{p,\infty}^{[r]}(\mu) \to \left(\bigoplus_{f \in \Gamma} L_{p,\infty}^{[1]}(\nu_f) \right)_\infty \qquad \text{by} \quad Tg = (S_f g)_{f \in \Gamma}.$$

Clearly, T is a lattice homomorphism and $\|T\| \leqslant 1$. Suppose that $g \in L_{p,\infty}^{[r]}(\mu)_+$ with $\|g\|_{L_{p,\infty}^{[r]}} = 1$. For any $\varepsilon > 0$, there exists a non-negative simple function h such that $g \geqslant h$ and $\|h\|_{L_{p,\infty}^{[r]}} \geqslant 1 - \varepsilon$. Write $h = \sum_{i=1}^{n} a_i \chi_{U_i}$. There exists $I \subseteq \{1, \dots, n\}$ so that

$$\left(\sum_{i \in I} \mu(U_i) \right)^{\frac{1}{p}-\frac{1}{r}} \left\| \sum_{i \in I} a_i \chi_{U_i} \right\|_r = \|h\|_{L_{p,\infty}^{[r]}} \geqslant 1 - \varepsilon.$$

Then $f := \sum_{i \in I} a_i \chi_{U_i} \in \Gamma$ with $C_f \geqslant 1 - \varepsilon$. Thus,

$$\|Tg\| \geqslant \|S_f g\| \geqslant \|S_f f\| \geqslant C_f^r \geqslant (1 - \varepsilon)^r,$$
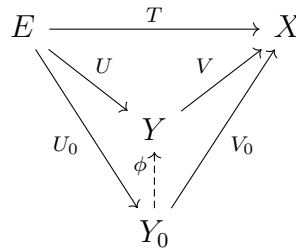
so that $\|Tg\| \geqslant 1$. This shows that T is an isometric embedding. $\qquad\square$

4. Factorization of convex and concave operators

There are several results in the literature that provide methods of factoring (p, q)-convex/concave operators through (p, q)-convex/concave Banach lattices. Namely, in [25, Proposition 5] it is shown that every q-concave operator factors through a q-concave Banach lattice, and the bi-adjoint of every p-convex operator factors through a p-convex Banach lattice. In [20, Theorem 2.8.7] one can find analogous statements for cone q-absolutely summing operators and q-superadditive Banach lattices

(i.e., $(q, 1)$-concave operators and Banach lattices with lower q-estimates) and for p-majorizing operators and p-subadditive Banach lattices (i.e., (p, ∞)-convex operators and Banach lattices with upper p-estimates). Finally, in [24], the approach of [20] was further explored to cover the setting of q-concavity and p-convexity, establishing the existence of canonical factorizations for q-concave and p-convex operators that satisfy certain maximality/minimality properties, as well as a duality theory for such factorizations.

The aim of this section is to extend these results to an abstract setting of operators satisfying general convexity and concavity conditions, recovering as a particular case the analogues of [24] for (p, ∞)-convex and $(q, 1)$-concave operators. We begin by considering operators whose domain or target space is a Banach lattice, which include concave and convex operators as particular cases, respectively. Namely, in Section 4.1 we identify suitable classes of factorizations for these operators in terms of the sets that they induce in the corresponding Banach lattice, and we devote Section 4.2 to introducing two types of convex sets associated to such operators and investigating their duality relations (see Theorem 4.10). Next, in Section 4.3, we define (τ, σ)-convex and concave operators and Banach lattices as a general version of Definition 2.1 and use the sets introduced in the previous subsection to establish that such operators admit factorizations through such Banach lattices. These factorizations also satisfy certain minimality and maximality properties with respect to the classes of factorizations introduced in Section 4.1, as was the case in [24, Section 2] (see Theorems 4.17 and 4.19). In particular, we improve [20, Theorem 2.8.7] in the following way:

Theorem 4.1. *Let E be a Banach space and X a Banach lattice.*

(1) *Let $T : E \to X$ be a (p, ∞)-convex operator. There exist a (p, ∞)-convex Banach lattice Y_0 with constant 1, and operators $U_0 : E \to Y_0$ and $V_0 : Y_0 \to X$, so that V_0 is an injective, interval preserving lattice homomorphism such that $V_0(B_{Y_0})$ is closed in X (i.e., V_0 is of Class $\mathcal{C}$) and $T = V_0 U_0$. Moreover, if there exist a (p, ∞)-convex Banach lattice Y and operators $U : E \to Y$ and $V : Y \to X$ such that V is of Class $\mathcal{C}$ and $T = VU$, then there is a Class $\mathcal{C}$ operator $\phi : Y_0 \to Y$ so that $U = \phi U_0$ and $V_0 = V\phi$.*

$$
\begin{array}{ccc}
E & \xrightarrow{\quad T \quad} & X \\
\end{array}
$$

with operators U, V, U_0, V_0, ϕ, and object Y, Y_0.

(2) *Let $T : X \to E$ be a $(p, 1)$-concave operator. There exist a $(p, 1)$-concave Banach lattice Y^0 with constant 1, and operators $U^0 : X \to Y^0$ and $V^0 : Y^0 \to E$, so that U^0 is a lattice homomorphism with dense range (i.e., U^0 is of Class $\mathcal{D}$) and $T = V^0 U^0$. Moreover, if there exist a $(p, 1)$-concave Banach lattice Y and operators $U : X \to Y$ and $V : Y \to E$ such that U is of Class $\mathcal{D}$ and $T = VU$, then there is a Class $\mathcal{D}$ operator $\phi : Y \to Y^0$ so that $U^0 = \phi U$ and $V^0 \phi = V$.*

$$
\begin{array}{ccc}
X & \xrightarrow{\quad T \quad} & E \\
\end{array}
$$
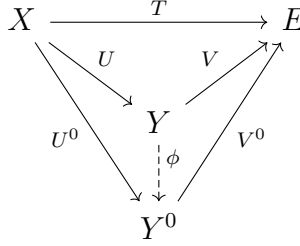
Moreover, we show in Section 4.4 that these canonical factorizations satisfy a duality relation similar to the one presented in [24, Theorem 5]. Finally, in Section 4.5, we apply the abstract theory developed in the previous sections to factor operators which are simultaneously (p, p_2)-convex and (q, q_2)-concave, for $p_2 \in \{p, \infty\}$ and $q_2 \in \{1, q\}$. The case $p_2 = p$ and $q_2 = q$ was already covered in [24, Theorem 15], whereas Theorems 4.27 and 4.31 cover the remaining three cases.

4.1. **Geometry of factorization.** We start by establishing suitable classes of factorizations for convex and concave operators. The definitions are motivated by results in [24]. Let X, Y be Banach lattices and let E be a Banach space. Recall that a positive linear operator $T : X \to Y$ is *interval preserving*, respectively *almost interval preserving*, if $T[0, x] = [0, Tx]$, respectively, $T[0, x]$ is norm dense in $[0, Tx]$, for any $x \in X_+$ [20, Definition 1.4.18]. A linear operator $T : X \to Y$ is of

(1) *Class $\mathcal{C}$* if T is an injective, interval preserving lattice homomorphism such that $T(B_X)$ is closed in Y. Note that T is necessarily bounded, and hence $T(B_X)$ is bounded in Y.

(2) *Class $\mathcal{D}$* if T is a lattice homomorphism with dense range. Note that in particular T is almost interval preserving.

Let $T : E \to X$ be a bounded linear operator. A triple (Y, U, V) is a *Class $\mathcal{C}$ factorization* of T if Y is a Banach lattice, $U : E \to Y$ and $V : Y \to X$ are bounded linear operators so that V is of class $\mathcal{C}$ and $T = VU$. A triple (Y, U, V) is a *Class $\mathcal{D}$ factorization* of a bounded linear operator $T : X \to E$ if Y is a Banach lattice, $U : X \to Y$ and $V : Y \to E$ are bounded linear operators so that U is of class $\mathcal{D}$ and $T = VU$. Two Class $\mathcal{C}$, respectively Class $\mathcal{D}$, factorizations (Y_i, U_i, V_i), $i = 1, 2$, of T are *equivalent* if there is a Banach lattice isomorphism $\iota : Y_1 \to Y_2$ so that $U_2 = \iota U_1$ and $V_1 = V_2 \iota$. Two closed bounded subsets B_1, B_2 of E are *equivalent* if there is a positive constant c so that $c^{-1} B_1 \subseteq B_2 \subseteq c B_1$. As we will see in this section, there is a correspondence between Class $\mathcal{C}$ factorizations of an operator $T : E \to X$ and subsets of X satisfying certain properties, and a similar situation operates for Class $\mathcal{D}$ factorizations. Before proceeding with the precise statements, let us recall the following well-known fact:

Lemma 4.2. *Let E be a Banach space and let B be a closed bounded convex circled subset of E. Then the Minkowski functional ρ of B defines a complete norm on $\operatorname{span} B$ so that B is the closed ball with respect to ρ.*

Proof. Let C be a finite constant so that $B \subseteq C B_E$. Recall that the Minkowski functional of B is given by $\rho(y) = \inf\{\lambda > 0 : y \in \lambda B\}$ for every $y \in E$ ($y \notin \operatorname{span} B$ if and only if $\rho(y) = \infty$). If $y \in \operatorname{span} B$ and $\rho(y) = 0$, then

$$
y \in \bigcap_{\lambda > 0} \lambda B \subseteq \bigcap_{\lambda > 0} \lambda C B_E = \{0\}.
$$

Hence ρ is a norm on $\operatorname{span} B$.

Suppose (y_n) is a sequence in span B so that $\sum \rho(y_n) < 1$. Then there exists a sequence (λ_n) so that $y_n \in \lambda_n B$ for all n and $\sum \lambda_n < 1$. In particular, $\|y_n\| \leqslant \lambda_n C$ and hence $\sum \|y_n\| < C$. So $\sum y_n$ converges in E to some y. Since

$$\sum_{n=1}^{m} y_n \in \sum_{n=1}^{m} \lambda_n \cdot B \subseteq B$$

for all m and B is closed, $y \in B$. If $k > m$,

$$\sum_{n=m+1}^{k} y_n \in \sum_{n=m+1}^{k} \lambda_n \cdot B \subseteq \sum_{n=m+1}^{\infty} \lambda_n \cdot B.$$

We take the limit as $k \to \infty$ in the space E. Then $y - \sum_{n=1}^{m} y_n \in \sum_{n=m+1}^{\infty} \lambda_n \cdot B$. Hence, $\rho(y - \sum_{n=1}^{m} y_n) \to 0$ as $m \to \infty$. This proves that ρ is complete.

If $y \in B$, then obviously $\rho(y) \leqslant 1$. Conversely, suppose that $\rho(y) \leqslant 1$. For any $\lambda > 1$, $\frac{y}{\lambda} \in B$. Since B is closed in E, $y \in B$, as required. $\qquad \square$

Regarding Class $\mathcal{C}$ factorizations, we have the following correspondence:

Proposition 4.3. *Let $T : E \to X$ be a nonzero bounded linear operator.*

(1) *The map $(Y, U, V) \mapsto V(B_Y)$ is a surjective correspondence between the set of Class $\mathcal{C}$ factorizations of T and the set of closed bounded convex solid subsets of X that contain $cT(B_E)$ for some $c > 0$.*

(2) *Let (Y_i, U_i, V_i), $i = 1, 2$, be two Class $\mathcal{C}$ factorizations of T. The following are equivalent:*

(a) *There is a bounded linear map $\phi : Y_1 \to Y_2$ such that $U_2 = \phi U_1$ and $V_1 = V_2 \phi$. In this case, ϕ is necessarily of Class $\mathcal{C}$.*

(b) *There is a positive constant c so that $V_1(B_{Y_1}) \subseteq cV_2(B_{Y_2})$.*

As a result, two Class $\mathcal{C}$ factorizations (Y_1, U_1, V_1) and (Y_2, U_2, V_2) are equivalent if and only if $V_1(B_{Y_1})$ and $V_2(B_{Y_2})$ are equivalent.

Proof. (1) If (Y, U, V) is a Class $\mathcal{C}$ factorization of T, then V is of Class $\mathcal{C}$ and hence $V(B_Y)$ is a closed bounded convex and solid subset of X. Since $T = VU$ is nonzero, $T(B_E) = VU(B_E) \subseteq \|U\|V(B_Y)$. Thus, $V(B_Y)$ contains $cT(B_E)$ with $c = \frac{1}{1+\|U\|} > 0$.

Conversely, suppose that B is a closed bounded convex solid subset of X that contains $cT(B_E)$ for some $c > 0$. Let $Y = \text{span } B$ and let ρ be the Minkowski functional of B on Y. Then (Y, ρ) is a Banach space by Lemma 4.2. Since B is solid, Y is a Banach lattice in the order inherited from X. Define $U : E \to Y$ by $Ux = Tx$. Since $T(E) \subseteq Y$, U is a well-defined linear operator. Moreover, $U(B_E) = T(B_E) \subseteq \frac{1}{c}B$. Hence U is bounded. Let $V : Y \to X$ be the formal inclusion. Then V is of class $\mathcal{C}$. Clearly, $T = VU$. Thus, (Y, U, V) is a Class $\mathcal{C}$ factorization of T.

We have shown that the correspondence $(Y, U, V) \mapsto V(B_Y)$ is well defined and surjective between the stated sets. Let us verify injectivity. Assume that (Y_i, U_i, V_i) are Class $\mathcal{C}$ factorizations so that $V_1(B_{Y_1}) = V_2(B_{Y_2})$. Then, since V_2 is an injective lattice homomorphism, the operator $\iota = V_2^{-1}V_1 : Y_1 \to Y_2$ is a well-defined lattice homomorphism and $\iota B_{Y_1} = B_{Y_2}$, so it is contractive. Similarly, $\iota^{-1} = V_1^{-1}V_2 : Y_2 \to Y_1$ is a well-defined and contractive lattice homomorphism. It follows that Y_1 and Y_2 are isometric Banach lattices, and moreover $U_2 = \iota U_1$ and $V_1 = V_2 \iota$, so both Class $\mathcal{C}$ factorizations are equivalent.

(2) (a) $\implies$ (b) From the assumption, we have $V_1(B_{Y_1}) = V_2\phi(B_{Y_1}) \subseteq \|\phi\|V_2(B_{Y_2})$. Let us verify that ϕ is of Class $\mathcal{C}$. Recall that V_2 is injective. For any $v \in Y_1$,

$$V_2|\phi v| = |V_2\phi v| = |V_1 v| = V_1|v| = V_2\phi|v|.$$

Hence $|\phi v| = \phi|v|$. Similarly, if $v \in (Y_1)_+$, then

$$V_2[0, \phi v] = [0, V_2\phi v] = [0, V_1 v] = V_1[0, v] = V_2(\phi[0, v]).$$

Thus $[0, \phi v] = \phi[0, v]$. This shows that ϕ is an interval preserving lattice homomorphism. It is injective since $V_1 = V_2\phi$ is injective. Finally, $\phi(B_{Y_1}) = V_2^{-1}(V_1(B_{Y_1}))$ is closed since $V_1(B_{Y_1})$ is closed. Therefore, ϕ is of Class $\mathcal{C}$.

(b) $\implies$ (a) Since $V_1(Y_1) \subseteq V_2(Y_2)$ and V_2 is injective, $\phi = V_2^{-1}V_1 : Y_1 \to Y_2$ is a well-defined linear operator. If $u \in B_{Y_1}$, then $\phi(u) = V_2^{-1}V_1 u \in V_2^{-1}(V_1(B_{Y_1})) \subseteq cB_{Y_2}$. Thus, ϕ is bounded. By the definition of ϕ, $V_1 = V_2\phi$. If $x \in E$, then

$$\phi U_1 x = V_2^{-1}V_1 U_1 x = V_2^{-1}T x = V_2^{-1}V_2 U_2 x = U_2 x.$$

Thus $\phi U_1 = U_2$. The final statement follows by applying (2) in both directions. $\qquad\square$

On the other hand, Class $\mathcal{D}$ factorizations can be characterized as follows:

Proposition 4.4. *Let $T : X \to E$ be a nonzero bounded linear operator.*
   (1) *The map $(Y, U, V) \mapsto U^{-1}(B_Y)$ is a surjective correspondence between the set of Class $\mathcal{D}$ factorizations of T and the set of closed convex solid subsets S of X so that $c^{-1}B_X \subseteq S \subseteq cT^{-1}(B_E)$ for some $c \geqslant 1$.*
   (2) *Let (Y_i, U_i, V_i), $i = 1, 2$, be two Class $\mathcal{D}$ factorizations of T. The following are equivalent:*
      (a) *There is a bounded linear map $\phi : Y_1 \to Y_2$ such that $U_2 = \phi U_1$ and $V_1 = V_2\phi$. In this case, ϕ is necessarily of Class $\mathcal{D}$.*
      (b) *There is a positive constant c so that $U_1^{-1}(B_{Y_1}) \subseteq cU_2^{-1}(B_{Y_2})$.*
   As a result, two Class $\mathcal{D}$ factorizations (Y_1, U_1, V_1) and (Y_2, U_2, V_2) are equivalent if and only if $U_1^{-1}(B_{Y_1})$ and $U_2^{-1}(B_{Y_2})$ are equivalent.

Proof. (1) Let (Y, U, V) be a Class $\mathcal{D}$ factorization of T. Since U is a lattice homomorphism, $U^{-1}(B_Y)$ is a closed convex solid subset of X. By the boundedness of U, $U(B_X) \subseteq \|U\|B_Y$. Hence $B_X \subseteq \|U\|U^{-1}(B_Y)$. Note that $T \neq 0$ implies that $U \neq 0$, so $\|U\| > 0$. Also, $T(U^{-1}(B_Y)) = VU(U^{-1}(B_Y)) = V(B_Y) \subseteq \|V\|B_E$. Thus $U^{-1}(B_Y) \subseteq \|V\|T^{-1}(B_E)$. Again, $\|V\| > 0$ since $V \neq 0$.

Conversely, suppose that B is a closed convex solid subset of X so that $c^{-1}B_X \subseteq B \subseteq T^{-1}(B_E)$ for some $c \geqslant 1$. The Minkowski functional ρ of B is a lattice seminorm on X. The kernel $I := \ker \rho$ is a vector lattice ideal in X; ρ induces a lattice norm $\hat{\rho}$ on the vector lattice X/I by $\hat{\rho}(Qu) = \rho(u)$, where $Q : X \to X/I$ is the quotient map. Let Y be the completion of X/I with respect to $\hat{\rho}$. Then Y is a Banach lattice. Let $U := Q : X \to X/I \subseteq Y$. Then U is a lattice homomorphism; in particular, U is bounded. By definition, U has dense range. Hence U is of Class $\mathcal{D}$. Define $V : X/I \to E$ by $V(Qu) = Tu$. Note that $u \in I$ implies that $u \in \lambda B$ for all $\lambda > 0$. Since $T(B) \subseteq B_E$, $Tu = 0$. Thus, V is well defined. If $\hat{\rho}(Qu) < 1$, then $u \in B$. Hence, $\|Tu\| \leqslant 1$. So, V is bounded. Therefore, V extends to a bounded linear operator from Y into E, still denoted by V. By definition, $VU = T$. Thus, (Y, U, V) is a Class $\mathcal{D}$ factorization of T.

To check the injectivity of the correspondence, assume that (Y_i, U_i, V_i), $i = 1, 2$, are two Class $\mathcal{D}$ factorizations of T such that $U_1^{-1}(B_{Y_1}) = U_2^{-1}(B_{Y_2})$, and let us define $\iota : U_1(X) \to Y_2$ by $\iota y = U_2 x$ for every $y = U_1 x \in U_1(X)$. Observe that

$$\ker U_1 = \bigcap_{\lambda > 0} \lambda U_1^{-1}(B_{Y_1}) = \bigcap_{\lambda > 0} \lambda U_2^{-1}(B_{Y_2}) = \ker U_2.$$

Therefore, whenever $U_1 x = U_1 x'$ it follows that $U_2 x = U_2 x'$, and hence ι is well defined. Moreover, if $y = U_1 x \in B_{Y_1}$, then $x \in U_1^{-1}(B_{Y_1}) = U_2^{-1}(B_{Y_2})$ and $\iota y = U_2 x \in B_{Y_2}$, so ι is contractive. Since U_1 and U_2 are lattice homomorphisms, $U_1(X)$ is a sublattice, and for every $y = U_1 x \in U_1(X)$, $|y| = U_1|x|$ and $\iota|y| = U_2|x| = |U_2 x| = |\iota y|$, so ι is a lattice homomorphism. Finally, U_1 has dense range, so ι extends to Y_1 as a contractive lattice homomorphism. Interchanging the roles of U_1 and U_2, we can show that ι has an inverse operator, so it is an onto isometry from Y_1 into Y_2. Clearly, $U_2 = \iota U_1$ and $V_1 = V_2 \iota$, so the factorizations are equivalent.

(2) (a) $\implies$ (b) If $u \in X$ and $U_1 u \in B_{Y_1}$, then $U_2 u = \phi U_1 u \in \phi(B_{Y_1}) \subseteq \|\phi\| B_{Y_2}$. Hence $U_1^{-1} B_{Y_1} \subseteq \|\phi\| U_2^{-1}(B_{Y_2})$. Let us verify that ϕ is of Class $\mathcal{D}$. For any $u \in X$,

$$|\phi U_1 v| = |U_2 v| = U_2|v| = \phi U_1|v| = \phi|U_1 v|.$$

Since $U_1(X)$ is dense in Y_1, this proves that ϕ is a lattice homomorphism. As $\phi(Y_1) \supseteq U_2(X)$, ϕ has dense range in Y_2. Therefore, ϕ is of Class $\mathcal{D}$.

(b) $\implies$ (a) If $u \in X$ and $U_1 u = 0$, then $u \in \lambda U_1^{-1}(B_{Y_1})$ for all $\lambda > 0$. Hence $u \in \lambda U_2^{-1}(B_{Y_2})$ for all $\lambda > 0$. Thus, $U_2 u = 0$. This shows that the map $\phi : U_1(X) \to Y_2$ given by $\phi(U_1 u) = U_2 u$ is well defined. Moreover, if $U_1 u \in B_{Y_1}$, then $u \in U_1^{-1}(B_{Y_1}) \subseteq c U_2^{-1}(B_{Y_2})$. Hence $\phi(U_1 u) = U_2 u \in c B_{Y_2}$. Thus, $\phi : U_1(X) \to Y_2$ is bounded. Denote its bounded extension to Y_1 by ϕ again. By definition, $\phi U_1 = U_2$. If $u \in X$, then $V_2 \phi U_1 u = V_2 U_2 u = Tu = V_1 U_1 u$. As $U_1(X)$ is dense in Y_1, $V_2 \phi = V_1$. This completes the proof of the proposition. $\qquad\square$

4.2. Duality of convex sets.

Krivine's functional calculus (cf. [17, Section 1.d]) allows us to define expressions such as $\left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$ for a finite quantity of elements $x_1, \ldots, x_n$ in an arbitrary Banach lattice X. However, we could replace the p-sum of n elements for an arbitrary $n \in \mathbb{N}$, by any norm σ on c_{00}, as the only requirements for functional calculus to exist is that the function on $\mathbb{R}^n$ that we want to represent is positively homogeneous and continuous. Therefore, we will use this fact to study convex and concave operators and Banach lattices in a more general sense.

Throughout the following subsections, let $\sigma : c_{00} \to \mathbb{R}$ be a lattice norm on c_{00} that is

(1) *Symmetric*: $\sigma((s_i)_i) = \sigma((s_{\pi(i)})_i)$ for any permutation π of $\mathbb{N}$.
(2) *Normalized*: $\sigma(1, 0, \ldots) = 1$.

Denote the norm on c_{00} dual to σ by σ^*. That is, for any $(s_i)_i \in c_{00}$,

$$\sigma^*((s_i)_i) = \sup \left\{ \sum_i s_i t_i : (t_i)_i \in c_{00}, \sigma((t_i)_i) \leq 1 \right\}.$$

It is clear that σ^* is a symmetric normalized lattice norm on c_{00}. Moreover, taking $\sigma^{**} = (\sigma^*)^*$, we have $\sigma^{**} = \sigma$. Let (e_i) be the unit vector basis of c_{00}. For each $n \in \mathbb{N}$, define $\sigma_n : \mathbb{R}^n \to \mathbb{R}$ by $\sigma_n(t_1, \ldots, t_n) = \sigma(\sum_{i=1}^{n} t_i e_i)$. Then σ_n is a monotone positively homogeneous continuous function. As a particular case, we can take σ to be the norm of ℓ_p for some $1 \leq p < \infty$ or the norm of c_0.

