MAXIMAL INEQUALITIES, FRAMES AND GREEDY ALGORITHMS

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Abstract. The aim of this article is to use Banach lattice techniques to study coordinate systems in function spaces. We begin by proving that the greedy algorithm of a basis is order convergent if and only if a certain maximal inequality is satisfied. We then show that absolute frames need not admit a reconstruction algorithm with respect to the usual order convergence, but do allow for reconstruction with respect to the order convergence inherited from the double dual. After this, we investigate the extent to which such coordinate systems affect the geometry of the underlying function space. Most notably, we prove that a Banach lattice X is lattice isomorphic to a closed sublattice of a $C(K)$-space if and only if every unconditional sequence in X is absolute.

Contents

1. Introduction

The study of coordinate systems in Banach spaces (Schauder bases, Markushevich bases, decompositions, frames, greedy algorithms) is a classical subject. However, in applications, the desirable coordinate systems often have additional structure, which may not even make sense in a generic Banach or Hilbert space. Common examples include wavelets, which make use of dilation and translation, and almost everywhere convergence systems. Evidently, to even define these concepts, one must be working in spaces with suitable symmetries, or possessing additional notions of convergence.

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Here, we will consider the interplay between coordinate systems and lattice structure. When considering a basic sequence (x_k) in a Banach lattice, there are many different ways to incorporate the partial ordering. The most common requirement is to place restrictions on the set $\{x_k\}$. For example, one may require that this set be contained in the positive cone, an order interval, or be pairwise disjoint. The issue with this approach is that bases in function spaces cannot be disjoint, and those that appear in practice are rarely contained in the positive cone. Indeed, it is non-trivial [22, 34] to even construct a positive basis in $L_1(\mathbb{R})$ and $L_2(\mathbb{R})$, and it is not known whether positive bases exist in $L_p(\mathbb{R})$ for $p \neq 1, 2$. Moreover, if (x_k) is a normalized basis for $L_1[0,1]$, then $\{x_k\}$ cannot be almost order bounded (equivalently, equiintegrable) [15, p. 74-75]. Hence, requiring that $\{x_k\}$ lie in an order interval is also very restrictive.

In contrast to the above, our approach in this article will be to utilize the lattice structure to define *maximal functions* and *order convergence*. For the moment, let us simply note that in any vector lattice X one may define notions of order convergence $f_k \xrightarrow{o} f$ and uniform convergence $f_k \xrightarrow{u} f$. Moreover, when X is a space of measurable functions, we have that $f_k \xrightarrow{o} f$ if and only if $f_k \xrightarrow{a.e.} f$ and there is a $g \in X$ with $|f_k| \leq g$ for all k. For this reason, one may view order convergence as a generalization of dominated almost everywhere convergence to vector lattices.

The starting point of our paper is a result of [65] which states that for a basic sequence (x_k) in a Banach lattice X, establishing order convergence of the basis expansions is equivalent to establishing boundedness of the associated maximal function. More precisely, letting $P_n(\sum_{k=1}^{\infty} a_k x_k) = \sum_{k=1}^{n} a_k x_k$ denote the n-th canonical basis projection, we have the following theorem.

Theorem 1.1 ([65] Theorem 3.1)**.** *Let (x_k) be a basic sequence in a Banach lattice X. Denote by $[x_k]$ its closed linear span and let $P_n : [x_k] \to [x_k]$ denote the n-th canonical basis projection. The following are equivalent.*

(i) For all $x \in [x_k]$, $P_n x \xrightarrow{u} x$;
(ii) For all $x \in [x_k]$, $P_n x \xrightarrow{o} x$;
(iii) For all $x \in [x_k]$, $|P_n x| \leq u$ for some $u \in X$ and all n;
(iv) For all $x \in [x_k]$, $(\bigvee_{n=1}^{m} |P_n x|)_m$ is norm bounded;
(v) There exists $M \geq 1$ such that for all $m \in \mathbb{N}$ and scalars $a_1, \dots, a_m$ one has

$$\left\| \bigvee_{n=1}^{m} \left| \sum_{k=1}^{n} a_k x_k \right| \right\| \leq M \left\| \sum_{k=1}^{m} a_k x_k \right\|. \tag{1.1}$$