In what follows, let X be a Banach lattice and E be a Banach space. Let τ be another symmetric normalized lattice norm on c_{00}. The following sets are the main objects of our considerations.

Definition 4.5. *(1) Let $T : E \to X$ be a bounded linear operator. Define*

$$C_T^{\tau,\sigma} = \left\{ u \in X : \exists (x_i)_{i=1}^n \subseteq E \text{ such that } |u| \leqslant \sigma_n(|Tx_1|, \ldots, |Tx_n|), \tau\left(\sum_{i=1}^n \|x_i\|e_i\right) \leqslant 1 \right\}.$$

Let $T : X \to E$ be a bounded linear operator. Define

$$D_{\tau,\sigma}^T = \left\{ u \in X : |u| \geqslant \sigma_n(|u_1|, \ldots, |u_n|), (u_i)_{i=1}^n \subseteq X \implies \tau\left(\sum_{i=1}^n \|Tu_i\|e_i\right) \leqslant 1 \right\}.$$

In the rest of the section, we consider the duality $\langle X, X^* \rangle$. In particular, all polars are taken with respect to this dual pair. Our aim is to show Theorem 4.10, which establishes that the sets defined above satisfy a duality relation. First, we need a linearization lemma. Let K be a compact Hausdorff space, and let μ be a regular Borel measure on K. Suppose that $(h_i)_{i=1}^n \subseteq C(K)$ and $(g_i)_{i=1}^n \subseteq L_1(\mu)$. It follows from the uniqueness of function calculus that

$$\sigma(h_1, \ldots, h_n)(t) = \sigma(h_1(t), \ldots, h_n(t)) \text{ for all } t \in K, \text{ and}$$
$$\sigma^*(g_1, \ldots, g_n)(t) = \sigma^*(g_1(t), \ldots, g_n(t)) \text{ for } \mu\text{-almost all } t \in K.$$

Lemma 4.6. *Let μ be a regular Borel measure on a compact Hausdorff space K. Given $(g_i)_{i=1}^n \subseteq L_1(\mu)$ and $\varepsilon > 0$, there is a sequence $(h_i)_{i=1}^n$ in $C(K)$ such that $\sigma_n(h_1(t), \ldots, h_n(t)) \leqslant 1$ for all $t \in K$, and*

$$\int \sigma_n^*(g_1, \ldots, g_n) \, d\mu \leqslant \sum_{i=1}^n \int h_i g_i \, d\mu + \varepsilon.$$

Proof. First, assume that the functions $(g_i)_{i=1}^n \subseteq L_1(\mu)$ are simple. Without loss of generality, we can assume that there is a Borel measurable partition $(A_j)_{j=1}^k$ so that each $g_i = \sum_{j=1}^k a_{ij}\chi_{A_j}$ for some a_{ij} (not necessarily positive). In this case, given $\varepsilon > 0$, by the regularity of the measure μ there exist closed sets $(C_j)_{j=1}^k$ and open sets $(U_j)_{j=1}^k$ such that $C_j \subseteq A_j \subseteq U_j$ and $\mu(U_j \backslash C_j) < \frac{\varepsilon}{4ks}$, where $s = \sup_j \sigma_n^*(a_{1j}, \ldots, a_{nj}) > 0$ (recall that σ_n^* is a lattice norm, so $\sigma_n^*(a_1, \ldots, a_n) = \sigma_n^*(|a_1|, \ldots, |a_n|)$). Since the sets $(C_j)_{j=1}^k$ are pairwise disjoint and K is compact and Hausdorff, and thus, normal, there exist pairwise disjoint open neighborhoods $(V_j)_{j=1}^k$ with $C_j \subseteq V_j$ for each j. Let $W_j = U_j \cap V_j$ and $\widetilde{g}_i = \sum_{j=1}^k a_{ij}\chi_{W_j}$. Then $\mu(W_j \Delta A_j) \leqslant \mu(U_j \backslash C_j)$, where $A \Delta B$ denotes the symmetric difference of A and B, so

$$\int |\sigma_n^*(g_1, \ldots, g_n) - \sigma_n^*(\widetilde{g}_1, \ldots, \widetilde{g}_n)| \, d\mu \leqslant \sum_{j=1}^k \sigma_n^*(a_{1j}, \ldots, a_{nj})\mu(W_j \Delta A_j) \leqslant \frac{\varepsilon}{4}.$$

Now, choose b_{ij} so that $b_{ij}a_{ij} \geqslant 0$, $\sigma_n(b_{1j}, \ldots, b_{nj}) \leqslant 1$ and

$$\sigma_n^*(a_{1j}, \ldots, a_{nj}) \leqslant \sum_{i=1}^n a_{ij}b_{ij} + \frac{\varepsilon}{4\mu(K)} \quad \text{for } 1 \leqslant j \leqslant k.$$

Since K is normal, there are functions $(u_j)_{j=1}^k \subseteq C(K)$ such that $0 \leqslant u_j \leqslant \chi_{W_j}$ and $u_j = 1$ on C_j. The functions $h_i = \sum_{j=1}^k b_{ij} u_j$ satisfy $|h_i| \leqslant \sum_{j=1}^k |b_{ij}| \chi_{W_j}$ and, moreover,

$$0 \leqslant \sum_{j=1}^k a_{ij} b_{ij} \mu(W_j) = \int \left(\sum_{j=1}^k b_{ij} \chi_{W_j} \right) \tilde{g}_i \, d\mu = \int h_i \tilde{g}_i \, d\mu + \int \left(\sum_{j=1}^k b_{ij} (\chi_{W_j} - u_j) \right) \tilde{g}_i \, d\mu$$

$$\leqslant \int h_i g_i \, d\mu + \int |h_i| |g_i - \tilde{g}_i| \, d\mu + \sum_{j=1}^k \int a_{ij} b_{ij} (\chi_{W_j} - u_j) \chi_{W_j \setminus C_j} \, d\mu$$

$$\leqslant \int h_i g_i \, d\mu + \sum_{j=1}^k a_{ij} b_{ij} (\mu(W_j \Delta A_j) + \mu(W_j \setminus C_j))$$

$$\leqslant \int h_i g_i \, d\mu + \frac{\varepsilon}{2ks} \sum_{j=1}^k a_{ij} b_{ij} \quad \text{for } 1 \leqslant i \leqslant n.$$

Observe that, if $t \in W_j$,

$$\sigma_n(h_1(t), \ldots, h_n(t)) \leqslant \sigma_n(b_{1j}, \ldots, b_{nj}) \leqslant 1,$$

so $\sigma(h_1(t), \ldots, h_n(t)) \leqslant 1$ for all $t \in K$. Finally,

$$\int \sigma^*(g_1, \ldots, g_n) \, d\mu \leqslant \int \sigma^*(\tilde{g}_1, \ldots, \tilde{g}_n) \, d\mu + \frac{\varepsilon}{4} = \sum_{j=1}^k \int \sigma_n^*(a_{1j}, \ldots, a_{nj}) \chi_{W_j} \, d\mu + \frac{\varepsilon}{4}$$

$$\leqslant \sum_{j=1}^k \left(\sum_{i=1}^n a_{ij} b_{ij} + \frac{\varepsilon}{4\mu(K)} \right) \mu(W_j) + \frac{\varepsilon}{4} \leqslant \sum_{i=1}^n \sum_{j=1}^k a_{ij} b_{ij} \mu(W_j) + \frac{\varepsilon}{2}$$

$$\leqslant \sum_{i=1}^n \int h_i g_i \, d\mu + \frac{\varepsilon}{2ks} \sum_{j=1}^k \sum_{i=1}^n a_{ij} b_{ij} + \frac{\varepsilon}{2}$$

$$\leqslant \sum_{i=1}^n \int h_i g_i \, d\mu + \frac{\varepsilon}{2ks} \sum_{j=1}^k \sigma_n^*(a_{1j}, \ldots, a_{nj}) + \frac{\varepsilon}{2} \leqslant \sum_{i=1}^n \int h_i g_i \, d\mu + \varepsilon.$$

This completes the proof for simple functions. Now, given general functions $(g_i)_{i=1}^n \subseteq L_1(\mu)$, for every $i = 1, \ldots, n$ there exists a sequence $(s_{im})_{m=1}^\infty$ of simple functions such that $s_{im}^+ \uparrow g_i^+$ and $s_{im}^- \uparrow g_i^-$ almost everywhere, so in particular $\sigma^*(s_{1m}, \ldots, s_{nm}) \uparrow \sigma^*(g_1, \ldots, g_n)$ almost everywhere. By the dominated convergence theorem, for every $\varepsilon > 0$ there is a natural number $M \in \mathbb{N}$ such that

$$\int |\sigma^*(g_1, \ldots, g_n) - \sigma^*(s_{1M}, \ldots, s_{nM})| \, d\mu \leqslant \frac{\varepsilon}{n+2}, \quad \text{and}$$

$$\int |g_i - s_{iM}| \, d\mu \leqslant \frac{\varepsilon}{n+2} \quad \text{for } 1 \leqslant i \leqslant n.$$

By the first part of the proof, for this M there exist some $(h_i)_{i=1}^n \subseteq C(K)$ such that we have $\sigma_n(h_1(t), \ldots, h_n(t)) \leqslant 1$ for all $t \in K$ and

$$\int \sigma_n^*(s_{1M}, \ldots, s_{nM}) \, d\mu \leqslant \sum_{i=1}^n \int h_i s_{iM} \, d\mu + \frac{\varepsilon}{n+2}.$$

In particular, $h_i \in B_{C(K)}$ and we have

$$\int |h_i||g_i - s_{iM}|\, d\mu \leqslant \int |g_i - s_{iM}|\, d\mu \leqslant \frac{\varepsilon}{n+2}$$

for every i. In order to obtain the result, we just use the previous inequalities:

$$\int \sigma^*(g_1, \ldots, g_n)\, d\mu \leqslant \int \sigma^*(s_{1M}, \ldots, s_{nM})\, d\mu + \frac{\varepsilon}{n+2}$$

$$\leqslant \sum_{i=1}^n \int h_i s_{iM}\, d\mu + \frac{2\varepsilon}{n+2} \leqslant \sum_{i=1}^n \int h_i g_i\, d\mu + \varepsilon.$$

$\square$

We will use the previous lemma to linearize expressions of the form $\langle |u|, \sigma_n^*(|u_1^*|, \ldots, |u_n^*|)\rangle$ for $u \in X$ and $u_1^*, \ldots, u_n^* \in X^*$.

Proposition 4.7. *The following statements hold:*

(1) Suppose that $u^* \in X^*$ and $(u_i)_{i=1}^n \subseteq X$. For any $\varepsilon > 0$, there are $(u_i^*)_{i=1}^n \subseteq X^*$ such that $\sigma_n^*(|u_1^*|, \ldots, |u_n^*|) \leqslant |u^*|$ and

$$\langle \sigma_n(|u_1|, \ldots, |u_n|), |u^*|\rangle \leqslant \sum_{i=1}^n \langle u_i, u_i^*\rangle + \varepsilon.$$

(2) Suppose that $u \in X$ and $(u_i^*)_{i=1}^n \subseteq X^*$. For any $\varepsilon > 0$, there are $(u_i)_{i=1}^n \subseteq X$ such that $\sigma_n(|u_1|, \ldots, |u_n|) \leqslant |u|$ and

$$\langle |u|, \sigma_n^*(|u_1^*|, \ldots, |u_n^*|)\rangle \leqslant \sum_{i=1}^n \langle u_i, u_i^*\rangle + \varepsilon.$$

Proof. It suffices to prove (2). (1) follows by applying (2) with X taken to be X^* and σ taken to be σ^*.

Let $u \in X$, $(u_i^*)_{i=1}^n \subseteq X^*$, and $\varepsilon > 0$ be given. There exists a compact Hausdorff space K and an interval preserving injective lattice homomorphism $\iota : C(K) \to X$ so that $\iota 1 = |u|$, where 1 denotes the constant 1 function on K. Set $u^* = \sigma_n^*(|u_1^*|, \ldots, |u_n^*|)$. Since $|u_i^*| \leqslant u^*$ and ι^* is a lattice homomorphism, $|\mu_i| := \iota^*|u_i^*| \leqslant \iota^* u^* =: \mu$. Hence, there are functions $g_i \in L_1(\mu)$ such that $\mu_i = g_i \cdot \mu$. Apply Lemma 4.6 to obtain the corresponding sequence $(h_i)_{i=1}^n$ in $C(K)$ and set $u_i = \iota h_i$. Then $u_i \in X$ and

$$\sigma_n(|u_1|, \ldots, |u_n|) = \sigma_n(\iota|h_1|, \ldots, \iota|h_n|) = \iota\sigma_n(|h_1|, \ldots, |h_n|) \leqslant \iota 1 = |u|.$$

Moreover,

$$\langle |u|, \sigma_n^*(|u_1^*|, \ldots, |u_n^*|)\rangle = \langle 1, \sigma_n^*(\iota^*|u_1^*|, \ldots, \iota^*|u_n^*|)\rangle = \langle 1, \sigma_n^*(\iota^* u_1^*, \ldots, \iota^* u_n^*)\rangle$$

$$= \int \sigma_n^*(g_1, \ldots, g_n)\, d\mu \leqslant \sum_{i=1}^n \int h_i g_i\, d\mu + \varepsilon$$

$$= \sum_{i=1}^n \langle h_i, \iota^* u_i^*\rangle + \varepsilon = \sum_{i=1}^n \langle u_i, u_i^*\rangle + \varepsilon.$$

This completes the proof. $\square$

Next, we show that $D_{\tau*,\sigma*}^{T*} \subseteq (C_T^{\tau,\sigma})^\circ$ (respectively, $D_{\tau,\sigma}^T \subseteq (C_{T*}^{\tau*,\sigma*})^\circ$) whenever $T : E \to X$ (respectively, $T : X \to E$) is bounded.

Proposition 4.8. *The following statements hold:*

(1) Let $T : E \to X$ be a bounded linear operator. Then $u^*(u) \leqslant 1$ for all $u \in C_T^{\tau,\sigma}$ and all $u^* \in D_{\tau*,\sigma*}^{T*}$.

(2) Let $T : X \to E$ be a bounded linear operator. Then $u^*(u) \leqslant 1$ for all $u \in D_{\tau,\sigma}^T$ and all $u^* \in C_{T*}^{\tau*,\sigma*}$.

Proof. (1) Suppose that $u \in C_T^{\tau,\sigma}$ and $u^* \in D_{\tau*,\sigma*}^{T*}$. There exists $(x_i)_{i=1}^n$ in E so that $|u| \leqslant \sigma_n(|Tx_1|, \ldots, |Tx_n|)$ and $\tau(\sum_{i=1}^n \|x_i\|e_i) \leqslant 1$. By Proposition 4.7(1), for any $\varepsilon > 0$, there exists $(u_i^*)_{i=1}^n \subseteq X^*$ so that $\sigma_n^*(|u_1^*|, \ldots, |u_n^*|) \leqslant |u^*|$ and

$$\langle |u|, |u^*| \rangle \leqslant \langle \sigma_n(|Tx_1|, \ldots, |Tx_n|), |u^*| \rangle \leqslant \sum_{i=1}^n \langle Tx_i, u_i^* \rangle + \varepsilon$$

$$= \sum_{i=1}^n \langle x_i, T^*u_i^* \rangle + \varepsilon \leqslant \tau\left(\sum_{i=1}^n \|x_i\|e_i\right) \cdot \tau^*\left(\sum_{i=1}^n \|T^*u_i^*\|e_i^*\right) + \varepsilon.$$

Since $u^* \in D_{\tau*,\sigma*}^{T*}$ and $|u^*| \geqslant \sigma_n^*(|u_1^*|, \ldots, |u_n^*|)$, $\tau^*(\sum_{i=1}^n \|T^*v_i^*\|e_i) \leqslant 1$. Therefore, $\langle |u|, |u^*| \rangle \leqslant 1$, as claimed.

The proof of (2) is similar. $\qquad\square$

The next lemma will help us show the reverse inclusions.

Lemma 4.9. *Let $(u_i)_{i=1}^n$ and $(u_i^*)_{i=1}^n$ be sequences in X and X^*, respectively. Then*

$$\sum_{i=1}^n \langle u_i, u_i^* \rangle \leqslant \langle \sigma_n(|u_1|, \ldots, |u_n|), \sigma_n^*(|u_1^*|, \ldots, |u_n^*|) \rangle.$$

Proof. Take a compact Hausdorff space K and an injective interval preserving lattice homomorphism $\iota : C(K) \to X$ so that $\iota 1 = \sigma_n(|u_1|, \ldots, |u_n|)$. Since $|u_i| \leqslant \sigma_n(|u_1|, \ldots, |u_n|)$, there exists $f_i \in C(K)_+$ so that $\iota f_i = |u_i|$. Set $\mu = \iota^* \sigma_n^*(|u_1^*|), \ldots, |u_n^*|)$ and $\mu_i = \iota^*(|u_i^*|)$. Then $0 \leqslant f_i \leqslant 1$ and $0 \leqslant \mu_i \leqslant \mu$. There are functions $g_i \in L_1(\mu)_+$ so that $\mu_i = g_i \cdot \mu$. Then

$$\sum_{i=1}^n \langle u_i, u_i^* \rangle \leqslant \sum_{i=1}^n \langle |u_i|, |u_i^*| \rangle = \sum_{i=1}^n \int f_i \, d\mu_i = \int \sum_{i=1}^n f_i(t)g_i(t) \, d\mu(t)$$

$$\leqslant \int \sigma_n(f_1(t), \ldots, f_n(t)) \cdot \sigma_n^*(g_1(t), \ldots, g_n(t)) \, d\mu$$

$$= \langle \sigma_n(f_1, \ldots, f_n), \sigma_n^*(\mu_1, \ldots, \mu_n) \rangle$$

$$= \langle \sigma_n(|u_1|, \ldots, |u_n|), \sigma_n^*(|u_1^*|, \ldots, |u_n^*|) \rangle.$$

$\qquad\square$

We are now ready to establish the main result of this subsection.

Theorem 4.10. *The following statements hold:*

(1) Let $T : E \to X$ be a bounded linear operator. Then $(C_T^{\tau,\sigma})^\circ = D_{\tau*,\sigma*}^{T*}$.

(2) Let $T : X \to E$ be a bounded linear operator. Then $(C_{T^*}^{\tau^*,\sigma^*})^\circ = D_{\tau,\sigma}^T$.

Proof. (1) It follows from Proposition 4.8(1) that $D_{\tau^*,\sigma^*}^{T^*} \subseteq (C_T^{\tau,\sigma})^\circ$. Conversely, suppose that $u^* \notin D_{\tau^*,\sigma^*}^{T^*}$. There is a sequence $(u_i^*)_{i=1}^n$ so that $|u^*| \geqslant \sigma_n^*(|u_1^*|, \ldots, |u_n^*|)$ and $\tau^*(\sum_{i=1}^n \|T^*u_i^*\|e_i) > 1$. Choose $x_i \in E$ with $\|x_i\| \leqslant 1$ and a real sequence $(\alpha_i)_{i=1}^n$ with $\tau(\sum_{i=1}^n \alpha_i e_i) \leqslant 1$ so that $\sum_{i=1}^n \alpha_i \langle x_i, T^*u_i^* \rangle > 1$. Set $u_i = \alpha_i T x_i$. Using Lemma 4.9, we have

$$1 < \sum_{i=1}^n \langle u_i, u_i^* \rangle \leqslant \langle \sigma_n(|u_1|, \ldots, |u_n|), \sigma_n^*(|u_1^*|, \ldots, |u_n^*|) \rangle \leqslant \langle \sigma_n(|u_1|, \ldots, |u_n|), |u^*| \rangle.$$

However, $\tau(\sum_{i=1}^n \|\alpha_i x_i\| e_i) \leqslant \tau(\sum_{i=1}^n \alpha_i e_i) \leqslant 1$. Hence, $\sigma_n(|u_1|, \ldots, |u_n|) \in C_T^{\tau,\sigma}$. This shows that $u^* \notin (C_T^{\tau,\sigma})^\circ$.

(2) can be proved in the same manner. $\qquad\square$

4.3. **Convexity and concavity.** Let σ and τ be symmetric and normalized lattice norms on c_{00}, X a Banach lattice and E a Banach space. As we advanced earlier, the following definition generalizes Definition 2.1.

Definition 4.11.

(1) *A bounded linear operator $T : E \to X$ is (τ, σ)-convex with constant K if for any sequence $(x_i)_{i=1}^n$ in E,*

$$\|\sigma_n(|Tx_1|, \ldots, |Tx_n|)\| \leqslant K\tau \left(\sum_{i=1}^n \|x_i\|e_i \right).$$

A Banach lattice X is (τ, σ)-convex with constant K if the identity operator on X is (τ, σ)-convex with constant K.

(2) *A bounded linear operator $T : X \to E$ is (τ, σ)-concave with constant K if for any sequence $(u_i)_{i=1}^n$ in X,*

$$\|\sigma_n(|u_1|, \ldots, |u_n|)\| \geqslant K^{-1}\tau \left(\sum_{i=1}^n \|Tu_i\|e_i \right).$$

A Banach lattice X is (τ, σ)-concave with constant K if the identity operator on X is (τ, σ)-concave with constant K.

It is a well-known fact (see, for instance, [6, Chapter 16]) that an operator T is (p, q)-convex if and only if its adjoint T^* is (p^*, q^*)-concave, and vice versa. Similarly, Theorem 4.10 leads to a duality theory for the convex and concave operators introduced in Definition 4.11.

Theorem 4.12. *Let $T : E \to X$ be a (τ, σ)-convex operator with constant K. Then T^* is a (τ^*, σ^*)-concave operator with constant K.*

Proof. Let $(u_i^*)_{i=1}^n$ be a sequence in X^* and let $\varepsilon > 0$ be given. Choose a sequence $(x_i^*)_{i=1}^n$ in B_E so that

$$\langle Tx_i, u_i^* \rangle = \langle x_i, T^*u_i^* \rangle \geqslant \|T^*u_i^*\| - \frac{K\varepsilon}{n} \text{ for any } i.$$

By definition of τ^*, there exists a positive real sequence $(a_i)_{i=1}^n$ so that $\tau(\sum_{i=1}^n a_i e_i) \leqslant 1$ and that

$$\tau^* \left(\sum_{i=1}^n \|T^*u_i^*\|e_i \right) \leqslant \sum_{i=1}^n a_i \|T^*u_i^*\| + K\varepsilon.$$

Since T is (τ, σ)-convex with constant K,

$$\|\sigma_n(a_1|Tx_1|, \ldots, a_n|Tx_n|)\| \leqslant K\tau\left(\sum_{i=1}^n a_i\|x_i\|e_i\right) \leqslant K\tau\left(\sum_{i=1}^n a_i e_i\right) \leqslant K.$$

Therefore,

$$\|\sigma_n^*(|u_1^*|, \ldots, |u_n^*|)\| \geqslant K^{-1}\langle\sigma_n(a_1|Tx_1|, \ldots, a_n|Tx_n|), \sigma_n^*(|u_1^*|, \ldots, |u_n^*|)\rangle$$

$$\geqslant K^{-1}\sum_{i=1}^n a_i\langle Tx_i, u_i^*\rangle \quad \text{by Lemma 4.9}$$

$$\geqslant K^{-1}\sum_{i=1}^n a_i\left(\|T^*u_i^*\| - \frac{K\varepsilon}{n}\right) \geqslant K^{-1}\sum_{i=1}^n a_i\|T^*u_i^*\| - \varepsilon$$

$$\geqslant K^{-1}\tau^*\left(\sum_{i=1}^n \|T^*u_i^*\|e_i\right) - 2\varepsilon.$$

Taking $\varepsilon \downarrow 0$ give the required inequality for (τ^*, σ^*)-concavity of T^* with constant K. $\square$

Theorem 4.13. *Let $T : X \to E$ be a (τ, σ)-concave operator with constant K. Then T^* is a (τ^*, σ^*)-convex operator with constant K.*

Proof. Let $(x_i^*)_{i=1}^n \subseteq E^*$, $\varepsilon > 0$ and $u \in B_X$ be given. By Proposition 4.7(2), there is a sequence $(u_i)_{i=1}^n \subseteq X$ such that $\sigma_n(|u_1|, \ldots, |u_n|) \leqslant |u|$ and

$$\langle|u|, \sigma_n^*(|T^*x_1^*|, \ldots, |T^*x_n^*|)\rangle \leqslant \sum_{i=1}^n\langle u_i, T^*x_i^*\rangle + \varepsilon = \sum_{i=1}^n\langle Tu_i, x_i^*\rangle + \varepsilon$$

$$\leqslant \tau\left(\sum_{i=1}^n \|Tu_i\|e_i\right)\tau^*\left(\sum_{i=1}^n \|x_i^*\|e_i\right) + \varepsilon \leqslant K\tau^*\left(\sum_{i=1}^n \|x_i^*\|e_i\right) + \varepsilon,$$

where the last inequality follows from the concavity of T, since

$$\tau\left(\sum_{i=1}^n \|Tu_i\|e_i\right) \leqslant K\|\sigma_n(|u_1|, \ldots, |u_n|)\| \leqslant K\|u\| \leqslant K.$$

Taking $\varepsilon \downarrow 0$ we conclude that

$$\|\sigma_n^*(|T^*x_1^*|, \ldots, |T^*x_n^*|)\| = \sup_{u \in B_X}\langle|u|, \sigma_n^*(|T^*x_1^*|, \ldots, |T^*x_n^*|)\rangle \leqslant K\tau^*\left(\sum_{i=1}^n \|x_i^*\|e_i\right),$$

so T^* is a (τ^*, σ^*)-convex operator with constant K. $\square$

Combining both results, we get:

Corollary 4.14.
    (1) $T : E \to X$ is (τ, σ)-convex with constant K if and only if $T^* : X^* \to E^*$ is (τ^*, σ^*)-concave with constant K.
    (2) $T : X \to E$ is (τ, σ)-concave with constant K if and only if $T^* : E^* \to X^*$ is (τ^*, σ^*)-convex with constant K.

Proof. (1) One implication was established in Theorem 4.12. For the reverse implication, we can apply Theorem 4.13 to obtain that T^{**} is (τ, σ)-convex with constant K. If we denote by J_X and J_E the canonical embeddings into the corresponding bidual spaces, we get that the mapping $J_X T = T^{**} J_E : E \to J_X(X) \subseteq X^{**}$ is also (τ, σ)-convex with the same constant. Since J_X is a lattice isometry, we get the claimed result by composing $J_X T$ with J_X^{-1}.

(2) Again, one implication was established in Theorem 4.13, and the converse follows by applying Theorem 4.12 to T^* to obtain that $T = J_E^{-1} J_E T = J_E^{-1} T^{**} J_X$ is (τ, σ)-concave with constant K. □

We say that a symmetric lattice norm ρ on c_{00} is *block convex with constant A*, respectively *block concave with constant A*, if $\rho(\sum_i y_i) \leqslant A\rho(\sum_i \rho(y_i)e_i)$, respectively $\rho(\sum_i y_i) \geqslant A^{-1}\rho(\sum_i \rho(y_i)e_i)$, for any finite disjoint sequence $(y_i)_i$ in c_{00}. It is easy to see that if a symmetric lattice norm ρ is block convex with constant A, respectively block concave with constant A, then its dual norm ρ^* is block concave, respectively block convex with the same constant. Observe that, for every $1 \leqslant p < \infty$, the norm in ℓ_p is both block concave and block convex with constant 1, and the same can be said about the norm in c_0.

The following proposition shows that, given an operator $T : E \to X$, the Banach lattice generated by the set $(C_T^{\tau,\sigma})^{\circ\circ}$ with the procedure described in Lemma 4.2 is (τ, σ)-convex.

Proposition 4.15. *Let $T : E \to X$ be a bounded linear operator, and assume that σ is block concave with constant A_σ and τ is block convex with constant A_τ. Let $(u_i)_{i=1}^n$ be a sequence in $(C_T^{\tau,\sigma})^{\circ\circ}$ and let $(\alpha_i)_{i=1}^n$ be a real sequence so that $\tau(\sum_{i=1}^n \alpha_i e_i) \leqslant 1$. Then $\sigma_n(|\alpha_1 u_1|, \ldots, |\alpha_n u_n|) \in A_\sigma A_\tau \cdot (C_T^{\tau,\sigma})^{\circ\circ}$.*

Proof. Let $u^* \in D_{\tau^*,\sigma^*}^{T^*}$ and let $\varepsilon > 0$ be given. By Proposition 4.7(1), there exists $(u_i^*)_{i=1}^n \subseteq X^*$ such that $\sigma_n^*(|u_1^*|, \ldots, |u_n^*|) \leqslant |u^*|$ and

$$\langle \sigma_n(|\alpha_1 u_1|, \ldots, |\alpha_n u_n|), |u^*| \rangle \leqslant \sum_{i=1}^n \langle |\alpha_i u_i|, |u_i^*| \rangle + \varepsilon.$$

Let $\lambda_i = \inf\{\lambda > 0 : |u_i^*| \in \lambda D_{\tau^*,\sigma^*}^{T^*}\}$. For each i, there exists $(u_{ij}^*)_{j=1}^{m_i} \subseteq X^*$ such that $|u_i^*| \geqslant \sigma_{m_i}^*(|u_{i1}^*|, \ldots, |u_{im_i}^*|)$ and

$$\tau^*(y_i) \geqslant (1-\varepsilon)\lambda_i, \quad \text{where } y_i = \sum_{j=1}^{m_i} \|T^* u_{ij}^*\| e_{j+\sum_{k=1}^{i-1} m_k}.$$

Using the fact that σ^* is block convex with constant A_σ, we find that

$$
\begin{aligned}
|u^*| &\geqslant \sigma_n^*(|u_1^*|, \ldots, |u_n^*|) \\
&\geqslant \sigma_n^*(\sigma_{m_1}^*(|u_{11}^*|, \ldots, |u_{1m_1}^*|), \ldots, \sigma_{m_n}^*(|u_{n1}^*|, \ldots, |u_{nm_n}^*|)) \\
&\geqslant (A_\sigma)^{-1} \sigma_r^*(|u_{11}^*|, \ldots, |u_{1m_1}^*|, \ldots, |u_{n1}^*|, \ldots, |u_{nm_n}^*|), \quad r = \sum_{i=1}^n m_i.
\end{aligned}
$$

Since $u^* \in D_{\tau^*,\sigma^*}^{T^*}$ and τ^* is block concave with constant A_τ,

$$1 \geqslant \tau^* \left(\sum_{i=1}^n \sum_{j=1}^{m_i} (A_\sigma)^{-1} \|T^* u_{ij}^*\| e_{j+\sum_{k=1}^{i-1} m_k} \right)$$

$$= (A_\sigma)^{-1}\tau^* \left(\sum_{i=1}^{n} y_i \right) \geqslant (A_\sigma A_\tau)^{-1}\tau^* \left(\sum_{i=1}^{n} \tau^*(y_i)e_i \right)$$

$$\geqslant (1-\varepsilon)(A_\sigma A_\tau)^{-1}\tau^* \left(\sum_{i=1}^{n} \lambda_i e_i \right).$$

As $|u_i| \in (C_T^{\tau,\sigma})^{\circ\circ}$ and $|u_i^*| \in \lambda_i D_{\tau*,\sigma*}^{T^*} = \lambda_i (C_T^{\tau,\sigma})^\circ$ by Theorem 4.10(1), $\langle |u_i|, |u_i^*|\rangle \leqslant \lambda_i$. Therefore,

$$\langle \sigma_n(|\alpha_1 u_1|, \ldots, |\alpha_n u_n|), |u^*|\rangle \leqslant \sum_{i=1}^{n} \langle |\alpha_i u_i|, |u_i^*|\rangle + \varepsilon$$

$$\leqslant \sum_{i=1}^{n} |\alpha_i \lambda_i| + \varepsilon \leqslant \tau \left(\sum_{i=1}^{n} \alpha_i e_i \right) \cdot \tau^* \left(\sum_{i=1}^{n} \lambda_i e_i \right) + \varepsilon$$

$$\leqslant \frac{A_\sigma A_\tau}{1-\varepsilon} + \varepsilon.$$

Taking $\varepsilon \downarrow 0$, we see that $\langle \sigma_n(|\alpha_1 u_1|, \ldots, |\alpha_n u_n|), |u^*|\rangle \leqslant A_\sigma A_\tau$. As this holds for all $u^* \in D_{\tau*,\sigma*}^{T^*}$, we conclude that $(A_\sigma A_\tau)^{-1}\sigma_n(|\alpha_1 u_1|, \ldots, |\alpha_n u_n|) \in (D_{\tau*,\sigma*}^{T^*})^\circ$. Since the last set coincides with $(C_T^{\tau,\sigma})^{\circ\circ}$ by Theorem 4.10(1), the proof is complete. $\qquad\square$

We will use the previous lemma to construct optimal factorizations for (τ,σ)-convex and (τ,σ)-concave operators, extending the results of [24, Section 2] for p-convex and p-concave operators. In particular, applying Theorem 4.17 with τ and σ being the norms on ℓ_p and c_0, respectively, we will obtain the first statement of Theorem 4.1. If we apply instead Theorem 4.19 with the norms on ℓ_q and ℓ_1 acting as τ and σ, we will recover the second part of Theorem 4.1.

4.3.1. *Minimal convex factorization.* In this subsection, let $T : E \to X$ be a (τ,σ)-convex operator with constant K and recall the sets $C_T^{\tau,\sigma}$ and $D_{\tau*,\sigma*}^{T^*}$ from Definition 4.5.

Proposition 4.16. *The set $(C_T^{\tau,\sigma})^{\circ\circ}$ is a closed convex solid set in X so that $T(B_E) \subseteq (C_T^{\tau,\sigma})^{\circ\circ} \subseteq K B_X$.*

Proof. The set $(C_T^{\tau,\sigma})^{\circ\circ}$ is the closed convex hull of $C_T^{\tau,\sigma}$, where the polars are taken with respect to the duality $\langle X, X^*\rangle$. Since $C_T^{\tau,\sigma}$ is solid, so is $(C_T^{\tau,\sigma})^{\circ\circ}$. Suppose that $u \in C_T^{\tau,\sigma}$. There exists $(x_i)_{i=1}^n$ in E so that $|u| \leqslant \sigma_n(|Tx_1|, \ldots, |Tx_n|)$ and $\tau(\sum_{i=1}^n \|x_i\|e_i) \leqslant 1$. By the convexity of T, $\|u\| \leqslant K$. Thus $C_T^{\tau,\sigma} \subseteq K B_X$ and hence $(C_T^{\tau,\sigma})^{\circ\circ} \subseteq K B_X$. Finally, if $x \in B_E$, then $|Tx| \leqslant \sigma_1(|Tx|)$ and $\tau(\|x\|e_1) = \|x\| \leqslant 1$. Hence, $Tx \in C_T^{\tau,\sigma} \subseteq (C_T^{\tau,\sigma})^{\circ\circ}$. $\qquad\square$

By Proposition 4.3, there is a Class $\mathcal{C}$ factorization (Y_0, U_0, V_0) of T so that $V_0 B_{Y_0} = (C_T^{\tau,\sigma})^{\circ\circ}$. The Banach lattice Y_0 induced by this factorization is (τ,σ)-convex.

Theorem 4.17. *Assume that σ is block concave with constant A_σ and τ is block convex with constant A_τ. Then, the Banach lattice Y_0 is (τ,σ)-convex with constant $A_\sigma A_\tau$. The operators U_0, V_0 satisfy $\|U_0\| \leqslant 1$, $\|V_0\| \leqslant K$. If (Y,U,V) is a Class $\mathcal{C}$ factorization of T so that Y is (τ,σ)-convex, then there is a linear operator $\phi : Y_0 \to Y$ of Class $\mathcal{C}$ so that $U = \phi U_0$ and $V_0 = V\phi$.*

Proof. By definition, $Y_0 = \operatorname{span}(C_T^{\tau,\sigma})^{\circ\circ}$ and $(C_T^{\tau,\sigma})^{\circ\circ}$ is the closed ball of Y_0 (Lemma 4.2). Since U_0 agrees with T formally and V_0 is the inclusion map from Y to X, by Proposition 4.16, $\|U_0\| \leqslant 1$ and $\|V_0\| \leqslant K$.

Let $(u_i)_{i=1}^n$ be a sequence in $B_{Y_0} = (C_T^{\tau,\sigma})^{\circ\circ}$ and let $(\alpha_i)_{i=1}^n$ be a scalar sequence satisfying $\tau(\sum_{i=1}^n \alpha_i e_i) \leqslant 1$. By Proposition 4.15, $\sigma_n(|\alpha_1 u_1|, \ldots, |\alpha_n u_n|) \in A_\sigma A_\tau \cdot B_{Y_0}$. Hence Y_0 is (τ,σ)-convex with constant $A_\sigma A_\tau$.

Now, suppose that (Y, U, V) is a Class $\mathcal{C}$ factorization of T so that Y is (τ,σ)-convex with constant K. Without loss of generality, we can assume that U is contractive. We claim that $\frac{1}{K}(V(B_Y))^\circ \subseteq D_{\tau*,\sigma*}^{T^*}$. Indeed, if $u^* \notin D_{\tau*,\sigma*}^{T^*}$, then there exists $(u_i^*)_{i=1}^n$ in X^* so that $|u^*| \geqslant \sigma_n^*(|u_1^*|, \ldots, |u_n^*|)$ and $\tau^*(\sum_{i=1}^n \|T^* u_i^*\| e_i) > 1$. Choose $(x_i)_{i=1}^n \subseteq B_E$ and a scalar sequence $(\alpha_i)_{i=1}^n$ so that $\tau(\sum_{i=1}^n \alpha_i e_i) \leqslant 1$ and

$$
\begin{aligned}
1 < \sum_{i=1}^n \alpha_i \langle x_i, T^* u_i^* \rangle &= \sum_{i=1}^n \langle T(\alpha_i x_i), u_i^* \rangle \quad \text{(Lemma 4.9)} \\
&\leqslant \langle \sigma_n(|T(\alpha_1 x_1)|, \ldots, |T(\alpha_n x_n)|), \sigma_n^*(|u_1^*|, \ldots, |u_n^*|) \rangle \\
&\leqslant \langle \sigma_n(|T(\alpha_1 x_1)|, \ldots, |T(\alpha_n x_n)|), |u^*| \rangle \\
&= \langle V \sigma_n(|U(\alpha_1 x_1)|, \ldots, |U(\alpha_n x_n)|), |u^*| \rangle.
\end{aligned}
$$

However, since Y is (τ,σ)-convex with constant K and U was assumed to be contractive,

$$
\|\sigma_n(|U(\alpha_1 x_1)|, \ldots, |U(\alpha_n x_n)|)\| \leqslant K\tau\left(\sum_{i=1}^n \alpha_i e_i\right) \leqslant K.
$$

Thus $V\sigma_n(|U(\alpha_1 x_1)|, \ldots, |U(\alpha_n x_n)|) \in KV(B_Y)$. This shows that $|u^*| \notin (KV(B_Y))^\circ = \frac{1}{K}(V(B_Y))^\circ$, completing the proof of the claim.

From the claim, Theorem 4.10(1), and the fact that V_0 is the formal inclusion, we obtain that

$$
V_0 B_{Y_0} = V_0((C_T^{\tau,\sigma})^{\circ\circ}) = (C_T^{\tau,\sigma})^{\circ\circ} = (D_{\tau*,\sigma*}^{T^*})^\circ \subseteq K(V(B_Y))^{\circ\circ} = KV(B_Y),
$$

where the last equality holds since $V(B_Y)$ is closed and convex. The existence of the desired map $\phi : Y_0 \to Y$ now follows from Proposition 4.3(2). □

Due to Theorem 4.17, we call (Y_0, U_0, V_0) the *minimal (τ,σ)-convex factorization* of $T : E \to X$.

4.3.2. *Maximal concave factorization.* Similar results for concave operators can be obtained via duality. Let $T : X \to E$ be a (τ,σ)-concave operator with constant K.

Proposition 4.18. *The set $D_{\tau,\sigma}^T$ is a closed convex solid set in X so that $\frac{1}{K}B_X \subseteq D_{\tau,\sigma}^T \subseteq T^{-1}(B_E)$.*

Proof. By Theorem 4.13, $T^* : E^* \to X^*$ is a (τ^*, σ^*)-convex operator. By Proposition 4.16,

$$
T^*(B_{E^*}) \subseteq (C_{T*}^{\tau^*,\sigma^*})^{\circ\circ} \subseteq KB_{X^*}.
$$

Easy calculations show that $(T^*(B_{E^*}))^\circ = T^{-1}(B_E)$ and $(KB_{X^*})^\circ = K^{-1}B_X$. Thus, taking polars in the string of inclusions above gives

$$
K^{-1}B_X \subseteq (C_{T*}^{\tau^*,\sigma^*})^\circ \subseteq T^{-1}(B_E).
$$

By Theorem 4.10(2), $(C_{T*}^{\tau^*,\sigma^*})^\circ = D_{\tau,\sigma}^T$. This concludes the proof of the proposition. □

By Proposition 4.4, there is a Class $\mathcal{D}$ factorization (Y^0, U^0, V^0) of T such that $(U^0)^{-1}(B_{Y^0}) = D_{\tau,\sigma}^T$. The next result follows from Theorem 4.17 by duality. We omit the details.

Theorem 4.19. *Assume that σ is block convex with constant A_σ and τ is block concave with constant A_τ. Then, the Banach lattice Y^0 is (τ, σ)-concave with constant $A_\sigma A_\tau$. The operators U^0, V^0 satisfy $\|U^0\| \leqslant K$, $\|V^0\| \leqslant 1$. If (Y, U, V) is a Class $\mathcal{D}$ factorization of T so that Y is (τ, σ)-concave, then there is a linear operator $\phi : Y \to Y^0$ of Class $\mathcal{D}$ so that $U^0 = \phi U$ and $V^0 \phi = V$.*

We call (Y^0, U^0, V^0) the *maximal (τ, σ)-concave factorization* of $T : X \to E$.

As a consequence, we have the following generalized version of a factorization result given in [25].

Theorem 4.20. *Let E, F be Banach spaces. Let $\tau, \sigma, \tau', \sigma'$ be normalized symmetric norms on c_{00} such that σ and τ' are block concave with constants A_σ and A_τ', and τ and σ' are block convex with constants A_τ and A_σ', respectively. Assume that a bounded linear operator $T : E \to F$ has a factorization $T = VU$, where X is a Banach lattice, $U : E \to X$ is (τ, σ)-convex with constant K and $V : X \to F$ is (τ', σ')-concave with constant K'. Then there are a (τ, σ)-convex Banach lattice Y with constant $A_\sigma A_\tau$ and a (τ', σ')-concave Banach lattice Z with constant $A_\sigma' A_\tau'$ and bounded linear operators $A : E \to Y$, $B : Y \to Z$, $C : Z \to F$ so that $T = CBA$, $\|A\| \leqslant K$, $\|C\| \leqslant K'$ and B is a contraction of Class $\mathcal{D}$.*

Proof. We may replace X by the closed ideal I generated by $U(E)$ in X and V by its restriction $V|_I$. In this way, we may assume without loss of generality that the ideal generated by $U(E)$ in X is dense. Apply Theorem 4.17 to U and Theorem 4.19 to V to find the following:

(1) A (τ, σ)-convex Banach lattice Y with constant $A_\sigma A_\tau$ and bounded linear operators $A : E \to Y$, $R : Y \to X$ so that R is of Class $\mathcal{C}$ and $U = RA$. Furthermore, $\|A\| \leqslant K$ and $\|R\| \leqslant 1$.

(2) A (τ', σ')-concave Banach lattice Z with constant $A_\sigma' A_\tau'$ and bounded linear operators $S : X \to Z$, $C : Z \to F$ so that S is of Class $\mathcal{D}$, $V = CS$, $\|S\| \leqslant K'$ and $\|C\| \leqslant 1$. We may replace S and C by S/K' and $K'C$, respectively. Thus, we may instead assume that $\|S\| \leqslant 1$ and $\|C\| \leqslant K'$.

Since $U = RA$, $U(E) \subseteq R(Y)$. Also, since R is of Class $\mathcal{C}$, $R(Y)$ is an ideal in X. By the comment at the beginning of the proof, $\overline{R(Y)} = X$. Let $B = SR : Y \to Z$. Then $T = CBA$, $\|A\| \leqslant K$, $\|C\| \leqslant K'$ and $\|B\| \leqslant 1$. Since both R and S are lattice homomorphisms with dense range, B is a lattice homomorphism with dense range. Thus, B is of Class $\mathcal{D}$. $\qquad\square$

4.4. **Duality of factorization.** Our next goal is to generalize [24, Theorem 5]. Let us start by studying the relations between the Banach lattices induced by a closed bounded convex solid set B and its polar B°, and their corresponding duals.

Proposition 4.21. *Let B be a closed bounded convex solid subset of X. Denote the Minkowski functional of B on X by ρ and the Minkowski functional of B° on X^* by ρ^*. Let $Y = (\operatorname{span} B, \rho)$. Let $Q : X^* \mapsto X^*/\ker \rho^*$ be the quotient map, and let Z be the completion of $(X^*/\ker \rho^*, \overline{\rho^*})$, where $\overline{\rho^*}(Qu^*) = \rho^*(u^*)$. Then Y and Z are Banach lattices, and there are unique bounded linear maps ϕ and ψ determined as follows:*

$$\phi : Y \to Z^*, \quad \langle Qu^*, \phi(u) \rangle = u^*(u),$$
$$\psi : Z \to Y^*, \quad \langle u, \psi(Qu^*) \rangle = u^*(u).$$

Moreover, ϕ and ψ are lattice isometric embeddings, the map ψ is of Class $\mathcal{C}$, and, if X has order continuous norm, then the map ϕ is of Class $\mathcal{C}$.

Proof. The fact that (Y, ρ) is a Banach lattice follows from Lemma 4.2. It is clear that Z is a Banach lattice. If $u \in Y$ and $u^* \in X^*$, then $|u^*(u)| \leqslant \rho(u) \cdot \rho^*(u^*) = \rho(u) \cdot \overline{\rho^*}(Qu^*)$. It follows that $\phi(u)$ determines a bounded linear functional on $(X^*/\ker \rho^*, \overline{\rho^*})$ and hence $\phi(u) \in Z^*$. Similarly, we have that ψ is a bounded linear map on $(X^*/\ker \rho^*, \overline{\rho^*})$ and hence extends uniquely to a bounded linear map on Z. Furthermore,

$$\|\phi(u)\|_{Z^*} = \sup_{\overline{\rho^*}(Qu^*) \leqslant 1} \langle Qu^*, \phi(u)\rangle = \sup_{\rho^*(u^*) \leqslant 1} u^*(u) = \rho(u).$$

Thus, ϕ is an isometric embedding. Similarly, it can be seen that ψ is an isometric embedding. If $u \in Y_+$ and $u^* \in X^*$, then

$$\langle u, |\psi(Qu^*)|\rangle = \sup_{|v| \leqslant u} |\langle v, \psi(Qu^*)\rangle| = \sup_{|v| \leqslant u} |u^*(v)| = |u^*|(u) = \langle u, \psi(|Qu^*|)\rangle.$$

Hence, $|\psi(Qu^*)| = \psi(|Qu^*|)$. Since $Q(X^*)$ is dense in Z, ψ is a lattice homomorphism. Similarly, ϕ is a lattice homomorphism.