The motivation for Theorem 1.1 stems from a well-known principle in harmonic analysis [11, 54, 59] which states that for a family of linear operators $(T_k)_{k \in \mathbb{N}}$ mapping $L_p(\Omega)$ to $L_0(\Omega)$, establishing almost everywhere convergence $T_k f \xrightarrow{a.e.} f$ for all $f \in L_p(\Omega)$ is equivalent to establishing an inequality of weak type (p,p) for the associated maximal operator $T^* f(x) := \sup_k |T_k f(x)|$. From this perspective, Theorem 1.1 states that – in the full generality of Banach lattices – establishing *dominated* almost everywhere convergence of the basis expansions is equivalent to establishing strong boundedness of the associated maximal operator.

A second motivation for Theorem 1.1 is that most of the important bases in martingale theory, harmonic analysis, probability, stochastic processes and orthogonal series do possess order convergent sums, although proving this is often a major result. Common examples include martingale difference sequences in $L_p(P)$ with $p > 1$ and P a probability measure (this is essentially Doob's inequality), the

Walsh basis (see [58]), and unconditional blocks of the Haar in $L_1[0,1]$ (this is essentially Burkholder-Davis-Gundy combined with Khintchine). The proof of the Carleson-Hunt theorem in [30] establishes (1.1) for the trigonometric basis. It is also worth mentioning the work of Bourgain [10] who used probabilistic techniques to make progress on the Kolmogorov and Garsia conjectures. In our language, these conjectures essentially ask whether every orthonormal basis of $L_2(P)$ admits a rearrangement satisfying Theorem 1.1. In [10], Bourgain was able to construct a rearrangement satisfying the inequality (1.1) with $M \sim \log(\log(m))$, which is the optimal dependence that one can achieve for random rearrangements. More generally, maximal inequalities such as (1.1) have a long history in analysis, and can be used to prove the Birkhoff ergodic theorem [49], a.e. convergence of the Schrödinger evolution back to the initial data [20], and even appear in the non-commutative setting [29, 35, 51, 56].

The objective of this paper is twofold. First, we wish to continue the study initiated in [7, 26, 65] of basic sequences satisfying Theorem 1.1. Secondly, we want to extend the theory to frames and greedy algorithms. Informally, frames can be thought of as redundant bases. The extra redundancy present in a frame allows for a more robust reconstruction with respect to the norm convergence, which is crucial in many applications. However, it also makes reconstruction with respect to order and uniform convergence much more subtle. Indeed, as we will see, Theorem 1.1 completely fails for frames, yet certain natural strengthenings of (1.1) do guarantee a reconstruction algorithm with respect to the order convergence inherited from the double dual.

In contrast, an analogue of Theorem 1.1 holds in complete generality for the greedy algorithm. Recall that the greedy algorithm is a nonlinear approximation scheme which approximates vectors using the coefficients of a basis with largest magnitude. Such algorithms appear in applications such as signal processing, where one needs to approximate a signal $x = \sum_k a_k x_k$ in an infinite or high dimensional Banach space using only a relatively small number of coordinates $\sum_{k=1}^n b_{m_k} x_{m_k}$. For orthornormal bases in Hilbert spaces, the most effective way to approximate x in norm using n coordinates is to choose the n coordinates of the basis expansion with the largest magnitude. That is, we set $\mathcal{G}_n(x) := \sum_{k=1}^n a_{m_k} x_{m_k}$ where $|a_{m_k}| \geq |a_i|$ for all $1 \leq k \leq n$ and $i \notin \{m_k\}_{1\leq k \leq n}$. Although it is easy to see that such a scheme provides an optimal approximation when (x_k) is orthonormal, it also turns out to be highly efficient for a large class of bases in Banach spaces. See, for example, [66] for a survey on the applications and effectiveness of the greedy algorithm.