Next, we claim that for any $u^* \in X_+^*$, $\psi[0, Qu^*] = [0, \psi Qu^*]$. Indeed, the inclusion $\subseteq$ is clear. Suppose that $x^* \in [0, \psi Qu^*]$. For any $u \in Y$,

$$|\langle u, x^*\rangle| \leqslant \langle |u|, x^*\rangle \leqslant \langle |u|, \psi Qu^*\rangle = u^*(|u|).$$

By the Hahn-Banach Theorem for positive functionals (cf. [1, Theorem 1.26]), there exists a functional v^* on X such that $v^*|_Y = x^*$ and $0 \leqslant v^*(u) \leqslant u^*(u)$ for all $u \in X_+$. Thus, $Qv^* \in [0, Qu^*]$ and for any $u \in Y$,

$$\langle u, \psi Qv^*\rangle = v^*(u) = \langle u, x^*\rangle.$$

Hence, $\psi Qv^* = x^*$. This completes the proof of the claim.

We extend the claim to show that ψ is interval preserving and thus ψ is of Class $\mathcal{C}$. Suppose that $z \in Z_+$ and $0 \leqslant x^* \leqslant \psi z$. For each $n \in \mathbb{N}$, choose $v_n^* \in X_+^*$ so that $\|z - Qv_n^*\|_Z < \frac{1}{n}$. Set $y_n^* = (x^* + \psi Qv_n^* - \psi z)^+$. Then $0 \leqslant y_n^* \leqslant \psi Qv_n^*$ and

$$\|y_n^* - x^*\| \leqslant \|\psi Qv_n^* - \psi z\| = \|Qv_n^* - z\|_Z < \frac{1}{n}.$$

By the claim, $y_n^* \in [0, \psi Qv_n^*] = \psi[0, Qv_n^*]$. Hence, there exists $w_n^* \in X^*$, $Qw_n^* \in [0, Qv_n^*]$, so that $y_n^* = \psi Qw_n^*$. Since (y_n^*) converges to x^* and ψ is an isometry, (Qw_n^*) converges in Z to some $z_0 \in Z$. Clearly, $0 \leqslant z_0 \leqslant \lim Qv_n^* = z$ and

$$\psi z_0 = \lim \psi Qw_n^* = \lim y_n^* = x^*.$$

This shows that $[0, \psi z] \subseteq \psi[0, z]$. The reverse inclusion is clear.

Finally, we show that ϕ is interval preserving if X has order continuous norm. Let $J_X : X \to X^{**}$ be the canonical inclusion. Then $J_X[-u, u] = [-J_X u, J_X u]$ for all $u \in X_+$. Suppose that $u \in Y_+$. Then $\phi[0, u] \subseteq [0, \phi u]$. Conversely, let $z^* \in Z$, $z^* \in [0, \phi u]$. Since $Q : X^* \to (X^*/\ker \rho^*, \overline{\rho^*}) \subseteq Z$ is bounded, $Q^* z^* \in X^{**}$. For any $u^* \in X^*$,

$$|\langle u^*, Q^* z^*\rangle| \leqslant \langle |Qu^*|, z^*\rangle \leqslant \langle Q|u^*|, \phi u\rangle = \langle u, |u^*|\rangle = \langle |u^*|, J_X u\rangle.$$

Hence, $|Q^* z^*| \leqslant J_X u$. Thus $Q^* z^* \in [-J_X u, J_X u] = J_X[-u, u]$. Let $v \in X$, $v \in [-u, u]$ be such that $J_X v = Q^* z^*$. Then $v \in Y$ and

$$\langle Qu^*, \phi v\rangle = u^*(v) = \langle u^*, Q^* z^*\rangle = \langle Qu^*, z^*\rangle$$

for all $u^* \in X^*$. Therefore, $\phi v = z^*$. This completes the proof that $\phi[0, u] = [0, \phi u]$.   □

Now, let $T : E \to X$ be a (τ, σ)-convex operator, and assume that σ is block concave and τ is block convex. By Theorem 4.17, there is a minimal Class $\mathcal{C}$ factorization (Y_0, U_0, V_0) of T so that $Y_0 = \operatorname{span}(C_T^{\tau,\sigma})^{\circ\circ}$ with norm given by the Minkowski functional of $(C_T^{\tau,\sigma})^{\circ\circ}$. By Theorem 4.12, $T^* : X^* \to E^*$ is (τ^*, σ^*)-concave, σ^* is block convex and τ^* is block concave. By Theorem 4.19, there is a maximal Class $\mathcal{D}$ factorization (Y^0, U^0, V^0) of T^*. Let ρ be the Minkowski functional of $D_{\tau^*,\sigma^*}^{T^*}$ and let $\overline{\rho}$ be the norm on $X^*/\ker \rho$ induced by $D_{\tau^*,\sigma^*}^{T^*}$. Then Y^0 is the completion of $(X^*/\ker \rho, \overline{\rho})$. From Theorem 4.10(1), we have $(C_T^{\tau,\sigma})^{\circ} = D_{\tau^*,\sigma^*}^{T^*}$. As a result, using Proposition 4.21, we obtain partial duality of the Banach lattices Y_0 and Y^0. Note that below we use the fact that Y_0 is a subset of X by construction.

Theorem 4.22. *Let $T : E \to X$ be a (τ, σ)-convex operator and assume that σ is block concave and τ is block convex. Let (Y_0, U_0, V_0) be the Class $\mathcal{C}$ factorization of T induced by the set $(C_T^{\tau,\sigma})^{\circ\circ}$ and let (Y^0, U^0, V^0) be the Class $\mathcal{D}$ factorization of T^* induced by the set $D_{\tau^*,\sigma^*}^{T^*}$. Let the maps ϕ and ψ be determined by*

$$\phi : Y_0 \to (Y^0)^*, \quad \langle U^0 u^*, \phi(u) \rangle = u^*(u),$$
$$\psi : Y^0 \to (Y_0)^*, \quad \langle u, \psi(U^0 u^*) \rangle = u^*(u).$$

Then ϕ and ψ are lattice isometric embeddings. The map ψ is of Class $\mathcal{C}$. If X has order continuous norm, then the map ϕ is of Class $\mathcal{C}$.

Proof. The proof is essentially immediate. Let us just check that ϕ is an into isometry. Denote the norm on $(Y^0)^*$ by $\overline{\rho}^*$. Note that U^0 is the quotient map $X^* \to X^*/\ker \rho$. Thus,

$$\overline{\rho}^*(\phi(u)) = \sup_{\overline{\rho}(U^0 u^*) \leqslant 1} \langle U^0 u^*, \phi(u) \rangle = \sup_{\rho(u^*) \leqslant 1} u^*(u) = \sup_{u^* \in D_{\tau^*,\sigma^*}^{T^*}} u^*(u) = \|u\|_{Y_0},$$

where the last equality holds because $\|\cdot\|_{Y_0}$ is the Minkowski functional of $(C_T^{\tau,\sigma})^{\circ\circ} = (D_{\tau^*,\sigma^*}^{T^*})^{\circ}$.   □

A similar result holds for concave operators and their adjoints. Let $T : X \to E$ be a (τ, σ)-concave operator, where σ is block convex and τ is block concave. Let θ be the Minkowski functional of the set $D_{\tau,\sigma}^T$ and let $\overline{\theta}$ be the induced norm on $X/\ker \theta$. Let Z^0 be the completion of $(X/\ker \theta, \overline{\theta})$. By Theorem 4.19, there is a maximal Class $\mathcal{D}$ factorization (Z^0, A^0, B^0) of T. Also, there is a minimal Class $\mathcal{C}$ factorization (Z_0, A_0, B_0) of T^*, where $Z_0 = \operatorname{span}(C_{T^*}^{\tau^*,\sigma^*})^{\circ\circ}$, normed by the Minkowski functional of $\operatorname{span}(C_{T^*}^{\tau^*,\sigma^*})^{\circ\circ}$. As a result of the duality $(C_{T^*}^{\tau^*,\sigma^*})^{\circ} = D_{\tau,\sigma}^T$ (Theorem 4.10(2)), we obtain the following duality theorem. The proof is similar to that of Theorem 4.22 and is omitted.

Theorem 4.23. *Let $T : X \to E$ be a (τ, σ)-concave operator and assume that σ is block convex and τ is block concave. Let (Z^0, A^0, B^0) be the Class $\mathcal{D}$ factorization of T induced by the set $D_{\tau,\sigma}^T$ and let (Z_0, A_0, B_0) be the Class $\mathcal{C}$ factorization of T^* induced by the set $(C_{T^*}^{\tau^*,\sigma^*})^{\circ\circ}$. Let the maps $\widetilde{\phi}$ and $\widetilde{\psi}$ be determined by*

$$\widetilde{\phi} : Z^0 \to (Z_0)^*, \quad \langle u^*, \widetilde{\phi}(A^0 u) \rangle = u^*(u),$$
$$\widetilde{\psi} : Z_0 \to (Z^0)^*, \quad \langle A^0 u, \widetilde{\psi}(u^*) \rangle = u^*(u).$$

Then $\widetilde{\phi}$ and $\widetilde{\psi}$ are lattice isometric embeddings. The map $\widetilde{\psi}$ is of Class $\mathcal{C}$. If X has order continuous norm, then the map $\widetilde{\phi}$ is of Class $\mathcal{C}$.

4.5. **Factorization of a simultaneously convex and concave operator.** Let X and Y be Banach lattices. In [24], a factorization for an operator $T : X \to Y$ that is both p-convex and q-concave was given. In this section, we adapt an argument of [25] to prove an analogous factorization result that allows us to replace the p-convexity or q-concavity assumptions by (p, ∞)-convexity and $(q, 1)$-concavity, respectively. We start with the following:

Proposition 4.24. *Let Z be a Banach lattice and let C_0, C_1 be bounded convex sets in Z. For any $0 < \theta < 1$, let*

(4.1)
$$C_\theta = \{v \in Z : |x| \leqslant |v_0|^\theta |v_1|^{1-\theta} \text{ for some } v_0 \in C_0, v_1 \in C_1\}.$$

Then C_θ is a bounded convex solid set in Z.

Proof. Suppose that a_0, a_1, b_0, b_1 are nonnegative real numbers. Let $0 < \alpha < 1$. Then

$$(1-\alpha)a_0^\theta a_1^{1-\theta} + \alpha b_0^\theta b_1^{1-\theta} \leqslant \|((1-\alpha)^\theta a_0^\theta, \alpha^\theta b_0^\theta)\|_{\frac{1}{\theta}} \cdot \|((1-\alpha)^{1-\theta} a_1^{1-\theta}, \alpha^{1-\theta} b_1^{1-\theta})\|_{\frac{1}{1-\theta}}$$
$$= ((1-\alpha)a_0 + \alpha b_0)^\theta \cdot ((1-\alpha)a_1 + \alpha b_1)^{1-\theta}.$$

Now, for $v, w \in C_\theta$, there are $v_0, w_0 \in C_0$ and $v_1, w_1 \in C_1$ so that $|v| \leqslant |v_0|^\theta |v_1|^{1-\theta}$ and $|w| \leqslant |w_0|^\theta |w_1|^{1-\theta}$. Hence,

$$|(1-\alpha)v + \alpha w| \leqslant (1-\alpha)|v| + \alpha|w| \leqslant (1-\alpha)|v_0|^\theta |v_1|^{1-\theta} + \alpha|w_0|^\theta |w_1|^{1-\theta}$$
$$\leqslant ((1-\alpha)|v_0| + \alpha|w_0|)^\theta \cdot ((1-\alpha)|v_1| + \alpha|w_1|)^{1-\theta}.$$

Therefore, $(1 - \alpha)v + \alpha w \in C_\theta$. It follows that C_θ is convex. Clearly, C_θ is also solid. By [17, Proposition 1.d.2], if $|v| \leqslant |v_0|^\theta |v_1|^{1-\theta}$, then $\|v\| \leqslant \|v_0\|^\theta \|v_1\|^{1-\theta}$. Thus, the boundedness of C_θ follows from that of C_0 and C_1. $\square$

4.5.1. *Factoring a (p, p_2)-convex and $(q, 1)$-concave operator.* In this subsection, let X, Y be Banach lattices, $1 \leqslant p, q \leqslant \infty$ and $p_2 \in \{p, \infty\}$. Suppose that $T : X \to Y$ is a bounded linear operator that is (p, p_2)-convex with constant K_p and $(q, 1)$-concave with constant K_q. As in Definition 4.5, let

$$C_T^{p,p_2} = \left\{ v \in Y : \exists (u_i)_{i=1}^n \subseteq X \text{ such that } |v| \leqslant \left(\sum_{i=1}^n |Tu_i|^{p_2} \right)^{\frac{1}{p_2}}, \sum_{i=1}^n \|u_i\|^p \leqslant 1 \right\},$$

which corresponds to the set $C_T^{\tau;\sigma}$ taking the norms $\sigma = \|\cdot\|_{p_2}$ and $\tau = \|\cdot\|_p$ on c_{00}, both of them block convex and block concave with constant 1. By Proposition 4.16, $C_0 := (C_T^{p,p_2})^{\circ\circ}$ is a closed convex solid set in Y so that $T(B_X) \subseteq C_0 \subseteq K_p B_Y$. By Proposition 4.15, if ρ_0 is the Minkowski functional of C_0, then

$$\rho_0\left(\left(\sum_{i=1}^n |u_i|^{p_2}\right)^{\frac{1}{p_2}}\right) \leqslant \left(\sum_{i=1}^n \rho_0(u_i)^p\right)^{\frac{1}{p}},$$

for $(u_i)_{i=1}^n \subseteq \operatorname{span} C_0$.

Proposition 4.25. *Let $C_0 = (C_T^{p,p_2})^{\circ\circ}$ and $C_1 = B_Y$. For any $0 < \theta < 1$, let p_θ be determined by the equation $\frac{1}{p_\theta} = \frac{\theta}{p} + (1 - \theta)$ and set $\overline{p_2} = p_\theta$ if $p_2 = p$, and $\overline{p_2} = \infty$ if $p_2 = \infty$. Define C_θ by (4.1) and let ρ_θ be the Minkowski functional of C_θ on $Z_\theta := \operatorname{span} C_\theta$. Then $T(B_X) \subseteq K_p^{1-\theta} C_\theta$ and (Z_θ, ρ_θ) is $(p_\theta, \overline{p_2})$-convex with constant 1.*

Proof. First, observe that if $K_p = 0$, then T is the null operator, $Z_\theta = \{0\}$, and the result becomes trivial. Therefore, we can assume that $K_p > 0$. Now, suppose that $u \in B_X$. Then $Tu \in C_0 \subseteq K_p C_1$, and hence

$$\frac{|Tu|}{K_p^{1-\theta}} \leqslant |Tu|^\theta \left(\frac{|Tu|}{K_p}\right)^{1-\theta} \in C_\theta.$$

Thus, $T(B_X) \subseteq K_p^{1-\theta} C_\theta$.

Now, suppose that $v_i \in \alpha_i C_\theta$, $1 \leqslant i \leqslant n$. There are $v_0^i \in C_0$, $v_1^i \in C_1$ so that $|v_i| \leqslant \alpha_i |v_0^i|^\theta |v_1^i|^{1-\theta}$. First, consider the case where $p_2 = p$. Note that $\frac{p}{\theta p_\theta}$ and $\frac{1}{(1-\theta)p_\theta}$ are conjugate indices. Thus, using Hölder's inequality in the second step below, we have

$$(4.2) \qquad \left(\sum_{i=1}^n |v_i|^{p_\theta}\right)^{\frac{1}{p_\theta}} \leqslant \left(\sum_{i=1}^n (\alpha_i^{\frac{p_\theta}{p}} |v_0^i|)^{\theta p_\theta} (\alpha_i^{p_\theta} |v_1^i|)^{(1-\theta)p_\theta}\right)^{\frac{1}{p_\theta}}$$

$$\leqslant \left(\left(\sum_{i=1}^n (\alpha_i^{\frac{p_\theta}{p}} |v_0^i|)^p\right)^{\frac{1}{p}}\right)^\theta \left(\sum_{i=1}^n \alpha_i^{p_\theta} |v_1^i|\right)^{(1-\theta)}.$$

Hence,

$$\rho_\theta\left(\left(\sum_{i=1}^n |v_i|^{p_\theta}\right)^{\frac{1}{p_\theta}}\right) \leqslant \rho_0\left(\left(\sum_{i=1}^n (\alpha_i^{\frac{p_\theta}{p}} |v_0^i|)^p\right)^{\frac{1}{p}}\right)^\theta \left\|\sum_{i=1}^n \alpha_i^{p_\theta} |v_1^i|\right\|_Y^{1-\theta}$$

$$\leqslant \left(\sum_{i=1}^n \alpha_i^{p_\theta}\right)^{\frac{\theta}{p}} \left(\sum_{i=1}^n \alpha_i^{p_\theta}\right)^{1-\theta} = \left(\sum_{i=1}^n \alpha_i^{p_\theta}\right)^{\frac{1}{p_\theta}}.$$

For the case $p_2 = \infty$, we use the inequality

$$(4.3) \qquad \bigvee_{i=1}^n |v_i| \leqslant \left(\bigvee_{i=1}^n (\alpha_i^{\frac{p_\theta}{p}} |v_0^i|)\right)^\theta \left(\bigvee_{i=1}^n (\alpha_i^{p_\theta} |v_1^i|)\right)^{1-\theta}$$

to obtain

$$\rho_\theta\left(\bigvee_{i=1}^n |v_i|\right) \leqslant \rho_0\left(\bigvee_{i=1}^n (\alpha_i^{\frac{p_\theta}{p}} |v_0^i|)\right)^\theta \left\|\sum_{i=1}^n \alpha_i^{p_\theta} |v_1^i|\right\|_Y^{1-\theta}$$

$$\leqslant \left(\sum_{i=1}^n \alpha_i^{p_\theta}\right)^{\frac{\theta}{p}} \left(\sum_{i=1}^n \alpha_i^{p_\theta}\right)^{1-\theta} = \left(\sum_{i=1}^n \alpha_i^{p_\theta}\right)^{\frac{1}{p_\theta}}.$$

In both cases, we may take α_i to be arbitrarily close to $\rho_\theta(v_i)$ to complete the proof of the proposition. $\qquad\square$

In Proposition 4.25 we have exploited the convexity of the operator T to build a $(p_\theta, \overline{p_2})$-convex normed vector lattice Z_θ. Next, we exploit the $(q, 1)$-concavity of T:

Proposition 4.26. *For $0 < \theta < 1$, set $q_\theta = \frac{q}{1-\theta}$. Let $S_\theta : X \to Z_\theta$ be the operator defined by $S_\theta u = Tu$. Then S_θ is $(q_\theta, 1)$-concave with constant $K_q^{1-\theta}$.*

Proof. First, note that S_θ maps into Z_θ by Proposition 4.25. Let $(u_i)_{i=1}^n \subseteq X$. Write ρ_1 for the original norm on Y. Note that

$$|S_\theta u_i| = |Tu_i| = \rho_0(Tu_i)^\theta \rho_1(Tu_i)^{1-\theta} \left(\frac{|Tu_i|}{\rho_0(Tu_i)} \right)^\theta \left(\frac{|Tu_i|}{\rho_1(Tu_i)} \right)^{1-\theta},$$

which implies that $\rho_\theta(S_\theta u_i) \leqslant \rho_0(Tu_i)^\theta \rho_1(Tu_i)^{1-\theta}$. Thus

$$\left(\sum_{i=1}^n \rho_\theta(S_\theta u_i)^{q_\theta} \right)^{\frac{1}{q_\theta}} \leqslant \left(\sum_{i=1}^n \rho_0(Tu_i)^{\theta q_\theta} \rho_1(Tu_i)^{(1-\theta)q_\theta} \right)^{\frac{1}{q_\theta}}$$

$$\leqslant \sup_i \rho_0(Tu_i)^\theta \left(\sum_{i=1}^n \rho_1(Tu_i)^q \right)^{\frac{1}{q_\theta}}.$$

Let $u = \sum_{i=1}^n |u_i|$. Since T is $(q,1)$-concave with constant K_q,

$$\|u\| = \left\| \sum_{i=1}^n |u_i| \right\| \geqslant K_q^{-1} \left(\sum_{i=1}^n \rho_1(Tu_i)^q \right)^{\frac{1}{q}}.$$

Since $T(B_X) \subseteq C_0$, $\rho_0(Tu_i) \leqslant \|u_i\| \leqslant \|u\|$. Hence,

$$\left(\sum_{i=1}^n \rho_\theta(S_\theta u_i)^{q_\theta} \right)^{\frac{1}{q_\theta}} \leqslant \|u\|^\theta (K_q \|u\|)^{\frac{q}{q_\theta}} = K_q^{1-\theta} \|u\|.$$

Therefore, S_θ is $(q_\theta, 1)$-concave with constant $K_q^{1-\theta}$.   □

The previous results yield an analogue of [24, Theorem 15] for operators that are simultaneously (p, p_2)-convex and $(q, 1)$-concave:

Theorem 4.27. *Let X, Y be Banach lattices and let $T : X \to Y$ be a bounded linear operator that is both (p, p_2)-convex ($p_2 \in \{p, \infty\}$) and $(q, 1)$-concave. Let $0 < \theta < 1$. Determine p_θ, q_θ by the equations $\frac{1}{p_\theta} = \frac{\theta}{p} + (1 - \theta)$ and $q_\theta = \frac{q}{1-\theta}$. Set $\overline{p_2} = p_\theta$ if $p_2 = p$, and $\overline{p_2} = \infty$ if $p_2 = \infty$. There exists*

(1) a $(p_\theta, \overline{p_2})$-convex Banach lattice $\mathcal{Z}_\theta$ and a $(q_\theta, 1)$-concave Banach lattice W_θ;
(2) an interval preserving injective lattice homomorphism $\Phi : \mathcal{Z}_\theta \to Y$ and a Class $\mathcal{D}$ operator $\Psi : X \to W_\theta$;
(3) a bounded linear operator $T_\theta : W_\theta \to \mathcal{Z}_\theta$,

so that $T = \Phi T_\theta \Psi$.

Proof. Let (Z_θ, ρ_θ) be as in Proposition 4.25 and let $S_\theta : X \to Z_\theta$ be defined by $S_\theta u = Tu$. By Proposition 4.25, (Z_θ, ρ_θ) is a $(p_\theta, \overline{p_2})$-convex normed vector lattice. Hence, its norm completion $\mathcal{Z}_\theta$ is a $(p_\theta, \overline{p_2})$-convex Banach lattice. Since C_θ is a bounded convex solid subset of Y, its Minkowski functional ρ_θ dominates the norm of Y. Thus, every ρ_θ-Cauchy sequence in Z_θ converges in Y. Hence $\mathcal{Z}_\theta$ may be identified (as a vector lattice) with

$$\{v \in Y : \exists \text{ a } \rho_\theta\text{-Cauchy sequence } (v_n)_{n=1}^\infty \text{ in } Z_\theta \text{ so that } (v_n)_{n=1}^\infty \text{ converges to } v \text{ in } Y\}.$$

In this way, $\mathcal{Z}_\theta$ is (identified with) a vector lattice ideal in Y. Hence, the inclusion map $\Phi : \mathcal{Z}_\theta \to Y$ is an interval preserving injective lattice homomorphism. Note that $\Phi S_\theta = T$ by the definition of Φ and S_θ.

By Proposition 4.26, $S_\theta : X \to Z_\theta \subseteq \mathcal{Z}_\theta$ is $(q_\theta, 1)$-concave. Hence, by Theorem 4.1(2), there is a $(q_\theta, 1)$-concave Banach lattice W_θ, a Class $\mathcal{D}$ operator $\Psi : X \to W_\theta$, and a bounded linear operator $T_\theta : W_\theta \to \mathcal{Z}_\theta$ so that $S_\theta = T_\theta \Psi$. Therefore, $T = \Phi S_\theta = \Phi T_\theta \Psi$, as claimed. $\qquad\square$

4.5.2. *Factoring a (p, ∞)-convex and (q, q_2)-concave operator.* In this subsection, let X, Y be Banach lattices, $1 \leqslant p, q \leqslant \infty$ and $q_2 \in \{q, 1\}$. Suppose that $T : X \to Y$ is a bounded linear operator that is (p, ∞)-convex with constant K_p and (q, q_2)-concave with constant K_q. As in Definition 4.5, let

$$C_{T^*}^{q^*, q_2^*} = \left\{ u^* \in X^* : \exists (v_i^*)_{i=1}^n \subseteq Y^* \text{ such that } |u^*| \leqslant \left(\sum_{i=1}^n |T^* v_i^*|^{q_2^*} \right)^{\frac{1}{q_2^*}}, \sum_{i=1}^n \|v_i\|^{q^*} \leqslant 1 \right\},$$

which corresponds to the set $C_{T^*}^{\tau, \sigma}$ taking the norms $\sigma = \| \cdot \|_{q_2^*}$ and $\tau = \| \cdot \|_{q^*}$ on c_{00}, both of them block convex and block concave with constant 1. Since T is (q, q_2)-concave with constant K_q, T^* is (q^*, q_2^*)-convex with constant K_q by Theorem 4.13. By Proposition 4.16, $C_0 := (C_{T^*}^{q^*, q_2^*})^{\circ\circ}$ is a closed convex solid set in X^* so that $T^* B_{Y^*} \subseteq C_0 \subseteq K_q B_{X^*}$. Fix $0 < \theta < 1$ and let

$$C_\theta = \{ u^* \in X^* : |u^*| \leqslant |u_0^*|^\theta |u_1^*|^{1-\theta}, u_0^* \in C_0, u_1^* \in B_{X^*} \}.$$

Let $Z_\theta = \text{span} \, C_\theta$ and let ρ_θ be the Minkowski functional of C_θ. Then (Z_θ, ρ_θ) is a normed vector lattice. By Proposition 4.25, $T^* B_{Y^*} \subseteq K_q^{1-\theta} C_\theta$ and (Z_θ, ρ_θ) is $(q_\theta^*, (\overline{q_2})^*)$-convex with constant 1, where

$$\frac{1}{q_\theta^*} = \frac{\theta}{q^*} + (1 - \theta) \quad \text{and} \quad \overline{q_2} = \begin{cases} q_\theta & \text{if } q_2 = q, \\ 1 & \text{if } q_2 = 1. \end{cases}$$

Note that the first condition is equivalent to taking $q_\theta = \frac{q}{\theta}$. Let $D \subseteq X$ be the set $(C_\theta)^\circ$, where the polar is taken with respect to the duality $\langle X, X^* \rangle$.

Proposition 4.28. *The set D is a closed convex solid subset of X that contains $K_p^{-\theta} B_X$. Let π be the Minkowski functional of D on X and let $Q : X \to X/\ker \pi$ be the quotient map. Define $\widehat{\pi}$ on $X/\ker \pi$ by $\widehat{\pi}(Qu) = \pi(u)$. Then $(X/\ker \pi, \widehat{\pi})$ is a normed vector lattice. The map $j : (X/\ker \pi, \widehat{\pi}) \to (Z_\theta, \rho_\theta)^*$ given by $\langle u^*, j(Qu) \rangle = \langle u, u^* \rangle$ is a well-defined lattice isometric embedding. In particular, $(X/\ker \pi, \widehat{\pi})$ is $(q_\theta, \overline{q_2})$-concave.*

Proof. Since $C_\theta \subseteq K_p^\theta B_{X^*}$ is a bounded solid subset of X^*, D is a closed convex solid subset of X that contains $K_p^{-\theta} B_X$. In particular, $\ker \pi$ is a vector lattice ideal in X and $\widehat{\pi}$ is a well-defined lattice norm on the vector lattice $X/\ker \pi$. If $Qu = 0$, then $u \in \lambda D$ for all $\lambda > 0$. Thus, $|\langle u, u^* \rangle| \leqslant \lambda$ for any $u^* \in C_\theta$ and $\lambda > 0$. Hence, $\langle u, u^* \rangle = 0$ for any $u^* \in C_\theta$. It follows that $\langle u, u^* \rangle = 0$ for any $u^* \in Z_\theta$. This shows that the map j is well defined. For any $u \in X$,

$$\|j(Qu)\| = \sup_{\rho_\theta(u^*) \leqslant 1} |\langle u^*, j(Qu) \rangle| = \sup_{u^* \in C_\theta} |\langle u, u^* \rangle|.$$

If $\widehat{\pi}(Qu) < 1$, then $u \in D$ and hence $|\langle u, u^* \rangle| \leqslant 1$ for any $u^* \in C_\theta$, since $D = (C_\theta)^\circ$. On the other hand, if $\widehat{\pi}(Qu) > 1$, then $u \notin D = (C_\theta)^\circ$. Hence, there exists $u^* \in C_\theta$ such that $\langle u, u^* \rangle > 1$. Thus, $\|j(Qu)\| > 1$ by the above equation. This completes the proof that j is an isometric embedding.

For any $u \in X$ and $u^* \in (Z_\theta)_+$,

$$\langle u^*, |j(Qu)| \rangle = \sup_{|w^*| \leqslant |u^*|} |\langle w^*, jQu \rangle| = \sup_{|w^*| \leqslant |u^*|} |\langle u, w^* \rangle| = \langle |u|, u^* \rangle = \langle u^*, jQ|u| \rangle.$$

Thus $|jQu| = jQ|u|$. Since Q is a lattice homomorphism, $Q|u| = |Qu|$. Therefore, $|jQu| = j|Qu|$. This shows that j is a lattice homomorphism.

Finally, since (Z_θ, ρ_θ) is $(q_\theta^*, (\overline{q_2})^*)$-convex, its dual is $(q_\theta, \overline{q_2})$-concave by Theorem 4.12. Thus, $(X/\ker \pi, \widehat{\pi})$, which is lattice isometric to a sublattice of $(Z_\theta, \rho_\theta)^*$, is $(q_\theta, \overline{q_2})$-concave. $\qquad\square$

The next step is to show that T factors through $(X/\ker \pi, \widehat{\pi})$ as $T = RQ$, with R a (p_θ, ∞)-convex operator.

Proposition 4.29. *Define* $R : (X/\ker \pi, \widehat{\pi}) \to Y$ *by* $R(Qu) = Tu$. *Then* R *is a well-defined bounded linear operator.*

Proof. Suppose that $u \in D$. Then $\langle u, u^* \rangle \leqslant 1$ for all $u^* \in C_\theta$. Since $T^* B_{Y*} \subseteq K_q^{1-\theta} C_\theta$, $|\langle Tu, v^* \rangle| \leqslant K_q^{1-\theta}$ for all $v^* \in B_{Y*}$. This proves that $\|Tu\| \leqslant K_q^{1-\theta} \pi(u)$. In particular, if $Qu = 0$, then $\pi(u) = 0$ and thus $Tu = 0$. Therefore, R is well defined. Also, the same inequality shows that

$$\|R(Qu)\| = \|Tu\| \leqslant K_q^{1-\theta} \pi(u) = K_q^{1-\theta} \widehat{\pi}(Qu).$$

Thus, R is bounded. $\qquad\square$

Define $S_\theta : Y^* \to Z_\theta$ by $S_\theta v^* = T^* v^*$. By Proposition 4.26, S_θ is $(p_\theta^*, 1)$-concave, where $p_\theta^* = \frac{p^*}{1-\theta}$ (equivalently, $\frac{1}{p_\theta} = \frac{1-\theta}{p} + \theta$).

Proposition 4.30. *Define* $\psi : (Z_\theta, \rho_\theta) \to (X/\ker \pi, \widehat{\pi})^*$ *by* $\langle Qu, \psi u^* \rangle = \langle u, u^* \rangle$. *Then* ψ *is a well-defined bounded linear operator so that* $R^* = \psi S_\theta$. *Consequently,* R *is* (p_θ, ∞)-convex.

Proof. Note that $|\langle u, u^* \rangle| \leqslant 1$ if $u \in D$ and $u^* \in C_\theta$. Hence $\langle u, u^* \rangle = 0$ if $Qu = 0$ and $u^* \in Z_\theta$. This shows that ψu^* is a well-defined functional on $X/\ker \pi$. Also, if $\widehat{\pi}(Qu) < 1$, then $u \in D$ and hence $|\langle Qu, \psi u^* \rangle| = |\langle u, u^* \rangle| \leqslant \rho_\theta(u^*)$ for any $u^* \in Z_\theta$. Hence, $\psi u^* \in (X/\ker \pi, \widehat{\pi})^*$ and $\|\psi u*\| \leqslant \rho_\theta(u^*)$. Therefore, ψ is a bounded linear operator.

For any $v^* \in Y^*$ and any $u \in X$,

$$\langle Qu, \psi S_\theta v^* \rangle = \langle u, S_\theta v^* \rangle = \langle u, T^* v^* \rangle = \langle Tu, v^* \rangle = \langle RQu, v^* \rangle = \langle Qu, R^* v^* \rangle.$$

Thus, $R^* = \psi S_\theta$. Finally, since S_θ is $(p_\theta^*, 1)$-concave, so is R^*. Hence R is (p_θ, ∞)-convex. $\qquad\square$

Finally, we can state the analogue of [24, Theorem 15] for operators that are simultaneously (p, ∞)-convex and (q, q_2)-concave.

Theorem 4.31. *Let* X, Y *be Banach lattices and let* $T : X \to Y$ *be a bounded linear operator that is both* (p, ∞)-convex *and* (q, q_2)-concave *($q_2 \in \{q, 1\}$). Let* $0 < \theta < 1$. *Determine* p_θ, q_θ *by the equations* $q_\theta = \frac{q}{\theta}$ *and* $\frac{1}{p_\theta} = \frac{\theta}{p} + 1 - \theta$. *Set* $\overline{q_2} = q_\theta$ *if* $q_2 = q$, *and* $\overline{q_2} = 1$ *if* $q_2 = 1$. *There exists*

    (1) a $(q_\theta, \overline{q_2})$-concave *Banach lattice* $\mathcal{Z}_\theta$ *and a* (p_θ, ∞)-convex *Banach lattice* W_θ,
    (2) a Class $\mathcal{C}$ *operator* $\Phi : W_\theta \to Y$ *and a Class* $\mathcal{D}$ *operator* $\Psi : X \to \mathcal{Z}_\theta$,
    (3) a bounded linear operator $T_\theta : \mathcal{Z}_\theta \to Y_\theta$,

so that $T = \Phi T_\theta \Psi$.

Proof. Let $\mathcal{Z}_\theta$ be the completion of $(X/\ker \pi, \widehat{\pi})$ and let $\mathcal{R} : \mathcal{Z}_\theta \to Y$ be the continuous extension to $\mathcal{Z}_\theta$ of the bounded linear operator $R : (X/\ker \pi, \widehat{\pi}) \to Y$. Also, let $\Psi : X \to \mathcal{Z}_\theta$ be given by $\Psi u = Qu$. Then Ψ is a Class $\mathcal{D}$ operator. By Proposition 4.29, $T = \mathcal{R}\Psi$. It follows from Proposition 4.30 that $\mathcal{R}$ is (p_θ, ∞)-convex. By Theorem 4.1(1), there is a (p_θ, ∞)-convex Banach lattice W_θ, a

Class $\mathcal{C}$ operator $\Phi : W_\theta \to Y$ and a bounded linear operator $T_\theta : \mathcal{Z}_\theta \to W_\theta$ so that $\mathcal{R} = \Phi T_\theta$. Therefore, $T = \Phi T_\theta \Psi$, as claimed. $\qquad\square$

5. Push-outs in classes of convex Banach lattices

Recall that in a given category, for objects X_0, X_1, X_2 and morphisms $T_i : X_0 \to X_i$, $i = 1, 2$, a *push-out diagram* is an object $PO = PO(T_1, T_2)$ together with morphisms $S_i : X_i \to PO$, $i = 1, 2$, making the following diagram commutative

$$\begin{array}{ccc} X_1 & \xrightarrow{\;S_1\;} & PO \\ \scriptstyle{T_1} \big\uparrow & & \big\uparrow \scriptstyle{S_2} \\ X_0 & \xrightarrow[\;T_2\;]{} & X_2 \end{array}$$

and with the universal property that if $S_i' : X_i \to Y$ are such that $S_1' T_1 = S_2' T_2$, then there is a unique $\gamma : PO \to Y$ such that $\gamma S_i = S_i'$ for $i = 1, 2$, as follows:

$$\begin{array}{ccc} & & Y \\ & {\scriptstyle \gamma} \nearrow & \\ X_1 & \xrightarrow{\;S_1\;} & PO \\ \scriptstyle{T_1} \big\uparrow & & \big\uparrow \scriptstyle{S_2} \\ X_0 & \xrightarrow[\;T_2\;]{} & X_2 \end{array}$$

In the category $\mathcal{BL}$ of Banach lattices and lattice homomorphisms, *isometric push-outs*, i.e., push-outs satisfying the additional conditions that $\max\{\|S_1\|, \|S_2\|\} \leqslant 1$ and $\|\gamma\| \leqslant \max\{\|S_1'\|, \|S_2'\|\}$, were shown to exist in [3]. The construction of such isometric push-outs was extended in [8] to certain classes of Banach lattices $\mathcal{C}$:

Theorem 5.1. *Let $\mathcal{C}$ be a class of Banach lattices which is closed under the operations of taking quotients and sublattices of ℓ_∞-sums of spaces in $\mathcal{C}$. Given Banach lattices X_0, X_1, X_2 in $\mathcal{C}$ and lattice homomorphisms $T_i : X_0 \to X_i$ for $i = 1, 2$, there is a Banach lattice $PO^\mathcal{C}$ in $\mathcal{C}$ and lattice homomorphisms S_1, S_2 so that the following is an isometric push-out diagram in $\mathcal{C}$:*

$$\begin{array}{ccc} X_1 & \xrightarrow{\;S_1\;} & PO^{\overline{\mathcal{C}}} \\ \scriptstyle{T_1} \big\uparrow & & \big\uparrow \scriptstyle{S_2} \\ X_0 & \xrightarrow[\;T_2\;]{} & X_2 \end{array}$$

Theorem 5.1 applies, in particular, when $\mathcal{C}$ is the class $\mathcal{C}_p$ of p-convex Banach lattices or the class $\mathcal{C}_{p,\infty}$ of Banach lattices with upper p-estimates (both with constant 1). For these classes, the problem of extending [3, Theorem 4.4] was also studied in [8]. This result establishes that isometric push-outs in the category $\mathcal{BL}$ preserve isometries. Namely, if in the definition of the push-out diagram above, $T_1 : X_0 \to X_1$ is contractive and $T_2 : X_0 \to X_2$ is an isometric embedding, it was shown in [3] that $S_1 : X_1 \to PO$ is an isometric embedding. For push-outs in the classes $\mathcal{C}_p$ and $\mathcal{C}_{p,\infty}$, it was shown in [8, Theorem 4.7] that S_1 is a $K_\mathcal{C}$-isomorphic embedding, with $K_\mathcal{C} = 2^{\frac{1}{p^*}}$ when $\mathcal{C} = \mathcal{C}_p$

and $K_{\mathcal{C}} = 2^{\frac{1}{p^*}} \gamma_p$ when $\mathcal{C} = \mathcal{C}_{p,\infty}$. The proof provided in [8] uses Maurey's and Pisier's Factorization Theorems. It turns out that both constants can be improved by using different techniques.

In the p-convex setting, we can obtain $K_{\mathcal{C}_p} = 1$, that is, a complete extension of [3, Theorem 4.4], using a p-concavification and p-convexification argument (see, for instance, [17, Section 1.d]):

Theorem 5.2. *Let $1 < p < \infty$ and let $\mathcal{C}_p$ be the class of p-convex Banach lattices (with constant 1). Let*

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\ S_1\ } & PO^{\mathcal{C}_p} \\
{\scriptstyle T_1}\uparrow & & \uparrow{\scriptstyle S_2} \\
X_0 & \xrightarrow[\ T_2\ ]{} & X_2
\end{array}
$$

be an isometric push-out diagram in $\mathcal{C}_p$. If T_1 is contractive and T_2 is an isometric embedding, then S_1 is an isometric embedding.

Proof. Since X_i, $i = 0, 1, 2$, are p-convex Banach lattices, we can p-concavify the spaces. Let $X_{i(p)}$ be their p-concavifications. The maps $T_i : X_{0(p)} \to X_{i(p)}$, $i = 1, 2$, will still be contractive lattice homomorphisms, since lattice homomorphisms preserve functional calculus, and moreover T_2 will still be an isometry:

$$
\|T_2 z\|_{X_{2(p)}} := \|T_2 z\|_{X_2}^p = \|z\|_{X_0}^p =: \|z\|_{X_{0(p)}}, \quad z \in X_{0(p)}.
$$

Let PO be a push-out in the category of all Banach lattices for this new diagram:

$$
\begin{array}{ccc}
X_{1(p)} & \xrightarrow{\ R_1\ } & PO \\
{\scriptstyle T_1}\uparrow & & \uparrow{\scriptstyle R_2} \\
X_{0(p)} & \xrightarrow[\ T_2\ ]{} & X_{2(p)}
\end{array}
$$

By [3, Theorem 4.4] we know that $R_1 : X_{1(p)} \to PO$ is an isometric embedding. If we p-convexify this diagram, we find that for every $i = 0, 1, 2$, $(X_{i(p)})^{(p)} = X_i$, as X_i has p-convexity constant 1. Arguing as before, the lattice homomorphisms $R_i : X_i \to PO^{(p)}$, $i = 1, 2$, are contractive, and R_1 is an isometry. Since $PO^{(p)}$ is p-convex with constant 1, we can apply the minimality property of $PO^{\mathcal{C}_p}$ to find a lattice homomorphism $\gamma : PO^{\mathcal{C}_p} \to PO^{(p)}$ such that $R_i = \gamma S_i$, $i = 1, 2$, and $\|\gamma\| \leqslant \max\{\|R_1\|, \|R_2\|\} \leqslant 1$:

$$
\begin{array}{ccc}
 & & PO^{(p)} \\
 & \overset{\gamma}{\nearrow} & \\
X_1 \xrightarrow{\ S_1\ } & PO^{\mathcal{C}_p} & \\
{\scriptstyle T_1}\uparrow \quad & \uparrow{\scriptstyle S_2} & \\
X_0 \xrightarrow[\ T_2\ ]{} & X_2 &
\end{array}
$$

It follows that

$$
\|y\|_{X_1} = \|R_1 y\|_{PO^{(p)}} = \|\gamma S_1 y\|_{PO^{(p)}} \leqslant \|S_1 y\|_{PO^{\mathcal{C}_p}} \leqslant \|y\|_{X_1},
$$

so S_1 is an isometry. $\qquad\square$

Unfortunately, this technique cannot be used in the upper p-estimates setting. However, we can adapt the abstract factorization techniques developed in Section 4 to improve the constant $K_{\mathcal{C}_{p,\infty}}$ obtained in [8].

Theorem 5.3. *Let $1 < p < \infty$ and let $\mathcal{C}_{p,\infty}$ be the class of Banach lattices with upper p-estimates (with constant 1). Let*

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\ S_1\ } & PO^{\mathcal{C}_{p,\infty}} \\
{\scriptstyle T_1}\big\uparrow & & \big\uparrow{\scriptstyle S_2} \\
X_0 & \xrightarrow[\ T_2\ ]{} & X_2
\end{array}
$$

be an isometric push-out diagram in $\mathcal{C}_{p,\infty}$. If T_1 is contractive and T_2 is an isometric embedding, then S_1 is a $2^{\frac{1}{p^*}}$-embedding.

Proof. We begin the proof by considering the isometric push-out diagram in the category $\mathfrak{BL}$ from [3, Theorem 4.4], so that R_1 is an isometric embedding:

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\ R_1\ } & PO \\
{\scriptstyle T_1}\big\uparrow & & \big\uparrow{\scriptstyle R_2} \\
X_0 & \xrightarrow[\ T_2\ ]{} & X_2
\end{array}
$$

Our aim is to construct a (p,∞)-convex Banach lattice Z with constant 1 and a lattice homomorphism $\psi : Z \to PO$ with $\|\psi\| \leqslant 2^{\frac{1}{p^*}} =: K_p$ such that for $i = 1, 2$, R_i factors through Z as $\psi \widehat{R}_i$, with $\widehat{R}_i : X_i \to Z$ contractive lattice homomorphisms. Once this is established, it follows that $\widehat{R}_1$ must be an isomorphism, since

$$\|y\|_{X_1} = \|R_1 y\|_{PO} \leqslant \|\psi\| \|\widehat{R}_1 y\|_Z \leqslant K_p \|\widehat{R}_1 y\|_Z$$

for every $y \in X_1$. Moreover, by the universal property of the push-out in $\mathcal{C}_{p,\infty}$ established in Theorem 5.1, there exists a contractive lattice homomorphism $\gamma : PO^{\mathcal{C}_{p,\infty}} \to Z$ that makes the following diagram commute:

$$
\begin{array}{ccc}
 & & Z \\
 & \nearrow^{\widehat{R}_1} \quad \nearrow^{\gamma} \nearrow & \\
X_1 \xrightarrow{\ S_1\ } & PO^{\mathcal{C}_{p,\infty}} & \\
{\scriptstyle T_1}\big\uparrow \qquad & \big\uparrow{\scriptstyle S_2} \quad \searrow^{\widehat{R}_2} & \\
X_0 \xrightarrow[\ T_2\ ]{} & X_2 &
\end{array}
$$

It follows that

$$\|y\|_{X_1} \leqslant K_p \|\widehat{R}_1 y\|_Z = K_p \|\gamma S_1 y\|_Z \leqslant K_p \|S_1 y\|_{PO^{\mathcal{C}_{p,\infty}}},$$

as we wanted to show.

Let us construct the Banach lattice Z. We will take $Z = \operatorname{span} B$, for $B \subseteq PO$ a certain closed, convex and solid set (so that Z endowed with the Minkowski functional is a Banach lattice by Lemma 4.2) such that Z is (p,∞)-convex with constant 1 and $R_i(B_{X_i}) \subseteq B \subseteq K_p B_{PO}$. The latter

will imply that R_i can be factored through Z as $R_i = \psi \widehat{R}_i$, with $\widehat{R}_i : X_i \to Z$ given by $\widehat{R}_i x = R_i x$ and $\psi : Z \to PO$ the formal identity, so that $\widehat{R}_i$ is contractive and $\|\psi\| \leqslant K_p$. Using the ideas from Section 4, let us choose $B = C^{\circ\circ}$, where

$$C = \left\{ u \in PO : \exists (v_j)_{j=1}^m \subseteq X_1, (w_j)_{j=m+1}^n \subseteq X_2 \text{ such that } |u| \leqslant \left(\bigvee_{j=1}^m |R_1 v_j| \right) \vee \left(\bigvee_{j=m+1}^n |R_2 w_j| \right) \right.$$

$$\left. \text{and } \left(\sum_{j=1}^m \|v_j\|^p + \sum_{j=m+1}^n \|w_j\|^p \right)^{\frac{1}{p}} \leqslant 1 \right\},$$

and the polars are taken with respect to $\langle PO, PO^* \rangle$. B is the closed convex hull of the solid set C, so it is closed, convex and solid. Clearly $R_i(B_{X_i}) \subseteq C \subseteq B$. Moreover, $C \subseteq K_p B_{PO}$. Indeed, if $u \in C$, there exist $(v_j)_{j=1}^m \subseteq X_1$ and $(w_j)_{j=m+1}^n \subseteq X_2$ such that

$$|u| \leqslant \left(\bigvee_{j=1}^m |R_1 v_j| \right) \vee \left(\bigvee_{j=m+1}^n |R_2 w_j| \right) \text{ and } \left(\sum_{j=1}^m \|v_j\|^p + \sum_{j=m+1}^n \|w_j\|^p \right)^{\frac{1}{p}} \leqslant 1.$$

Therefore, using that the X_i satisfy upper p-estimates with constant 1, and hence R_i are (p, ∞)-convex with constant 1, we obtain that

$$\|u\| \leqslant \left\| \left(\bigvee_{j=1}^m |R_1 v_j| \right) \vee \left(\bigvee_{j=m+1}^n |R_2 w_j| \right) \right\| \leqslant \left\| \bigvee_{j=1}^m |R_1 v_j| \right\| + \left\| \bigvee_{j=m+1}^n |R_2 w_j| \right\|$$

$$\leqslant \left(\sum_{j=1}^m \|v_j\|^p \right)^{\frac{1}{p}} + \left(\sum_{j=m+1}^n \|w_j\|^p \right)^{\frac{1}{p}} \leqslant 2^{\frac{1}{p^*}}.$$

Since B_{PO} is already closed and convex, it follows that $B \subseteq K_p B_{PO}$. It remains to show that Z satisfies an upper p-estimate (equivalently, it is (p, ∞)-convex) with constant 1. This can be done by reproducing the results from Sections 4.2 and 4.3 for the (p, ∞)-convex case, that is, choosing σ to be the c_0-norm and τ the ℓ_p-norm, and replacing the roles of the sets $C_T^{\tau,\sigma}$ and $D_{\tau^*,\sigma^*}^{T^*}$ by C and

$$D = \left\{ u^* \in PO^* : |u^*| \geqslant \left(\sum_{j=1}^n |u_j^*|^{p_2^*} \right)^{\frac{1}{p_2^*}} \text{ implies that} \right.$$

$$\left. \forall J \subseteq \{1, \ldots, n\} \left(\sum_{j \in J} \|R_1^* u_j^*\|^{p^*} + \sum_{j \notin J} \|R_2^* u_j^*\|^{p^*} \right)^{\frac{1}{p^*}} \leqslant 1 \right\},$$

respectively. More specifically, one needs to show that $C^\circ = D$ following the proofs of Proposition 4.8(1) and Theorem 4.10(1), and then adapt the argument of Proposition 4.15 to show that for every sequence $(u_i)_{i=1}^n \subseteq B = C^{\circ\circ}$ and $(\alpha_i)_{i=1}^n \subseteq \mathbb{R}$ such that $\sum_{i=1}^n |\alpha_i|^p \leqslant 1$, it follows that $\bigvee_{i=1}^n |\alpha_i u_i| \in B$, i.e., that the Minkowski functional of B induces a (p, ∞)-convex norm with constant 1 over Z. We omit the details.   □

The contrast between Theorems 5.2 and 5.3 makes us wonder whether the constant $K_p = 2^{\frac{1}{p^*}}$ in the latter result is just an artifact of the proof or, on the contrary, it is a fundamental difference between p-convexity and upper p-estimates. It should be noted that if K_p could be taken to be 1 in Theorem 5.3, this would allow us to adapt other categorical constructions from [3], such as Banach

lattices of universal disposition or separably injective Banach lattices, to the category of Banach lattices with upper p-estimates.