Given the interest in determining whether the greedy algorithm provides an effective approximation in norm, it is also of interest to know how well it approximates a vector in order. As it turns out, by considering the convergence of the greedy sums in Theorem 1.1 rather than the convergence of the partial sums over initial segments of the basis, we are able to obtain the following nonlinear variant of the equivalence between establishing order convergence and maximal function estimates.

Theorem 1.2. *Let (x_k) be a semi-normalized basic sequence in a Banach lattice X and let $[x_k]$ denote its closed linear span. The following are equivalent.*

- *(i) For all $x \in [x_k]$, $\mathcal{G}_n(x) \xrightarrow{u} x$;*
- *(ii) For all $x \in [x_k]$, $\mathcal{G}_n(x) \xrightarrow{o} x$;*
- *(iii) For all $x \in [x_k]$, $|\mathcal{G}_n(x)| \leq u$ for some $u \in X$ and all n;*
- *(iv) For all $x \in [x_k]$, $(\bigvee_{n=1}^m |\mathcal{G}_n(x)|)_m$ is norm bounded;*

(v) There exists $C \geq 1$ such that for all $x \in [x_k]$ and $m \in \mathbb{N}$,

$$\left\| \bigvee_{n=1}^{m} |\mathcal{G}_n(x)| \right\| \leq C\|x\|.$$

One of the main goals of this article is to study the implications of Theorem 1.2. For now, we note that Theorem 1.2 is a theorem about basic sequences which can have implications outside the context of Banach lattices. That is, it is possible to study different properties of a basic sequence (x_k) by embedding (x_k) into different Banach lattices and then applying Theorem 1.2. In particular, since $[x_k]$ is separable, we may consider $[x_k]$ as a subspace of $C[0,1]$. In $C[0,1]$, uniform convergence agrees with norm convergence and $\||x| \vee |y|\| = \|x\| \vee \|y\|$ for all $x, y \in C[0,1]$. Thus, choosing $X = C[0,1]$ in Theorem 1.2 we recover the following key theorem about quasi-greedy bases from [69] which, a priori, makes no mention of Banach lattices.

Theorem 1.3 ([69] Theorem 1; [4] Theorem 4.1)**.** *Let E be a Banach space with a semi-normalized Schauder basis (x_k). The following are equivalent.*

(i) For all $x \in E$, $\mathcal{G}_n(x) \xrightarrow{\|\cdot\|} x$;
(ii) There exists $C \geq 1$ such that for all $x \in E$ and $n \in \mathbb{N}$, $\|\mathcal{G}_n(x)\| \leq C\|x\|$.

Although Theorem 1.2 reduces to Theorem 1.3 when $X = C[0,1]$, when $X = L_p(\Omega)$ it is often highly non-trivial to verify whether a basis satisfies Theorem 1.2. Several examples and structural properties of such bases will be given throughout the paper, but for now let us note that Tao [61] proved that all compactly supported wavelet bases satisfy Theorem 1.2 whereas Körner [38, 39] proved that the Fourier and Walsh bases do not. In other words, even though the Fourier and Walsh bases satisfy Theorem 1.1 and are optimally approximated by the greedy algorithm in norm, there are $f \in L_2([0,1])$ for which $(\mathcal{G}_n f)_{n=1}^{\infty}$ fails to converge to f almost everywhere. In fact, Körner could construct $f \in L_2([0,1])$ such that the greedy algorithm for f diverges at almost every point.

1.1. **Outline of the paper.** We now briefly summarize the paper. In Section 2 we recall the necessary vector lattice and Banach space terminology. Then, in Section 3 we combine the ideas of [63, 65] with those from greedy theory to generalize several results on quasi-greedy bases to the lattice setting. More specifically, in Section 3.1 we prove Theorem 1.2, in Section 3.2 we discuss the extent to which Theorem 1.1 holds for frames (and, as a corollary of our methods, solve a problem from [27]), and in Section 3.3 we verify that most of the fundamental results on quasi-greedy bases extend to uniformly quasi-greedy bases. Since the results in Sections 3.1 and 3.3 must recover the classical results on quasi-greedy bases when $X = C[0,1]$, there is some inevitable overlap between the new and classical proofs – our exposition in these two subsections will loosely follow [5, Chapter 10].

The heart of the paper is Section 4 where we aim to develop a qualitatively new perspective on coordinate systems in Banach lattices by proving results which have no direct analogues in the classical theories of Schauder or quasi-greedy bases. We begin in Section 4.1 with the goal of characterizing when a "good" basis of a Banach space E can be embedded into a Banach lattice X so that it inherits "bad" lattice properties. As already mentioned, by choosing $X = C[0,1]$ one can always embed a Schauder or quasi-greedy basis into a lattice so that Theorems 1.1 and 1.2 hold. In Section 4.1.1 we construct a Banach lattice X containing a copy of the Lindenstrauss basis which simultaneously fails Theorems 1.1 and 1.2. Then, in Section 4.1.2 we construct, for each $p > 1$, a basis (u_i) of ℓ_p which is equivalent to the canonical basis of ℓ_p yet simultaneously fails Theorems 1.1 and 1.2. Finally, we show in Section 4.1.3 that a normalized basis can be embedded into a Banach lattice in such a way that it fails to be absolute (i.e., so that the maximal inequality for unconditional bases fails) if and only if it is not equivalent to the canonical ℓ_1 basis.

In Section 4.2 we move from embeddings to blockings. This is a topic where bibases and greedy bases behave very differently. Indeed, as we will demonstrate in Section 4.2.1, it is quite easy to block a basis so that it satisfies Theorem 1.1 or its unconditional variant. On the other hand, in Section 4.2.2 we show that the canonical basis of ℓ_p is the only subsymmetric basis for which every blocking is greedy.

In Section 4.3 we characterize (up to a lattice isomorphism) AM-spaces as the only Banach lattices for which every unconditional sequence is absolute. We also show that if every basic sequence in a Banach lattice X satisfies Theorem 1.1 then X must be p-convex for all finite p.

In Section 4.4 we prove that complemented absolute sequences behave very much like disjoint sequences and use this to give a Banach lattice proof of the well-known fact that the only complemented subspace of $C[0,1]$ with an unconditional basis is c_0. In Section 4.5 we prove that every unconditional basis of L_p can be rearranged to fail Theorem 1.1 and that σ-order complete Banach lattices which embed into the span of an absolute FDD must be purely atomic.

Throughout, we make use of standard facts and notation – see e.g. [5, 44, 45] for Banach spaces in general, [47] for Banach lattices specifically. The field of scalars is $\mathbb{R}$, unless otherwise specified.

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2. Background

In this section, we provide the necessary vector lattice and Banach space background – background on frames and greedy algorithms will appear throughout the paper.

If (y_α) is a net in a vector lattice X then $y_\alpha \downarrow 0$ means that y_α is decreasing and $\inf\{y_\alpha\} = 0$. A vector lattice X is called *Archimedean* if $n^{-1}e \downarrow 0$ for every positive vector $e \in X$. We will be working with the following three classical notions of sequential convergence of a sequence $(x_n)_{n=1}^\infty$ in an Archimedean vector lattice X.