6. Searching for a universal separable Banach lattice with upper p-estimates

As we have seen in the previous section, certain universal constructions in the category of Banach lattices can be easily generalized to the p-convex setting via a p-concavification and p-convexification argument, but this process does not carry over to the class of Banach lattices with upper p-estimates. Another example concerns injectively universal Banach lattices. Recall that in [16] it has been shown that $C(\Delta, L_1[0,1])$ (Δ denotes the Cantor set) is injectively universal for the class of all separable Banach lattices, that is, every separable Banach lattice embeds lattice isometrically into $C(\Delta, L_1[0,1])$. Using p-concavification and p-convexification, we can show the following:

Theorem 6.1. $C(\Delta, L_p[0,1])$ *is injectively universal for the class of separable p-convex Banach lattices (with constant 1). That is, if X is a p-convex Banach lattice with constant 1, then there exists a lattice isometric embedding from X into $C(\Delta, L_p[0,1])$.*

Proof. The proof is composed of two claims: First, we show that the p-convexification of the Banach lattice $C(\Delta, L_1[0,1])$ is injectively universal for the class of separable p-convex Banach lattices, and then we show that this space can be identified with $C(\Delta, L_p[0,1])$.

To show the first claim, let X be a p-convex Banach lattice with constant 1, and consider its p-concavification $X_{(p)}$, which is also a Banach lattice. By [16, Theorem 1.1], there exists a lattice isometric embedding $T : X_{(p)} \to C(\Delta, L_1[0,1])$. If we p-convexify both spaces, we find that T is again a lattice isometry from $X = (X_{(p)})^{(p)}$ into $C(\Delta, L_1[0,1])^{(p)}$. Therefore, $C(\Delta, L_1[0,1])^{(p)}$ is injectively universal for the class of separable p-convex Banach lattices.

Next, we identify $C(\Delta, L_1[0,1])^{(p)}$ with $C(\Delta, L_p[0,1])$. Let $\Phi : C(\Delta, L_p[0,1]) \to C(\Delta, L_1[0,1])^{(p)}$ be the map that sends any function $f \in C(\Delta, L_p[0,1])$ to f^p. This map is well defined: Given any $t \in \Delta$, $f(t)^p \in L_1[0,1]$, and clearly f^p is continuous as a function from Δ into $L_1[0,1]$, so f^p belongs to $C(\Delta, L_1[0,1])$, which coincides as a set with $C(\Delta, L_1[0,1])^{(p)}$. If we denote by $\oplus$ and $\odot$ the sum and product by scalars defined in $C(\Delta, L_1[0,1])^{(p)}$ (see [17, Section 1.d]), it can be easily checked that $\Phi(f+g) = \Phi f \oplus \Phi g$ and $\Phi(\lambda f) = \lambda \odot \Phi f$ for any $f, g \in C(\Delta, L_p[0,1])$ and $\lambda \in \mathbb{R}$, so Φ is a linear map from $C(\Delta, L_p[0,1])$ to $C(\Delta, L_1[0,1])^{(p)}$. It is straightforward to check that Φ is a lattice homomorphism. We also observe that

$$\|\Phi f\|_{C(\Delta, L_1[0,1])^{(p)}} = \|f^p\|_{C(\Delta, L_1[0,1])}^{\frac{1}{p}} = \|f\|_{C(\Delta, L_p[0,1])},$$

so Φ is a lattice isometry. Finally, we can argue similarly to check that the map that sends every $h \in C(\Delta, L_1[0,1])^{(p)}$ to $h^{\frac{1}{p}}$ is the inverse map of Φ, so in particular Φ is an onto isometry. $\square$

As we discussed earlier, there is no analogue of p-convexification and p-concavification for upper p-estimates, so the proof given above does not directly adapt to show the existence of an injectively universal Banach lattice for separable Banach lattices with upper p-estimates (with constant 1). In analogy with Theorem 6.1, one could expect that if such a Banach lattice existed, it should be of the form $C(\Delta, Z)$ for some separable Banach lattice with upper p-estimates Z that is somehow canonical in the setting of upper p-estimates. A reasonable candidate could be the order continuous part of $L_{p,\infty}$. Given a measure space (Ω, Σ, μ), the *order continuous part of $L_{p,\infty}(\mu)$*, denoted $L_{p,\infty}^\circ(\mu)$, is

defined as the closure in $L_{p,\infty}(\mu)$ of the simple functions. For instance, the prototypical example of a function that belongs to $L_{p,\infty}[0,1]$ but not to $L_{p,\infty}^{\circ}[0,1]$ is $t^{-\frac{1}{p}}$. The following characterizes when a function belongs to $L_{p,\infty}^{\circ}(\mu)$. We include its proof for the sake of completeness.

Proposition 6.2. *Let $f \in L_{p,\infty}(\mu)$. Then $f \in L_{p,\infty}^{\circ}(\mu)$ if and only if*

$$\lim_{\mu(A)\to 0} \mu(A)^{-\frac{1}{p^*}} \int_A |f|\, d\mu = 0 \quad and \quad \lim_{\mu(A)\to\infty} \mu(A)^{-\frac{1}{p^*}} \int_A |f|\, d\mu = 0.$$

Proof. Let us denote by

$$Y := \left\{ f \in L_{p,\infty}(\mu) : \lim_{\mu(A)\to 0} \mu(A)^{-\frac{1}{p^*}} \int_A |f|\, d\mu = 0 \quad and \quad \lim_{\mu(A)\to\infty} \mu(A)^{-\frac{1}{p^*}} \int_A |f|\, d\mu = 0 \right\}.$$

It is clear that Y is an ideal in $L_{p,\infty}(\mu)$. Moreover, it is closed. Indeed, if $(f_n)_n \subseteq Y$ is a sequence that converges to $f \in L_{p,\infty}(\mu)$, then for every $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that $\|f - f_N\|_{L_{p,\infty}} < \frac{\varepsilon}{2}$. For this function $f_N \in Y$, there are $\delta > 0$ and $M > 0$ such that for every measurable set A with $\mu(A) < \delta$ or $\mu(A) > M$, it follows that $\mu(A)^{-\frac{1}{p^*}} \int_A |f_N|\, d\mu < \frac{\varepsilon}{2}$. Therefore,

$$\mu(A)^{-\frac{1}{p^*}} \int_A |f|\, d\mu \leqslant \mu(A)^{-\frac{1}{p^*}} \int_A |f_N|\, d\mu + \mu(A)^{-\frac{1}{p^*}} \int_A |f - f_N|\, d\mu < \frac{\varepsilon}{2} + \|f - f_N\|_{L_{p,\infty}} < \varepsilon,$$

so $f \in Y$. Now, in order to show that $L_{p,\infty}^{\circ}(\mu) \subseteq Y$, it suffices to check that every characteristic function belongs to Y. Let B be a measurable subset with $0 < \mu(B) < \infty$ and $\varepsilon > 0$. Then, for every measurable set A with $\mu(A) < \varepsilon^p$ we have that

$$\mu(A)^{-\frac{1}{p^*}} \int_A \chi_B\, d\mu = \mu(A)^{-\frac{1}{p^*}} \mu(A \cap B) \leqslant \mu(A)^{\frac{1}{p}} < \varepsilon.$$

On the other hand, if $\mu(A) > (\mu(B)/\varepsilon)^{p^*}$, it follows that

$$\mu(A)^{-\frac{1}{p^*}} \int_A \chi_B\, d\mu = \mu(A)^{-\frac{1}{p^*}} \mu(A \cap B) \leqslant \mu(A)^{-\frac{1}{p^*}} \mu(B) < \varepsilon.$$

We conclude that $\chi_B \in Y$.

To show the reverse inclusion, let us fix $f \in Y$ and $\varepsilon > 0$, and find $\delta > 0$ and $M > 0$ satisfying $\mu(A)^{-\frac{1}{p^*}} \int_A |f|\, d\mu < \frac{\varepsilon}{3}$ for every measurable set A with $\mu(A) < \delta$ or $\mu(A) > M$. Since $f \in L_{p,\infty}(\mu)$, $\mu(\{|f| > n\})$ decreases to 0 as n grows to infinity (see the definition of the quasinorm $\|| \cdot \||_{L_{p,\infty}}$), so there is $n \in \mathbb{N}$ such that $\mu(\{|f| > n\}) < \delta$ and hence $\|f\chi_{\{|f|>n\}}\|_{L_{p,\infty}} < \frac{\varepsilon}{3}$. If we also fix $m \in \mathbb{N}$ such that $\frac{M^{\frac{1}{p}}}{m} < \frac{\varepsilon}{3}$, we have that for every measurable subset A, either $\mu(A) > M$, so that

$$\mu(A)^{-\frac{1}{p^*}} \int_A |f\chi_{\{|f|\leqslant\frac{1}{m}\}}|\, d\mu \leqslant \mu(A)^{-\frac{1}{p^*}} \int_A |f|\, d\mu < \frac{\varepsilon}{3},$$

or $\mu(A) \leqslant M$, and hence

$$\mu(A)^{-\frac{1}{p^*}} \int_A |f\chi_{\{|f|\leqslant\frac{1}{m}\}}|\, d\mu \leqslant \frac{\mu(A)^{\frac{1}{p}}}{m} < \frac{\varepsilon}{3}.$$

In follows that $\|f\chi_{\{|f|\leqslant\frac{1}{m}\}}\|_{L_{p,\infty}} < \frac{\varepsilon}{3}$, so that $\|f - f\chi_{\{\frac{1}{m}<|f|\leqslant n\}}\|_{L_{p,\infty}} < \frac{2\varepsilon}{3}$. Note that $g = f\chi_{\{m<|f|\leqslant n\}}$ is bounded and has finite support, so there exists a simple function s supported in supp g such that

$\|g - s\|_{L_\infty} < \varepsilon / \mu(\operatorname{supp} g)^{\frac{1}{p}}$ and hence $\|g - s\|_{L_{p,\infty}} < \frac{\varepsilon}{3}$. Therefore, for every $\varepsilon > 0$ we have found a simple function s such that $\|f - s\|_{L_{p,\infty}} < \varepsilon$, so $f \in L_{p,\infty}^\circ(\mu)$. $\qquad\square$

It turns out that the space $L_{p,\infty}^\circ$ plays a distinguished role in the class of separable Banach lattices with upper p-estimates. Indeed, Theorem 2.4 can be improved so that the (p, ∞)-convex operator T factors through $L_{p,\infty}^\circ$ when E is separable.

Proposition 6.3. *Let E be a separable Banach space, $1 < p < \infty$ and $T : E \to L_1(\mu)$ a (p, ∞)-convex operator with constant C. There exists a normalized $h \in L_1(\mu)_+$ such that $\{h = 0\} \subseteq \{Tx = 0\}$ for every $x \in E$, and $T = MR$, where $R : E \to L_{p,\infty}^\circ(h \cdot \mu)$ given by $Rx = h^{-1}Tx$ is bounded with $\|R\| \leqslant K^{\frac{1}{p^*}}\gamma_p C$ for some $1 \leqslant K \leqslant 2$, and $M : L_{p,\infty}^\circ(h\mu) \to L_1(\mu)$ is the multiplication operator by h.*

Proof. By Pisier's factorization Theorem 2.4, there exists a normalized function $g \in L_1(\mu)_+$ such that $\{g = 0\} \subseteq \{Tx = 0\}$ for every $x \in E$ and

$$\left\| \frac{Tx}{g} \right\|_{L_{p,\infty}(g\mu)} \leqslant \gamma_p C \|x\|.$$

Let us take a dense subset $(x_n)_{n=1}^\infty \subset S_E$ and assume without loss of generality that $Tx_n \neq 0$. We define

$$f_n = \frac{|Tx_n|}{\|Tx_n\|_{L_1(\mu)}} \quad \text{and} \quad h = \frac{1}{K}\left[\left(\sum_n \frac{f_n}{2^n} \right) \vee g \right],$$

where K is chosen so that $\|h\|_{L_1(\mu)} = 1$, so in particular $1 \leqslant K \leqslant 2$. Clearly h is positive and $\{h = 0\} \subseteq \{g = 0\}$. Moreover, for every $x \in E$ and measurable subset A we have

$$\left(\int_A h \, d\mu \right)^{-\frac{1}{p^*}} \int_A |Tx| \, d\mu \leqslant K^{\frac{1}{p^*}} \left(\int_A g \, d\mu \right)^{-\frac{1}{p^*}} \int_A |Tx| \, d\mu \leqslant K^{\frac{1}{p^*}} \left\| \frac{Tx}{g} \right\|_{L_{p,\infty}(g\mu)} \leqslant K^{\frac{1}{p^*}}\gamma_p C \|x\|,$$

so the operator $Rx = \frac{Tx}{h}$ is bounded from E to $L_{p,\infty}(h \cdot \mu)$ with $\|R\| \leqslant K^{\frac{1}{p^*}}\gamma_p C$. Let us show that the range of R is contained in $L_{p,\infty}^\circ(h \cdot \mu)$. To do so, fix $n \in \mathbb{N}$ and consider a measurable subset A, so that

$$\left(\int_A h \, d\mu \right)^{-\frac{1}{p^*}} \int_A |Tx_n| \, d\mu \leqslant (2^n K)^{\frac{1}{p^*}} \left(\int_A f_n \, d\mu \right)^{-\frac{1}{p^*}} \|Tx_n\|_{L_1(\mu)} \int_A f_n \, d\mu$$

$$\leqslant C(2^n K)^{\frac{1}{p^*}} \left(\int_A f_n \, d\mu \right)^{\frac{1}{p}} \leqslant C 2^n K \left(\int_A h \, d\mu \right)^{\frac{1}{p}}.$$

It is clear that $\lim_{\mu(A) \to 0} \int_A h \, d\mu = 0$, so using Proposition 6.2 we conclude that $Rx_n \in L_{p,\infty}^\circ(h \cdot \mu)$ for every $n \in \mathbb{N}$. Since R is continuous and $(x_n)_n$ is dense in E, it follows that $R(E) \subseteq L_{p,\infty}^\circ(h \cdot \mu)$. Finally, it is straightforward to check that $T = MR$. $\qquad\square$

In particular, it follows from the above that every positive operator from a separable Banach lattice with upper p-estimates to L_1 factors through some $L_{p,\infty}^\circ$. One could try to apply this result in order to obtain an (isomorphic) analogue of Theorem 6.1 for upper p-estimates by using the following scheme: If X is a Banach lattice satisfying an upper p-estimate with constant 1, let $T : X \to C(\Delta, L_1[0, 1])$ be the isometric lattice embedding given by [16, Theorem 1.1], and

consider for each $t \in \Delta$ the point evaluation $\delta_t : C(\Delta, L_1[0,1]) \to L_1[0,1]$, which is a lattice homomorphism. Then, $\delta_t T : X \to L_1[0,1]$ is a (p, ∞)-convex operator, so by Proposition 6.3 it factors through $L_{p,\infty}^\circ(h_t)$ as $\delta_t T = M_t R_t$ for some normalized $h_t \in L_1[0,1]_+$, with $\|R_t\| \leq 2^{\frac{1}{p^*}} \gamma_p$. Since $([0,1], \mathfrak{m}, h \cdot \lambda)$, where λ denotes the Lebesgue measure and $\mathfrak{m}$ the σ-algebra of measurable sets, is a separable, non-atomic probability measure space, by Carathéodory's Isomorphism Theorem ([4], see also [12, Section 41]) it must be measure isomorphic to the unit interval with the Lebesgue measure. In particular, $L_{p,\infty}^\circ(h_t)$ can be identified isometrically with $L_{p,\infty}^\circ[0,1]$. Therefore, each of the maps $\delta_t T : X \to L_1[0,1]$ can be factored through $L_{p,\infty}^\circ[0,1]$ with a uniform bound on the norm of the factors. If one could show that the above process of choosing the density h_t could be carried out continuously with respect to $t \in \Delta$, then it would produce a lattice isomorphic embedding $R : X \to C(\Delta, L_{p,\infty}^\circ[0,1])$ that sends every $x \in X$ to the function $Rx(t) = R_t x$, $t \in \Delta$. However, the way the density is obtained in the original proof of Theorem 2.4 (see [23]) makes it difficult to deduce whether such a choice of density preserving continuity can be made.

Another unclear aspect of this approach is whether the candidate for the universal space should be $C(\Delta, L_{p,\infty}^\circ[0,1])$ or $C(\Delta, L_{p,\infty}^\circ[0,\infty))$. Clearly, if the former satisfies the universal property, the latter will as well, as $L_{p,\infty}^\circ[0,1]$ embeds lattice isometrically into $L_{p,\infty}^\circ[0,\infty)$. However, the converse might not be true. Indeed, $L_{p,\infty}^\circ[0,1]$ and $L_{p,\infty}^\circ[0,\infty)$ behave very differently, as the following results show.

Proposition 6.4. *If μ is a separable measure over Ω, $L_{p,\infty}^\circ(\mu)$ embeds lattice isomorphically into $L_{p,\infty}^\circ[0,\infty)$.*

Proof. Let us decompose μ into an atomic part μ_a and a non-atomic part μ_{na}, so that $L_{p,\infty}^\circ(\mu)$ is lattice isomorphic to $L_{p,\infty}^\circ(\mu_a) \oplus L_{p,\infty}^\circ(\mu_{na})$. Depending on whether it has a finite or infinite number of atoms, $L_{p,\infty}^\circ(\mu_a)$ is lattice isometric to $\ell_{p,\infty}^n(\omega)$ or $L_{p,\infty}^\circ(\mathbb{N}, \omega)$, where ω is a weighted counting measure over the set $\{1, \ldots, n\}$ or $\mathbb{N}$, respectively. These spaces are in turn lattice isometric to the span in $L_{p,\infty}^\circ[0,\infty)$ of characteristic functions of disjoint sets A_k of measure ω_k ($k = 1, \ldots, n$ or $k \in \mathbb{N}$, respectively). On the other hand, μ_{na} is measure isomorphic to the Lebesgue measure on $[0, r)$, where $r = \mu_{na}(\Omega)$. Hence, $L_{p,\infty}^\circ(\mu)$ is isomorphic to a sublattice of $L_{p,\infty}^\circ[0,\infty)$. $\qquad\square$

Therefore, $L_{p,\infty}^\circ[0,\infty)$ contains any other separable $L_{p,\infty}^\circ(\mu)$ as a sublattice. On the other hand, $L_{p,\infty}^\circ[0,1]$ does not contain $\ell_{p,\infty}^\circ$ nor $L_{p,\infty}^\circ[0,\infty)$ lattice isomorphically:

Proposition 6.5. *The Banach lattice $\ell_{p,\infty}^\circ$ does not lattice embed into $L_{p,\infty}^\circ[0,1]$.*

One should compare this result with [15], where it is established that $\ell_{p,\infty}$ embeds as a complemented sublattice of $L_{p,\infty}[0,1]$, with a positive projection. The proof of Proposition 6.5 uses the following lemma (cf. [5]):

Lemma 6.6. *Let $(f_n)_n \subseteq L_{p,\infty}^\circ[0,1]$ be a positive, disjoint and normalized (with respect to the quasinorm $\|\cdot\|_{L_{p,\infty}}$) sequence. Then $(f_n)_n$ has a subsequence equivalent to the c_0 basis.*

Proof. Since simple functions are dense in $L_{p,\infty}^\circ[0,1]$, after a small perturbation, we can assume that $f_n \in L_\infty[0,1]$ for every $n \in \mathbb{N}$. Let $r_n = \lambda(\operatorname{supp} f_n)$, where λ denotes the Lebesgue measure. Then $r_n \to 0$, and passing to a subsequence if necessary, we can assume that there is a positive sequence $(m_n)_n \subseteq \mathbb{R}$ increasing to ∞ such that $\|f_n\|_{L_\infty} \leq m_n$ and $\sum_{k=n+1}^\infty r_k \leq m_n^{-p}$ for all $n \in \mathbb{N}$. Let f be $\sum_n f_n$ almost everywhere (note that, since the f_n are pairwise disjoint, this coincides almost

everywhere with the supremum). To prove the statement, it suffices to show that $f \in L_{p,\infty}[0,1]$. Take any $t \in \mathbb{R}_+$, and choose n so that $m_{n-1} \leqslant t < m_n$, where $m_0 = 0$. Since $\|f_k\|_{L_\infty} \leqslant m_k$, $\{f_k > t\} = \varnothing$ for $1 \leqslant k < n$. Thus,

$$\lambda\{f > t\} = \sum_{k=n}^{\infty} \lambda\{f_k > t\} \leqslant \lambda\{f_n > t\} + \sum_{k=n+1}^{\infty} r_k \leqslant t^{-p} + m_n^{-p} \leqslant 2t^{-p}.$$

As $t \in \mathbb{R}_+$ was arbitrary and $f \geqslant 0$, we conclude that $f \in L_{p,\infty}[0,1]$, as desired. $\qquad \square$

Proof of Proposition 6.5. If such a lattice embedding $T : \ell_{p,\infty}^\circ \to L_{p,\infty}^\circ[0,1]$ existed, then $(Te_n)_n \subseteq L_{p,\infty}^\circ[0,1]$ would contain a subsequence equivalent to the basis of c_0, where $(e_n)_n$ denotes the canonical basis of $\ell_{p,\infty}^\circ$. In particular, $(e_n)_n$ would also contain a subsequence equivalent to the c_0 basis, which is impossible, since every subsequence of $(e_n)_n$ is equivalent to itself. $\qquad \square$

As a consequence of the above, we get:

Corollary 6.7. $\ell_{p,\infty}^\circ$, $L_{p,\infty}^\circ[0,1]$ and $L_{p,\infty}^\circ[0,\infty)$ *are pairwise non-isomorphic as Banach lattices.*

Proof. Clearly, $\ell_{p,\infty}^\circ$ cannot be lattice isomorphic to any of the other two spaces, as it is purely atomic. $L_{p,\infty}^\circ[0,1]$ and $L_{p,\infty}^\circ[0,\infty)$ are not lattice isomorphic, as the latter contains a lattice isometric copy of $\ell_{p,\infty}^\circ$ (the span of the characteristic functions on the intervals $[n-1,n)$), whereas the first one does not by Proposition 6.5. $\qquad \square$

These facts point towards $C(\Delta, L_{p,\infty}^\circ[0,\infty))$ as a more suitable candidate for a universal Banach lattice for upper p-estimates. Finally, it should be noted that none of these candidates will be *isometrically* universal for Banach lattices with upper p-estimates with constant 1, as both embed lattice isometrically into an ℓ_∞-sum of $L_{p,\infty}$-spaces, and we have shown in Example 3.4 that there are Banach lattices with upper p-estimates that cannot be embedded isometrically into any space of this form.