- We say that $(x_n)_{n=1}^\infty$ *uniformly converges* to x and write $x_n \xrightarrow{u} x$ if there exists $e \in X_+$ and a sequence $\epsilon_m \downarrow 0$ in $\mathbb{R}$ satisfying:

$$\forall m \ \exists n_m \ \forall n \geq n_m, \ |x_n - x| \leq \epsilon_m e.$$

- We say that $(x_n)_{n=1}^\infty$ *order converges* to x (with respect to a sequence) and write $x_n \xrightarrow{o_1} x$ if there exists a sequence $y_m \downarrow 0$ in X satisfying:

$$\forall m \ \exists n_m \ \forall n \geq n_m, \ |x_n - x| \leq y_m.$$

- We say that $(x_n)_{n=1}^\infty$ *order converges* to x (with respect to a net) and write $x_n \xrightarrow{o} x$ if there exists a net $y_\beta \downarrow 0$ in X satisfying:

$$\forall \beta \ \exists n_\beta \ \forall n \geq n_\beta, \ |x_n - x| \leq y_\beta.$$

Evidently, $x_n \xrightarrow{u} x \Rightarrow x_n \xrightarrow{o_1} x \Rightarrow x_n \xrightarrow{o} x$; however, neither converse implication holds in general. In a Banach lattice one also has norm convergence, and although u-convergence implies norm convergence, o_1-convergence need not. It is also not true that norm convergent sequences are o-convergent, but they do at least have u-convergent subsequences.

Recall that a *(Schauder) basis* of a Banach space X is a sequence (x_k) in X such that for every vector x in X there is a unique sequence of scalars (a_k) satisfying $x = \sum_{k=1}^{\infty} a_k x_k$. Here, of course, the series converges in norm. For each n, we define the n-th basis projection $P_n : X \to X$ via $P_n\big(\sum_{k=1}^{\infty} a_k x_k\big) = \sum_{k=1}^{n} a_k x_k$, so that $P_n x \xrightarrow{\|\cdot\|} x$. It is known that the projections P_n are uniformly bounded; the number $K := \sup_n \|P_n\|$ is called the basis constant of (x_k). A sequence (x_k) in X is called a *(Schauder) basic sequence* if it is a Schauder basis of its closed linear span $[x_k]$. In this case, the P_n's are defined on $[x_k]$. It is a standard fact that a sequence (x_k) of non-zero vectors is Schauder basic iff there exists a constant $K \geq 1$ such that

$$\bigvee_{n=1}^{m} \Big\| \sum_{k=1}^{n} a_k x_k \Big\| \leq K \Big\| \sum_{k=1}^{m} a_k x_k \Big\| \qquad \text{for all } m \in \mathbb{N} \text{ and all scalars } a_1, \dots, a_m. \tag{2.1}$$

Moreover, the least value of the constant K is the basis constant of (x_k). A sequence (x_k) is called *unconditional* if every permutation of (x_k) is basic. As is well-known, unconditionality of a basic sequence is characterized by the existence of a constant $C \geq 1$ such that

$$\bigvee_{\epsilon_k = \pm 1} \Big\| \sum_{k=1}^{m} \epsilon_k a_k x_k \Big\| \leq C \Big\| \sum_{k=1}^{m} a_k x_k \Big\| \qquad \text{for all } m \in \mathbb{N} \text{ and all scalars } a_1, \dots, a_m. \tag{2.2}$$

Suppose now that X is a Banach lattice. By interchanging the order of the supremum and the norm, one obtains the "maximal inequality" variants of (2.1) and (2.2). More formally, a sequence (x_k) of non-zero vectors in a Banach lattice X is called *bibasic* if there exists a constant $M \geq 1$ such that

$$\Big\| \bigvee_{n=1}^{m} \Big| \sum_{k=1}^{n} a_k x_k \Big| \Big\| \leq M \Big\| \sum_{k=1}^{m} a_k x_k \Big\| \qquad \text{for all } m \in \mathbb{N} \text{ and all scalars } a_1, \dots, a_m \tag{2.3}$$

and *absolute* if there exists a constant $A \geq 1$ such that

$$\Big\| \sum_{k=1}^{m} |a_k x_k| \Big\| \leq A \Big\| \sum_{k=1}^{m} a_k x_k \Big\| \qquad \text{for all } m \in \mathbb{N} \text{ and all scalars } a_1, \dots, a_m. \tag{2.4}$$

Note that in obtaining the inequality (2.4) from (2.2) we used the standard fact that

$$\bigvee_{\epsilon_k = \pm 1} \Big| \sum_{k=1}^{m} \epsilon_k a_k x_k \Big| = \sum_{k=1}^{m} |a_k x_k|, \tag{2.5}$$

which holds in any Archimedean vector lattice. Clearly, every bibasic sequence is basic. By Theorem 1.1, a basic sequence is bibasic if and only if, for each $x \in [x_k]$, $P_n x \xrightarrow{o} x$.

When X is a $C(K)$-space it is easy to see that every basic sequence is bibasic and every unconditional sequence is absolute. One of the main objectives of this paper is to prove a converse statement; namely, that a Banach lattice X is lattice isomorphic to a closed sublattice of a $C(K)$-space if and only if every unconditional sequence is absolute.

3. Reconstruction algorithms for coordinate systems in Banach lattices

In this section, we begin our goal of infusing lattice technique into the study of frames and greedy bases.

3.1. A greedy version of the bibasis theorem.

We begin by proving Theorem 1.2. To fix the notation, X will be a Banach lattice, E a closed subspace of X, and $\mathcal{B} = (e_n)$ a semi-normalized basis of E. The biorthogonal functionals of (e_n) will be denoted by $e_n^* \in E^*$.

We first recall some of the essentials on greedy algorithms. Fix $x \in E$. A *greedy ordering* of x is an injective map $\pi : \mathbb{N} \to \mathbb{N}$ such that $\{n : e_n^*(x) \neq 0\} \subseteq \pi(\mathbb{N})$ and $(|e_{\pi(n)}^*(x)|)$ is non-increasing. The *m-th greedy sum of x* associated to π is given by $G_{\pi,m}(x) := \sum_{n=1}^m e_{\pi(n)}^*(x)e_{\pi(n)}$, and the sequence $(G_{\pi,m}(x))_{m=1}^\infty$ is called a *greedy approximation of x*. A *strictly greedy sum of x of order m* is an element $G_{\pi,m}(x)$ for which there is no ambiguity in defining the m-th greedy sum of x, i.e., $G_{\pi,m}(x) = G_{\sigma,m}(x)$ for all greedy orderings σ. It is well-known that the choice of greedy ordering is not particularly important, and hence we will usually use the ordering induced by the basis. More specifically, we define $\rho : \mathbb{N} \to \mathbb{N}$ so that $\{n : e_n^*(x) \neq 0\} \subseteq \rho(\mathbb{N})$ and if $j < k$ then either $|e_{\rho(j)}^*(x)| > |e_{\rho(k)}^*(x)|$ or $|e_{\rho(j)}^*(x)| = |e_{\rho(k)}^*(x)|$ and $\rho(j) < \rho(k)$. We define the *m-th natural greedy sum of x* as $\mathcal{G}_m(x) := \sum_{n=1}^m e_{\rho(n)}^*(x)e_{\rho(n)}$ and call the basis (e_n) *quasi-greedy* if, for all $x \in E$, $\mathcal{G}_m(x) \xrightarrow{\|\cdot\|} x$. By Theorem 1.3, (e_n) is quasi-greedy iff there exists a constant $C \geq 1$ such that for all $x \in E$ and $m \in \mathbb{N}$, $\|\mathcal{G}_m(x)\| \leq C\|x\|$.