Question 6.8. Is there a separable Banach lattice with upper p-estimates (with constant 1) that is (isomorphically or isometrically) universal for this class? That is, a separable Banach lattice Y satisfying an upper p-estimate with constant 1 such that for every separable Banach lattice X satisfying an upper p-estimate with constant 1 there exists a lattice (isometric or isomorphic) embedding of X into Y? Do $C(\Delta, L_{p,\infty}^\circ[0,1])$ or $C(\Delta, L_{p,\infty}^\circ[0,\infty))$ have this isomorphic property?

7. Operators factoring through a (p, p_2)-convex and a (q, q_2)-concave operator

In [25], the author studies the ideal of operators that factor through a p-convex and a q-concave operator, which is denoted $M_{p,q}$. This ideal is shown to be perfect, that is, it coincides with its double adjoint, and can be identified with the ideal $I_{p,q}$ of operators factoring through a diagonal operator from $L_p(\mu)$ to $L_q(\mu)$ defined in [10]. In this section, we provide analogous results for the ideal of operators factoring through a (p, p_2)-convex and a (q, q_2)-concave operator, where $1 < p, q < \infty$, $p \leqslant p_2 \leqslant \infty$ and $1 \leqslant q_2 \leqslant q$. Note that in light of [6, Corollary 16.7], up to constants, the only relevant cases are $p_2 \in \{p, \infty\}$ and $q_2 \in \{1, q\}$.

Throughout this section, we will use the definitions and notation for tensor products and operator ideals given in [10]. For instance, recall that an *operator ideal* is a class A of operators between Banach spaces such that for each pair of Banach spaces E and F the set $A(E, F) = A \cap \mathcal{L}(E, F)$ satisfies that:

(1) If $x^* \in E^*$ and $y \in F$, then $x^* \otimes y \in A(E, F)$, where $x^* \otimes y(x) = x^*(x)y$ is a rank one operator.

(2) $A(E, F)$ is a linear subspace of $\mathcal{L}(E, F)$.

(3) If X and Y are Banach spaces, $U \in \mathcal{L}(X, E)$, $T \in A(E, F)$ and $V \in \mathcal{L}(F, Y)$, then $VTU \in A(X, Y)$.

Moreover, a function $\alpha : A \to \mathbb{R}_+$ is an *ideal norm* if one has:

(4) $\alpha(x^* \otimes y) = \|x^*\|\|y\|$ for every $x^* \in E^*$ and $y \in F$.

(4) $\alpha(S + T) \leqslant \alpha(S) + \alpha(T)$ for every $S, T \in A(E, F)$.

(4) If X and Y are Banach spaces, $U \in \mathcal{L}(X, E)$, $T \in A(E, F)$ and $V \in \mathcal{L}(F, Y)$, then $\alpha(VTU) \leqslant \|V\|\alpha(T)\|U\|$.

A *normed (operator) ideal* $[A, \alpha]$ is an ideal A with an ideal norm α, and we say that it is a *Banach ideal* if each $A(E, F)$ is a Banach space when endowed with α.

Given a normed ideal $[A, \alpha]$, the *adjoint ideal* $[A, \alpha]^* = [A^*, \alpha^*]$ is defined as follows (observe that here $*$ denotes the adjoint of an operator ideal instead of the dual of a Banach space): For each pair of Banach spaces E and F, $A^*(E, F)$ is the set of all $T \in \mathcal{L}(E, F)$ such that there exists $\rho > 0$ satisfying that for all finite-dimensional Banach spaces X, Y and all $U \in A(Y, X)$, $V \in \mathcal{L}(X, E)$ and $W \in \mathcal{L}(F, Y)$,

$$|\operatorname{tr} WTVU| \leqslant \rho\|W\|\|V\|\alpha(U),$$

where tr denotes the trace of an operator defined from a finite-dimensional Banach space to itself. The norm $\alpha^*(T)$ is given by the infimum of all ρ satisfying the above. α^* is always a complete ideal norm. An operator ideal $[A, \alpha]$ is *perfect* if $[A, \alpha] = [A, \alpha]^{**}$. This notion is equivalent to the ideal being *finitely generated*, *maximal* or being the adjoint of some other ideal (cf. [26, Theorem 8.11]).

Regarding tensor products, recall that if E and F are Banach spaces, a norm η on $E \otimes F$ is a *reasonable crossnorm* if it satisfies the following properties:

(1) $\eta(x \otimes y) \leqslant \|x\|\|y\|$ for every $x \in E$ and $y \in F$.

(2) For every $x^* \in E^*$ and $y^* \in F^*$, the linear functional $x^* \otimes y^*$ on $E \otimes F$ that takes values $x^* \otimes y^*(x \otimes y) = x^*(x)y^*(y)$ on the elementary products $x \otimes y \in E \otimes F$ is bounded, and $\|x^* \otimes y^*\| \leqslant \|x^*\|\|y^*\|$.

A *uniform crossnorm* is an assignment to each pair of Banach spaces of a reasonable crossnorm on their tensor product in such a way that whenever E, F, G and H are Banach spaces and $S : E \to G$ and $T : F \to H$ are operators, then the operator $S \otimes T : E \otimes F \to G \otimes H$ is bounded and $\|S \otimes T\| \leqslant \|S\|\|T\|$. It is easy to check that every uniform crossnorm induces an ideal norm over the operator ideal $\mathcal{F}$ of all finite rank operators via the identification $\mathcal{F}(E, F) = E^* \otimes F$ described in property (1) in the definition of an operator ideal. We will denote this normed ideal by $[\mathcal{F}, \eta]$. A uniform crossnorm η is *finitely generated* (see [26, p. 129], this is also referred to as a $\otimes$-norm in [25]), if for all $u \in E \otimes F$,

$$\eta(u) = \inf \eta(u; M, N),$$

where the infimum is taken over all finite-dimensional subspaces $M \subseteq E$ and $N \subseteq F$ such that $u \in M \otimes N$, and $\eta(u; M, N)$ denotes the η-norm of u as an element of $M \otimes N$.

Let us fix $1 < p, q < \infty$, $p \leqslant p_2 \leqslant \infty$ and $1 \leqslant q_2 \leqslant q$. We introduce the following definition, which generalizes the ideal $[M_{p,q}, \mu_{p,q}]$ of operators that factor through a p-convex and a q-concave operator introduced in [25, Section 2]:

Definition 7.1. *Let E and F be Banach spaces. An operator $T : E \to F$ belongs to the class $M_{(p,p_2;q,q_2)}(E,F)$ if there exists a Banach lattice X, a (p,p_2)-convex operator $U : E \to X$ and a (q,q_2)-concave operator $V : X \to F^{**}$ such that $J_F T = VU$, where $J_F : F \to F^{**}$ denotes the canonical embedding:*

$$
\begin{array}{ccc}
E \xrightarrow{\ T\ } F \xrightarrow{\ J_F\ } F^{**} \\
{\scriptstyle U} \searrow \quad \nearrow {\scriptstyle V} \\
X
\end{array}
$$

We define $\mu_{(p,p_2;q,q_2)}(T) = \inf\{K^{(p,p_2)}(U)K_{(q,q_2)}(V)\}$, where the infimum is taken over all possible factorizations above.

As is customary, when $p_2 = p$ we will simply write p instead of (p,p_2), and similarly when $q_2 = q$. Our first goal is to show that $M_{(p,p_2;q,q_2)}$ is a perfect ideal. This will be done by showing that it is the adjoint of $[\mathcal{F}, \eta_{(p,p_2;q,q_2)}]^*$, where $\eta_{(p,p_2;q,q_2)}$ is a uniform crossnorm defined as follows:

Definition 7.2. *Given $u \in E \otimes F$, we define*

$$
\eta_{(p,p_2;q,q_2)}(u) = \inf\left\{\theta_{(p,p_2;q,q_2)}(\{x_i \otimes y_i\}_{i=1}^n) : (x_i)_{i=1}^n \subseteq E, (y_i)_{i=1}^n \subseteq F, u = \sum_{i=1}^n x_i \otimes y_i\right\},
$$

where

$$
\theta_{(p,p_2;q,q_2)}(\{x_i \otimes y_i\}_{i=1}^n) = \sup\left\{\sum_{i=1}^n \left(\sum_{k=1}^\infty |x_k^*(x_i)|^{p_2}\right)^{\frac{1}{p_2}}\left(\sum_{j=1}^\infty |y_j^*(y_i)|^{q_2^*}\right)^{\frac{1}{q_2^*}} : \right.
$$

$$
\left. \|(x_k^*)_{k=1}^\infty\|_{\ell_p(E^*)} \leqslant 1, \|(y_j^*)_{j=1}^\infty\|_{\ell_{q^*}(F^*)} \leqslant 1\right\}
$$

and q^* and q_2^* denote the conjugate exponents of q and q_2, respectively (when $p_2 = \infty$ or $q_2 = 1$, the corresponding p_2 or q_2^*-sum must be understood as a supremum).

Following the steps of [25], the first thing to show is that:

Proposition 7.3. $\eta_{(p,p_2;q,q_2)}$ *is finitely generated.*

Proof. It suffices to check that whenever we fix a representation $u = \sum_{i=1}^n x_i \otimes y_i$ with $(x_i)_{i=1}^n \subseteq M$ and $(y_i)_{i=1}^n \subseteq N$ for some finite-dimensional subspaces $M \subseteq E$ and $N \subseteq F$, then

$$
\theta_{(p,p_2;q,q_2)}(\{x_i \otimes y_i\}_{i=1}^n; E, F) = \theta_{(p,p_2;q,q_2)}(\{x_i \otimes y_i\}_{i=1}^n; M, N).
$$

Clearly, $\theta_{(p,p_2;q,q_2)}(\{x_i \otimes y_i\}_{i=1}^n; E, F) \leqslant \theta_{(p,p_2;q,q_2)}(\{x_i \otimes y_i\}_{i=1}^n; M, N)$, since every pair of sequences $(x_k^*)_{k=1}^\infty \in B_{\ell_p(E^*)}$ and $(y_j^*)_{j=1}^\infty \in B_{\ell_{q^*}(F^*)}$ can be restricted to M and N, respectively, to induce sequences in $B_{\ell_p(M^*)}$ and $B_{\ell_{q^*}(N^*)}$, and the values of the sums considered in the computation of $\theta_{(p,p_2;q,q_2)}$ remain unchanged.

On the other hand, the inequality $\theta_{(p,p_2;q,q_2)}(\{x_i \otimes y_i\}_{i=1}^n; E, F) \geqslant \theta_{(p,p_2;q,q_2)}(\{x_i \otimes y_i\}_{i=1}^n; M, N)$ is a consequence of the Hahn–Banach Theorem: Every pair of sequences $(v_k^*)_{k=1}^\infty \in B_{\ell_p(M^*)}$ and $(w_j^*)_{j=1}^\infty \in B_{\ell_{q^*}(N^*)}$ can be extended to sequences in $B_{\ell_p(E^*)}$ and $B_{\ell_{q^*}(F^*)}$ without altering their values on $(x_i)_{i=1}^n$ and $(y_i)_{i=1}^n$. $\qquad\square$

The next proposition shows the connection between the $\eta_{(p,p_2;q,q_2)}$-norm and (p,p_2)-convex and (q,q_2)-concave operators.

Proposition 7.4. *For every $u \in E^* \otimes F = \mathcal{F}(E,F) \subseteq L(E,F)$,*

$$\eta_{(p,p_2;q,q_2)}(u) = \inf \left\{ K^{(p,p_2)}(R) K_{(q,q_2)}(S) \right\},$$

where the infimum is taken over all possible factorizations $u = SR$ with Z a finite-dimensional Banach space with a 1-unconditional basis (endowed with the order induced by the basis), $R : E \to Z$ and $S : Z \to F$.

Proof. The proof is just an adaptation of [25, Proposition 1.3] with the corresponding changes in the exponents. We include it for the sake of completeness.

Let $u \in E^* \otimes F$ and assume that it factors as $u = RS$ with Z a finite-dimensional Banach space with a 1-unconditional basis, $R : E \to Z$ and $S : Z \to F$. Let $(e_i, e_i^*)_{i=1}^n$ be the corresponding basis of Z and its biorthogonal functionals. We define $x_i^* = R^* e_i^*$ and $y_i = S e_i$, so that $u = SR = \sum_{i=1}^n x_i^* \otimes y_i$. By definition,

$$K^{(p,p_2)}(R) = \sup \left\{ \left\| \sum_{i=1}^n \left(\sum_k |x_i^*(x_k)|^{p_2} \right)^{\frac{1}{p_2}} e_i \right\| : \|(x_k)_{k=1}^\infty\|_{\ell_p(E)} \leqslant 1 \right\},$$

$$K_{(q,q_2)}(S) = K^{(q^*,q_2^*)}(S^*) = \sup \left\{ \left\| \sum_{i=1}^n \left(\sum_k |y_j^*(y_i)|^{q_2^*} \right)^{\frac{1}{q_2^*}} e_i^* \right\| : \|(y_j^*)_{k=1}^\infty\|_{\ell_{q*}(F^*)} \leqslant 1 \right\}.$$

Therefore, given $\varepsilon > 0$, we can choose $(x_k)_{k=1}^\infty \in \ell_p(E)$ and $(y_j^*)_{k=1}^\infty \in \ell_{q*}(F^*)$ that ε-almost attain the supremum in the definition of $\theta_{(p,p_2;q,q_2)}(\{x_i^* \otimes y_i\}_{i=1}^n)$, so that

$$\theta_{(p,p_2;q,q_2)}(\{x_i^* \otimes y_i\}_{i=1}^n) \leqslant \sum_{i=1}^n \left(\sum_{k=1}^\infty |x_k^*(x_i)|^{p_2} \right)^{\frac{1}{p_2}} \left(\sum_{j=1}^\infty |y_j^*(y_i)|^{q_2^*} \right)^{\frac{1}{q_2^*}} + \varepsilon$$

$$\leqslant \left\langle \sum_{i=1}^n \left(\sum_k |x_i^*(x_k)|^{p_2} \right)^{\frac{1}{p_2}} e_i, \sum_{i=1}^n \left(\sum_k |y_j^*(y_i)|^{q_2^*} \right)^{\frac{1}{q_2^*}} e_i^* \right\rangle + \varepsilon$$

$$\leqslant K^{(p,p_2)}(R) K_{(q,q_2)}(S) + \varepsilon.$$

Hence, $\eta_{(p,p_2;q,q_2)}(u) \leqslant \inf\{K^{(p,p_2)}(R) K_{(q,q_2)}(S)\}$.

For the reverse inequality, let $\varepsilon > 0$ and suppose that $u = \sum_{i=1}^n x_i^* \otimes y_i$ is a representation such that

$$\theta_{(p,p_2;q,q_2)}(\{x_i^* \otimes y_i\}_{i=1}^n) \leqslant \eta_{(p,p_2;q,q_2)}(u) + \varepsilon.$$

Let Z be $\mathbb{R}^n$ endowed with the norm

$$\|z\| = \sup \left\{ \sum_{i=1}^n \left| z_i \left(\sum_k |y_j^*(y_i)|^{q_2^*} \right)^{\frac{1}{q_2^*}} \right| : \|(y_j^*)_{k=1}^\infty\|_{\ell_{q*}(F^*)} \leqslant 1 \right\}$$

for $z = (z_i)_{i=1}^n \in \mathbb{R}^n$. Note that the unit vector basis $(e_i)_{i=1}^n$ of $\mathbb{R}^n$ is 1-unconditional for this norm, as $\|z\|$ is only determined by $(|z_i|)$. Let us define $R : E \to Z$ by $R = \sum_{i=1}^n x_i^* \otimes e_i$ and $S : Z \to F$

by $Sz = \sum_{i=1}^{n} z_i y_i$. It follows that $u = SR$,

$$K^{(p,p_2)}(R) = \sup\left\{\left\|\sum_{i=1}^{n}\left(\sum_k |x_i^*(x_k)|^{p_2}\right)^{\frac{1}{p_2}} e_i\right\| : \|(x_k)_{k=1}^{\infty}\|_{\ell_p(E)} \leqslant 1\right\}$$

$$= \theta_{(p,p_2;q,q_2)}(\{x_i^* \otimes y_i\}_{i=1}^{n}) \leqslant \eta_{(p,p_2;q,q_2)}(u) + \varepsilon,$$

and

$$K_{(q,q_2)}(S) = K^{(q^*,q_2^*)}(S^*) = \sup\left\{\left\|\sum_{i=1}^{n}\left(\sum_k |y_j^*(y_i)|^{q_2^*}\right)^{\frac{1}{q_2^*}} e_i^*\right\| : \|(y_j^*)_{k=1}^{\infty}\|_{\ell_{q^*}(F^*)} \leqslant 1\right\}$$

$$= \sup\left\{\sum_{i=1}^{n}\left|z_i\left(\sum_k |y_j^*(y_i)|^{q_2^*}\right)^{\frac{1}{q_2^*}}\right| : \|z\| \leqslant 1, \|(y_j^*)_{k=1}^{\infty}\|_{\ell_{q^*}(F^*)} \leqslant 1\right\}$$

$$= \sup\{\|z\| : \|z\| \leqslant 1\} = 1.$$

Therefore, $\inf\{K^{(p,p_2)}(R)K_{(q,q_2)}(S)\} \leqslant \eta_{(p,p_2;q,q_2)}(u)$. $\square$

Next, we show that the ideal $M_{(p,p_2;q,q_2)}$ defined in Definition 7.1 is perfect.

Proposition 7.5. *The following properties hold:*

(1) $[M_{(p,p_2;q,q_2)}, \mu_{(p,p_2;q,q_2)}]$ *is a perfect normed ideal of operators.*
(2) $[M_{(p,p_2;q,q_2)}, \mu_{(p,p_2;q,q_2)}] = [\mathcal{F}, \eta_{(p,p_2;q,q_2)}]^{**}$.

Proving Proposition 7.5 requires some previous lemmata. Namely, we need [25, Lemma 2], which is also valid for the case $p = \infty$ even though it was originally stated for $1 \leqslant p < \infty$, and an analogue of [25, Lemma 3]:

Lemma 7.6. *Let G be a finite-dimensional subspace of a Dedekind complete Banach lattice Y. Given $\delta > 0$ there are pairwise disjoint $(y_i)_{i=1}^{n} \subseteq Y_+$ and an operator $W : G \to \mathrm{span}(y_i)_{i=1}^{n}$ such that $\|x - Wx\| \leqslant \delta\|x\|$ for every $x \in G$, and moreover we have that*

$$\left\|\left(\sum_{j=1}^{m} |x_j|^p\right)^{\frac{1}{p}} - \left(\sum_{j=1}^{m} |Wx_j|^p\right)^{\frac{1}{p}}\right\| \leqslant \delta\left(\sum_{j=1}^{m} \|x_j\|^p\right)^{\frac{1}{p}}$$

for all $(x_j)_{j=1}^{m} \subseteq G$ and $1 \leqslant p \leqslant \infty$.

Lemma 7.7. *Assume that E or F are finite-dimensional and $T \in \mathcal{L}(E, F)$ is such that $J_F T = VU$ with X a Banach lattice, $U : E \to X$ a (p, p_2)-convex operator and $V : X \to F^{**}$ a (q, q_2)-concave operator. Then, for every $\varepsilon > 0$ there exists a finite-dimensional Z with a 1-unconditional basis and a factorization*

$$
\begin{array}{ccc}
E & \xrightarrow{\ \ T\ \ } & F \\
& R \searrow \quad \nearrow S & \\
& Z &
\end{array}
$$

with $K^{(p,p_2)}(R)K_{(q,q_2)}(S) \leqslant (1 + \varepsilon)K^{(p,p_2)}(U)K_{(q,q_2)}(V)$.

Proof. Let us first assume that E is finite-dimensional. By Theorem 4.19, $V : X \to F^{**}$ factors through a (q, q_2)-concave Banach lattice Y with constant 1 as $V = V_2 V_1$, where $V_1 : X \to Y$ is a lattice homomorphism with $\|V_1\| \leqslant K_{(q,q_2)}(V)$ and $V_2 : Y \to F^{**}$ is contractive. Without loss of generality, we can assume that Y is Dedekind complete. Otherwise, we replace Y by Y^{**}, V_1 by $J_Y V_1$ and V_2 by $J_{F*}^* V_2^{**}$. Let us fix $\varepsilon > 0$, and consider $\delta > 0$ to be chosen later. We can apply Lemma 7.6 to the finite-dimensional subspace $G = V_1 U(E) \subseteq Y$ to find a finite-dimensional sublattice $Z \subseteq Y$ and an isomorphism $W : G \to Z$ such that for all $(x_j)_{j=1}^m \subseteq G$,

$$\left\| \left(\sum_{j=1}^m |x_j|^{p_2} \right)^{\frac{1}{p_2}} - \left(\sum_{j=1}^m |Wx_j|^{p_2} \right)^{\frac{1}{p_2}} \right\| \leqslant \delta \left(\sum_{j=1}^m \|x_j\|^{p_2} \right)^{\frac{1}{p_2}}.$$

Observe that $V_1 U$ is (p, p_2)-convex with constant $K^{(p,p_2)}(U) K_{(q,q_2)}(V)$, as V_1 is a lattice homomorphism, and Z is also (q, q_2)-concave with constant 1. It follows that for all $(u_j)_{j=1}^m \subseteq E$

$$\left\| \left(\sum_{j=1}^m |WV_1 U u_j|^{p_2} \right)^{\frac{1}{p_2}} \right\| \leqslant \left\| \left(\sum_{j=1}^m |V_1 U u_j|^{p_2} \right)^{\frac{1}{p_2}} \right\| + \left\| \left(\sum_{j=1}^m |V_1 U u_j|^{p_2} \right)^{\frac{1}{p_2}} - \left(\sum_{j=1}^m |WV_1 U u_j|^{p_2} \right)^{\frac{1}{p_2}} \right\|$$

$$\leqslant K^{(p,p_2)}(U) K_{(q,q_2)}(V) \left(\sum_{j=1}^m \|u_j\|^p \right)^{\frac{1}{p}} + \delta \left(\sum_{j=1}^m \|V_1 U u_j\|^{p_2} \right)^{\frac{1}{p_2}}$$

$$\leqslant K^{(p,p_2)}(U) K_{(q,q_2)}(V) \left(\sum_{j=1}^m \|u_j\|^p \right)^{\frac{1}{p}} + \delta \|V_1 U\| \left(\sum_{j=1}^m \|u_j\|^{p_2} \right)^{\frac{1}{p_2}}$$

$$\leqslant K^{(p,p_2)}(U) K_{(q,q_2)}(V)(1 + \delta) \left(\sum_{j=1}^m \|u_j\|^p \right)^{\frac{1}{p}},$$

where, in the last step, we have used the fact that $p_2 \geqslant p$. Therefore, $R = WV_1 U : E \to Z$ is (p, p_2)-convex with $K^{(p,p_2)}(R) \leqslant K^{(p,p_2)}(U) K_{(q,q_2)}(V)(1 + \delta)$.