All of the above definitions are valid in Banach, and even quasi-Banach spaces. Since we require X to be a lattice, for each $x \in E$ and greedy ordering π we may define

$$G_{\pi,m}^\vee(x) := \bigvee_{n=1}^m |G_{\pi,n}(x)| = \bigvee_{n=1}^m \left|\sum_{k=1}^n e_{\pi(k)}^*(x)e_{\pi(k)}\right|.$$

In particular, we may consider the map $x \mapsto \mathcal{G}_m^\vee(x) := \bigvee_{n=1}^m |\mathcal{G}_n(x)|$, and because it is absolutely homogeneous we may define $\|\mathcal{G}_m^\vee\| := \sup_{x\in S_E} \|\mathcal{G}_m^\vee(x)\|$. We say that $G_{\pi,m}(x)$ is a *lattice strictly greedy sum of x* of order m if $|e_{\pi(i)}^*(x)| \neq |e_{\pi(j)}^*(x)|$ for each distinct $i, j \in \{1, \dots, m\}$ for which $e_{\pi(i)}^*(x) \neq 0$.

To begin, we extend the fact that the choice of greedy ordering is not particularly important to the lattice setting. In the proof, we need to take some care since $G_{\pi,m}^\vee(x)$ may depend on π, even when $G_{\pi,m}(x)$ is strictly greedy.

Lemma 3.1. *Let (e_n) be a semi-normalized basic sequence in a Banach lattice X and let $C > 0$ be a constant. The following are equivalent.*

- *(i) For all $x \in E$, all $m \in \mathbb{N}$ and all greedy orderings π, $\|G_{\pi,m}^\vee(x)\| \leq C\|x\|$.*
- *(ii) For all $x \in E$ and all $m \in \mathbb{N}$, $\|\mathcal{G}_m^\vee(x)\| \leq C\|x\|$.*
- *(iii) For all $x \in E$ and all $m \in \mathbb{N}$ there exists a greedy ordering π such that $\|G_{\pi,m}^\vee(x)\| \leq C\|x\|$.*
- *(iv) We have $\|\mathcal{G}_{|supp(x)|}^\vee(x)\| \leq C\|x\|$ whenever $x \in E$ has finite support and $\mathcal{G}_{|supp(x)|}(x)$ is lattice strictly greedy.*
- *(v) We have $\|\mathcal{G}_{|supp(x)|}^\vee(x)\| \leq C\|x\|$ for all $x \in E$ of finite support.*

Furthermore, the least such C so that the above estimates hold is $C = \sup_m \|\mathcal{G}_m^\vee\|$.

Proof. (i)⇒(ii)⇒(iii)⇒(iv) is clear.

(iv)⇒(v): Let $x \in E$ have finite support, say, r. Write $x = \mathcal{G}_r(x) = \sum_{n=1}^r e^*_{\rho(n)}(x)e_{\rho(n)}$. Fix $\varepsilon > 0$ and find $\delta_1, \dots, \delta_r$ with $|e^*_{\rho(1)}(x) + \delta_1| > \cdots > |e^*_{\rho(r)}(x) + \delta_r| > \max_{i \notin \{\rho(1),\dots,\rho(r)\}} |e^*_i(x)|$ and $\|\delta_n e_{\rho(n)}\| < \frac{\varepsilon}{Cm}$ for all $n = 1, \dots, r$. We consider the element $y := x + \sum_{n=1}^r \delta_n e_{\rho(n)}$. It is easy to see that $y = \mathcal{G}_r(y)$ is a lattice strictly greedy sum, so (iv) implies that

$$\left\| \bigvee_{i=1}^r \left| \sum_{n=1}^i (e^*_{\rho(n)}(x) + \delta_n) e_{\rho(n)} \right| \right\| \le C\|y\| \le C\|x\| + \varepsilon.$$

Hence,

$$\begin{aligned} \left\| \bigvee_{i=1}^r \left| \sum_{n=1}^i e^*_{\rho(n)}(x) e_{\rho(n)} \right| \right\| &\le \left\| \bigvee_{i=1}^r \left| \sum_{n=1}^i (e^*_{\rho(n)}(x) + \delta_n) e_{\rho(n)} \right| \right\| \\ &+ \left\| |\delta_1 e_{\rho(1)}| + \cdots + |\delta_r e_{\rho(r)}| \right\| \\ &\le C\|x\| + 2\varepsilon, \end{aligned}$$

which yields that $\|\mathcal{G}^\vee_m(x)\| \le C\|x\|$.