To construct S, observe that W satisfies that $(1 - \delta)\|x\| \leqslant \|Wx\| \leqslant (1 + \delta)\|x\|$ for every $x \in G$. Denote by $k = \dim G$, which is independent of ε and δ, and $n = \dim Z \geqslant k$, and let $(e_j)_{j=1}^k$ be a normalized basis of G. Then $(We_j)_{j=1}^k$ is a seminormalized basis of $W(G) \subseteq Z$. Let $(z_j^*)_{j=1}^k \subseteq Z^*$ be the Hahn–Banach extensions to Z of a set of seminormalized functionals which are biorthogonal to $(We_j)_{j=1}^k$, so that $\|z_j^*\| \leqslant \frac{1}{1-\delta}$ for every $j = 1, \ldots, k$. If we write $K = \bigcap_{j=1}^k \ker z_j^*$, then $Z = W(G) \oplus K$. Then, every $z \in Z$ can be represented as $z = \sum_{j=1}^k c_j We_j + y$ with $y \in K$ and $c_j = z_j^*(z)$ for every j. Let us define $M : Z = W(G) \oplus K \to Y$ that sends every $z = \sum_{j=1}^k c_j We_j + y$ to $Mz = \sum_{j=1}^k c_j e_j + y$, so that

$$\|Mz - z\| = \left\| \sum_{j=1}^k c_j e_j - \sum_{j=1}^k c_j We_j \right\| \leqslant \delta \left\| \sum_{j=1}^k c_j e_j \right\|$$

$$= \delta \left\| \sum_{j=1}^k z_j^*(z) e_j \right\| \leqslant \delta \sum_{j=1}^k \|z_j^*\| \|z\| \leqslant \frac{\delta k}{1 - \delta} \|z\|.$$

It follows that M is an isomorphism onto its image such that $\|M\| \leqslant 1 + \frac{\delta k}{1-\delta}$ and $MWx = x$ for every $x \in G$. Let us now compose M with V_2 to obtain a finite-dimensional subspace of F^{**}, to which we can apply the Local Reflexivity Principle (cf. [7, Theorem 6.3 and Exercise 1.76]) to get an isomorphism onto its image $Q : V_2 M(Z) \to F$ such that $Q = J_F^{-1}$ on $V_2 M(Z) \cap J_F(F)$ and $\|Q\| \leqslant \frac{1+\delta}{1-\delta}$. It follows that the operator $S = QV_2 M : Z \to F$ satisfies

$$SRu = QV_2 MWV_1 Uu = QV_2 V_1 Uu = QVUu = QJ_F Tu = Tu$$

for every $u \in E$, and moreover it is (q, q_2)-concave, since Z was already (q, q_2)-concave, with constant

$$K_{(q,q_2)}(S) = \|S\| \leqslant \|Q\|\|V_2\|\|M\| \leqslant \frac{1+\delta}{1-\delta}\left(1 + \frac{\delta k}{1-\delta}\right).$$

Taking δ small enough, we can achieve $K^{(p,p_2)}(R)K_{(q,q_2)}(S) \leqslant (1 + \varepsilon)K^{(p,p_2)}(U)K_{(q,q_2)}(V)$, as we wanted to show.

Now, if we assume that F is finite-dimensional, then $F = F^{**}$, so we can write $T = VU$. Then, it suffices to apply the first part of the proof to the factorization

$$F^* \xrightarrow{T^*} E^* \xrightarrow{J_{E*}} E^{***}$$

$$V^* \searrow \quad \nearrow J_{E*}U^*$$

$$Z$$

and take adjoints in the new factorization to obtain the desired result. $\qquad\square$

Now, we are ready to prove Proposition 7.5:

Proof of Proposition 7.5. The first statement follows from the second one by [26, Theorem 8.11].

Let us first show that $[M_{(p,p_2;q,q_2)}, \mu_{(p,p_2;q,q_2)}] \subseteq [\mathcal{F}, \eta_{(p,p_2;q,q_2)}]^{**}$. Let E and F be Banach spaces and $T \in M_{(p,p_2;q,q_2)}(E, F)$. Assume that $J_F T$ factors as in Definition 7.1. By Lemma 7.7, for every finite-dimensional $G \subseteq E$ and $H \subseteq F^*$ (denote by $i : G \to E$ and $j : H \to F^*$ the embeddings) the operator $j^* J_F Ti$ admits a factorization

$$G \xrightarrow{j^* J_F Ti} H^*$$

$$R \searrow \quad \nearrow S$$

$$Z$$

with Z finite-dimensional with a 1-unconditional basis, such that $K^{(p,p_2)}(R)K_{(q,q_2)}(S) \leqslant (1 + \varepsilon)K^{(p,p_2)}(U)K_{(q,q_2)}(V)$. Since G and H^* are finite-dimensional, using [22, Proposition 9.2.2] and Proposition 7.4 we get

$$\eta^{**}_{(p,p_2;q,q_2)}(j^* J_F Ti) = \eta_{(p,p_2;q,q_2)}(j^* J_F Ti) \leqslant K^{(p,p_2)}(R)K_{(q,q_2)}(S) \leqslant (1 + \varepsilon)K^{(p,p_2)}(U)K_{(q,q_2)}(V).$$

Observe that $[\mathcal{F}, \eta_{(p,p_2;q,q_2)}]^{**}$ is already a perfect ideal, as it is the adjoint of $[\mathcal{F}, \eta_{(p,p_2;q,q_2)}]^*$. Therefore, by [26, Theorem 8.11], it is a finitely generated ideal, so

$$\eta^{**}_{(p,p_2;q,q_2)}(T) = \sup\{\eta^{**}_{(p,p_2;q,q_2)}(j^* J_F Ti) : G \subseteq E, H \subseteq F^* \text{ finite-dimensional}\}$$

$$\leqslant (1 + \varepsilon)K^{(p,p_2)}(U)K_{(q,q_2)}(V).$$

It now follows that $\eta^{**}_{(p,p_2;q,q_2)}(T) \leqslant \mu_{(p,p_2;q,q_2)}(T)$.

In order to show that $[M_{(p,p_2;q,q_2)}, \mu_{(p,p_2;q,q_2)}] \supseteq [\mathcal{F}, \eta_{(p,p_2;q,q_2)}]^{**}$ we will make use of a standard ultraproduct argument, so we will omit some of the details. We refer to [13] for basic terminology and notation on ultraproducts. Again, let us fix Banach spaces E and F and an operator in $[\mathcal{F}(E;F), \eta_{(p,p_2;q,q_2)}]^{**}$. First of all, we make use of the fact that every Banach space embeds isometrically into the ultraproduct of a collection of its finite-dimensional subspaces (see [13, Proposition 6.2]). In particular, there exist index sets K and L, ultrafilters $\mathfrak{U}_E$ and $\mathfrak{U}_{F*}$ over K and L (respectively), collections $\{G_k\}_{k \in K} \subseteq \{G : G \subseteq E \text{ finite-dimensional}\}$ and $\{H_l\}_{l \in L} \subseteq \{H : H \subseteq F^* \text{ finite-dimensional}\}$ and isometric embeddings $i_E : E \to (G_k)_{\mathfrak{U}_E}$ and $j_{F*} : F^* \to (H_l)_{\mathfrak{U}_{F*}}$. It follows that

$$\mathfrak{U}_\times = \{C \subseteq K \times L : \exists A \in \mathfrak{U}_E, B \in \mathfrak{U}_{F*} \text{ such that } A \times B \subseteq C\}$$

defines a filter on $K \times L$, so by [13, Proposition 1.1] there exists an ultrafilter $\mathfrak{U}$ that dominates $\mathfrak{U}_\times$. If we put $G_{k,l} = G_k$ for every $l \in L$ and $H_{k,l} = H_l$ for every $k \in K$, there exist canonical isometric embeddings $\hat{i} : (G_k)_{\mathfrak{U}_E} \to (G_{k,l})_{\mathfrak{U}}$ and $\hat{j} : (H_l)_{\mathfrak{U}_{F*}} \to (H_{k,l})_{\mathfrak{U}}$. Finally, note that by [13, Section 7] there exists a canonical isometric embedding $\varphi : (H_{k,l}^*)_{\mathfrak{U}} \to (H_{k,l})_{\mathfrak{U}}^*$.

Now, let us fix $\varepsilon > 0$ and consider for each pair $(k,l) \in K \times L$ the finite-dimensional subspaces $G_{k,l} \subseteq E$ and $H_{k,l} \subseteq F^*$ (this time we denote by $i_{k,l} : G_{k,l} \to E$ and $j_{k,l} : H_{k,l} \to F^*$ the embeddings). By [22, Proposition 9.2.2], $T_{k,l} = j_{k,l}^* J_F T i_{k,l} \in [\mathcal{F}(G_{k,l}, H_{k,l}^*), \eta_{(p,p_2;q,q_2)}]^{**} = [\mathcal{F}(G_{k,l}, H_{k,l}^*), \eta_{(p,p_2;q,q_2)}]$, so we can use Proposition 7.4 to find a finite-dimensional $Z_{k,l}$ with a 1-unconditional basis and operators $R_{k,l} : G_{k,l} \to Z_{k,l}$ and $S_{k,l} : Z_{k,l} \to H_{k,l}$ such that $T_{k,l}$ factors as $S_{k,l} R_{k,l}$ and

$$K^{(p,p_2)}(R_{k,l}) K_{(q,q_2)}(S_{k,l}) \leqslant (1+\varepsilon) \eta_{(p,p_2;q,q_2)}(j_{k,l}^* J_F T i_{k,l}) \leqslant (1+\varepsilon) \eta_{(p,p_2;q,q_2)}^{**}(T)$$

(the last inequality holds because $[\mathcal{F}, \eta_{(p,p_2;q,q_2)}]^{**}$ is a finitely generated ideal, see [26, Theorem 8.11]). By rescaling the norm of $Z_{k,l}$ we can assume without loss of generality that $K_{(q,q_2)}(S_{k,l}) = 1$. It follows from [13, Proposition 3.2] that $X = (Z_{k,l})_{\mathfrak{U}}$ is a Banach lattice, and it is easy to check that $R = (R_{k,l})_{\mathfrak{U}} : (G_{k,l})_{\mathfrak{U}} \to X$ is (p,p_2)-convex with $K^{(p,p_2)}(R) \leqslant \sup_{k,l} K^{(p,p_2)}(R_{k,l}) \leqslant (1+\varepsilon)\eta_{(p,p_2;q,q_2)}^{**}(T)$ and $S = (S_{k,l})_{\mathfrak{U}} : X \to (H_{k,l}^*)_{\mathfrak{U}}$ is (q,q_2)-concave with $K_{(q,q_2)}(S) \leqslant \sup_{k,l} K_{(q,q_2)}(S_{k,l}) = 1$. Therefore, we have obtained the following commutative diagram:

$$E \xrightarrow{i_E} (G_k)_{\mathfrak{U}_E} \xrightarrow{\hat{i}} (G_{k,l})_{\mathfrak{U}} \xrightarrow{(T_{k,l})_{\mathfrak{U}}} (H_{k,l}^*)_{\mathfrak{U}} \xrightarrow{\varphi} (H_{k,l})_{\mathfrak{U}}^* \xrightarrow{\hat{j}^*} (H_l)_{\mathfrak{U}_{F*}}^* \xrightarrow{j_{F*}^*} F^{**}$$

with R, S, X

A careful computation shows that $j_{F*}^* \hat{j}^* \varphi (T_{k,l})_{\mathfrak{U}} \hat{i} i_E = J_F T$, so for every $\varepsilon > 0$ we have found a factorization of $J_F T$ through a Banach lattice X with operators $U = R\hat{i} i_E$ and $V = j_{F*}^* \hat{j}^* \varphi S$ such that

$$\mu_{(p,p_2;q,q_2)}(T) \leqslant K^{(p,p_2)}(R) K_{(q,q_2)}(S) \leqslant (1+\varepsilon) \eta_{(p,p_2;q,q_2)}^{**}(T).$$

$\square$

We devote the last part of the section to identifying the ideal $M_{(p,p_2;q,q_2)}$. First, observe that Theorem 4.20 allows us to redefine the ideal $M_{(p,p_2;q,q_2)}$ in an alternative way:

Proposition 7.8. *Let $T : E \to F$. Then $T \in M_{(p,p_2;q,q_2)}(E,F)$ if and only if $J_F T$ factors as*

$$
\begin{array}{ccccc}
E & \xrightarrow{\;T\;} & F & \xrightarrow{\;J_F\;} & F^{**} \\
{\scriptstyle U}\downarrow & & & & \uparrow{\scriptstyle V} \\
X & & \xrightarrow{\quad Q \quad} & & Y
\end{array}
$$

where X is a (p,p_2)-convex Banach lattice, Y is a (q,q_2)-concave Banach lattice, both with constant 1, U and V are bounded operators and Q is an (almost interval preserving) lattice homomorphism. Moreover, $\mu_{(p,p_2;q,q_2)}(T) = \inf \|U\|\|Q\|\|V\|$, where the infimum is computed over all possible factorizations.

From now on, we consider only $p_2 \in \{p, \infty\}$ and $q_2 \in \{1, q\}$, as they are the most relevant cases. The following definition is an analogue in our setting of the ideal $I_{p,q}$ of *operators factoring through a diagonal* $D : L_p(\mu) \to L_q(\mu)$ introduced in [10].

Definition 7.9. *An operator $T : E \to F$ belongs to $I_{(p,p_2;q,q_2)}(E,F)$ (i.e. factors through a multiplication operator from $L_{p,p_2}(g_1 \cdot \mu)$ to $L_{q,q_2}(g_2 \cdot \mu)$) if there exists a positive measure μ, functions $g_1, g_2 \in L_1(\mu)$ and operators $A : E \to L_{p,p_2}(g_1 \cdot \mu)$, $D : L_{p,p_2}(g_1 \cdot \mu) \to L_{q,q_2}(g_2 \cdot \mu)$ and $B : L_{q,q_2}(g_2 \cdot \mu) \to F^{**}$ such that $J_F T = BDA$ and D is a multiplication operator, that is, $Df = gf$ for some measurable function g:*

$$
\begin{array}{ccccc}
E & \xrightarrow{\;T\;} & F & \xrightarrow{\;J_F\;} & F^{**} \\
{\scriptstyle A}\downarrow & & & & \uparrow{\scriptstyle B} \\
L_{p,p_2}(g_1 \cdot \mu) & & \xrightarrow{\quad D \quad} & & L_{q,q_2}(g_2 \cdot \mu)
\end{array}
$$

We define $i_{(p,p_2;q,q_2)}(T) = \inf \{\|A\|\|D\|\|B\|\}$, where the infimum is taken over all possible factorizations above.

Theorem 7.10. *Assume $p > q$. Then $[M_{(p,p_2;q,q_2)}, \mu_{(p,p_2;q,q_2)}]$ and $[I_{(p,p_2;q,q_2)}, i_{(p,p_2;q,q_2)}]$ are isomorphic ideals.*

Proof. $[M_{(p,p_2;q,q_2)}, \mu_{(p,p_2;q,q_2)}] \supseteq [I_{(p,p_2;q,q_2)}, i_{(p,p_2;q,q_2)}]$: Assume that $J_F T$ factors as in Definition 7.9. Since $L_{q,q_2}(g_2 \cdot \mu)$ is (q, q_2)-concave with constant 1, it follows that B is (q, q_2)-concave with constant $\|B\|$. On the other hand, observe that $L_{q,q_2}(g_2 \cdot \mu)$ is Dedekind complete, so the operator D has a modulus $|D|$, which is given by $|D|f = |g|f$ for every $f \in L_{p,p_2}(g_1 \cdot \mu)$. Moreover, $\||D|\| = \|D\|$ and $|D|$ is a lattice homomorphism. Therefore,

$$
\left\| \left(\sum_{i=1}^n |Df_i|^{p_2} \right)^{\frac{1}{p_2}} \right\|_{q,q_2} = \left\| \left(\sum_{i=1}^n ||D|f_i|^{p_2} \right)^{\frac{1}{p_2}} \right\|_{q,q_2}
$$

$$
\leqslant \||D|\| \left\| \left(\sum_{i=1}^n |f_i|^{p_2} \right)^{\frac{1}{p_2}} \right\|_{p,p_2} \leqslant \|D\| \left(\sum_{i=1}^n \|f_i\|_{p,p_2}^{p_2} \right)^{\frac{1}{p_2}}
$$

and D is (p, p_2)-convex with constant $\|D\|$. It follows that DA is (p, p_2)-convex with constant $\|A\|\|D\|$, so $T \in M_{(p,p_2;q,q_2)}(E,F)$ and $\mu_{(p,p_2;q,q_2)}(T) \leqslant \|A\|\|D\|\|B\|$.

$[M_{(p,p_2;q,q_2)}, \mu_{(p,p_2;q,q_2)}] \subseteq [I_{(p,p_2;q,q_2)}, i_{(p,p_2;q,q_2)}]$: We start by considering a factorization $J_F T = VQU$ as in Proposition 7.8 and fixing $q < r < p$. The case $p_2 = p$, $q_2 = q$ was established in [25]. Here, we show the other three cases, which are summarized in the commutative diagrams below:

(1) **Case 1:** $p_2 = \infty$, $q_2 = q$. X is (p, ∞)-convex, so it is r-convex with constant $C_{p,r}$ [17, Theorem 1.f.7]. Let $\widetilde{X}$ be a lattice renorming of X so that the r-convexity constant is 1 [17, Proposition 1.d.8]. By rescaling, we can assume that $\widetilde{Q} : \widetilde{X} \to Y$, the composition of Q with the renorming isomorphism, is a contractive lattice homomorphism. Moreover, Y is already r-concave with constant 1, so by Krivine's Factorization Theorem (cf. [14, Théorème 2] or [24, Lemma 17]), $\widetilde{Q}$ can be factored through some $L_r(\mu)$ with contractive lattice homomorphisms $R : \widetilde{X} \to L_r(\mu)$ and $S : L_r(\mu) \to Y$. Note that S is q-concave, so, by Theorem 2.5, it factors through $L_q(g_2 \cdot \mu)$ for some change of density $0 \leqslant g_2 \in S_{L_1(\mu)}$, with contractive factors $S_1 : L_r(\mu) \to L_q(g_2 \cdot \mu)$ and $S_2 : L_q(g_2 \cdot \mu) \to Y$, where S_1 is the operator of multiplication by $g_2^{-\frac{1}{r}}$. Similarly, by Theorem 2.4, there exists a change of density $0 \leqslant g_1 \in S_{L_1(\mu)}$ such that the composition of the renorming of X with R (which is (p, ∞)-convex) factors through $L_{p,\infty}^{[r]}(g_1 \cdot \mu)$ with $R_1 : X \to L_{p,\infty}^{[r]}(g_1 \cdot \mu)$ of norm bounded by $(1 - \frac{r}{p})^{\frac{1}{p} - \frac{1}{r}} C_{p,r} \|Q\|$ and $R_2 : L_{p,\infty}^{[r]}(g_1 \cdot \mu) \to L_r(\mu)$ the contractive operator of multiplication by $g_1^{\frac{1}{r}}$. Composing the renorming isomorphism from $L_{p,\infty}^{[1]}(g_1 \cdot \mu)$ into $L_{p,\infty}^{[r]}(g_1 \cdot \mu)$ given by (2.5) with the multiplication operators R_2 and S_1, we obtain the operator $D : L_{p,\infty}^{[1]}(g_1 \cdot \mu) \to L_q(g_2 \cdot \mu)$ of multiplication by $\left(\frac{g_1}{g_2}\right)^{\frac{1}{r}} \in L_r(g_2 \cdot \mu)_+$, which has norm bounded by $\left(\frac{p}{p-r}\right)^{\frac{1}{r}}$. Putting everything together, we conclude that $T \in I_{(p,\infty;q)}(E, F)$ and, since the factorization through Q and the exponent $r \in (q, p)$ were arbitrary, we can take the infimum over all possibilities to get that
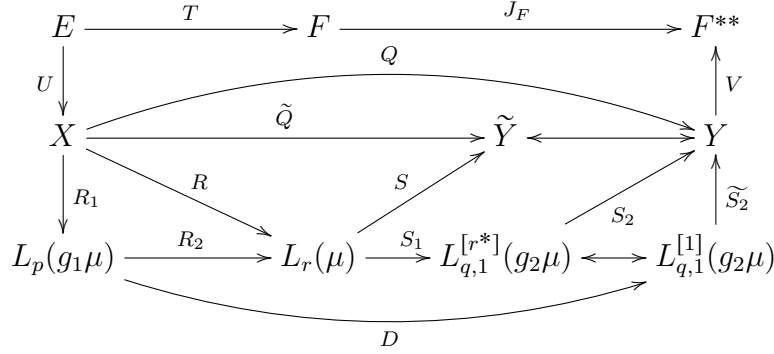
$$i_{(p,\infty;q)}(T) \leqslant \inf_{q < r < p} \left\{ \left(\frac{p}{p-r}\right)^{\frac{2}{r} - 1} C_{p,r} \right\} \mu_{(p,\infty;q)}(T).$$
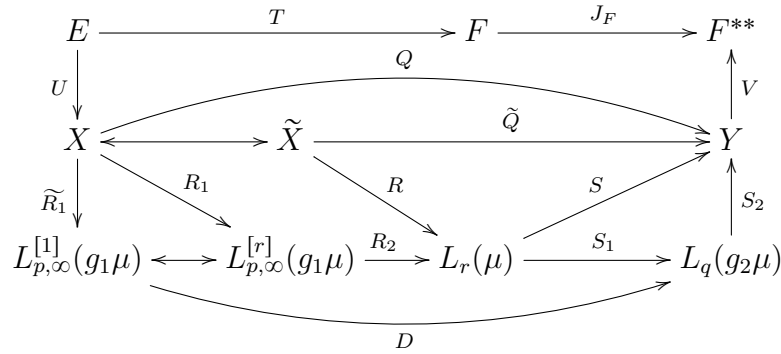
(2) **Case 2:** $p_2 = p$, $q_2 = 1$. The procedure is similar, but this time, we need to renorm Y with distortion $C_{q*,r*}$ so that it becomes r-concave with constant 1, and after factoring through $L_r(\mu)$ with Krivine's Theorem, we need to apply Theorem 2.3 and Theorem 2.6, and renorm

$L_{q,1}^{[r^*]}(g_2 \cdot \mu)$ to recover the standard norm of $L_{q,1}^{[1]}(g_2 \cdot \mu)$. Finally, we get that

$$i_{(p;q,1)}(T) \leqslant \inf_{q<r<p} \left\{ \left(\frac{q^*}{q^* - r^*} \right)^{\frac{2}{r^*}-1} C_{q^*,r^*} \right\} \mu_{(p;q,1)}(T).$$

(3) **Case 3:** $p_2 = \infty$, $q_2 = 1$. This time, both X and Y need to be renormed, introducing distortions $C_{p,r}$ and C_{p^*,r^*}, respectively, and, after applying Theorems 2.4 and 2.6, we need to renorm the corresponding $L_{p,\infty}^{[r]}(g_1 \cdot \mu)$ and $L_{q,1}^{[r^*]}(g_2 \cdot \mu)$ to obtain a multiplication operator D from $L_{p,\infty}^{[1]}(g_1 \cdot \mu)$ into $L_{q,1}^{[1]}(g_2 \cdot \mu)$. This time, the relation between the ideal norms is given by

$$i_{(p,\infty;q,1)}(T) \leqslant \inf_{q<r<p} \left\{ \left(\frac{p}{p-r} \right)^{\frac{2}{r}-1} \left(\frac{q^*}{q^* - r^*} \right)^{\frac{2}{r^*}-1} C_{p,r} C_{q^*,r^*} \right\} \mu_{(p,\infty;q,1)}(T).$$
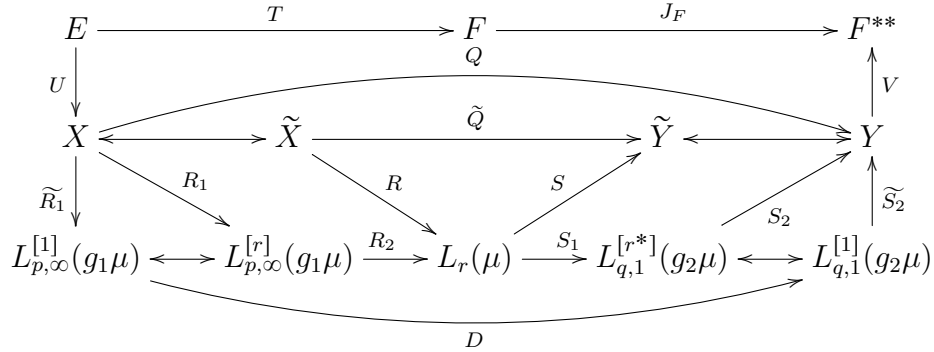
$\square$

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