(v)⇒(i): By a similar argument as (iv)⇒(v), the condition (v) implies that $\|G^\vee_{\pi,|\mathrm{supp}(x)|}(x)\| \le C\|x\|$ for all $x \in E$ of finite support and all greedy orderings π of x. We now fix $m \in \mathbb{N}$ and let π be a greedy ordering of x. Note that for any $n \in \mathbb{N}$ satisfying $n \ge \max(\pi(1), ..., \pi(m))$ we have

$$\|G^\vee_{\pi,m}(x)\| = \|G^\vee_{\pi,m}(P_n x)\| \le \|G^\vee_{\pi,n}(P_n x)\| \le C\|P_n x\|. \tag{3.1}$$

On the other hand, since (e_n) is a basic sequence, we have $x = \lim_{n\to\infty} P_n(x)$. Therefore, passing to the limit in (3.1), we conclude that $\|G^\vee_{\pi,m}(x)\| \le C\|x\|$ for all $m \in \mathbb{N}$, as desired.

□

In order to prove Theorem 1.2, we need the following lemma.

Lemma 3.2. *Suppose that (v) in Lemma 3.1 fails. Then for every $C \ge 1$ and every finite set $A \subseteq \mathbb{N}$, there exists $x \in E$ with $|supp(x)| < \infty$ and $supp(x) \cap A = \emptyset$ such that $\|\mathcal{G}^\vee_{|supp(x)|}(x)\| > C\|x\|$.*

Proof. Fix C and A as above and define $M := \sum_{n\in A} \|e^*_n\|\|e_n\|$. By assumption, there exists a finitely supported $y \in E$ such that

$$\|\mathcal{G}^\vee{}_{|\mathrm{supp}(y)|}(y)\| > (C(1+M) + M)\|y\|. \tag{3.2}$$

Set $r = |\mathrm{supp}(y)|$ and define $x = (I - P_A)(y)$, where for $z \in E$, $P_A(z) := \sum_{n\in A} e^*_n(z)e_n$. Since $x \in E$ has finite support disjoint from A, we must simply show that $\|\mathcal{G}^\vee_m(x)\| > C\|x\|$, where m is the cardinality of the support of x. For this, notice that $\mathcal{G}^\vee_r(y) \le \mathcal{G}^\vee_m(x) + \sum_{k\in A} |e^*_k(y)e_k|$, so that

$$\|\mathcal{G}^\vee_m(x)\| \ge \|\mathcal{G}^\vee_r(y)\| - \|\sum_{k\in A} |e^*_k(y)e_k|\| > (C(1+M) + M)\|y\| - M\|y\|.$$

Inserting the inequality $\|x\| \le (1+M)\|y\|$ into the above, we conclude that $\|\mathcal{G}^\vee_m(x)\| > C\|x\|$, as desired. □

We may now establish the greedy analogue of Theorem 1.1.

Theorem 3.3. *For a semi-normalized basic sequence $\mathcal{B} = (e_n)$ in a Banach lattice X, the following are equivalent.*

(i) For all $x \in E$, $\mathcal{G}_m(x) \xrightarrow{u} x$;
(ii) For all $x \in E$, $\mathcal{G}_m(x) \xrightarrow{o} x$;
(iii) For all $x \in E$, $|\mathcal{G}_m(x)| \leq u$ for some $u \in X$ and all m;
(iv) For all $x \in E$, $(\bigvee_{n=1}^m |\mathcal{G}_n(x)|)_m$ is norm bounded;
(v) There exists $C \geq 1$ such that for all $x \in E$ and $m \in \mathbb{N}$,

$$\left\| \bigvee_{n=1}^{m} |\mathcal{G}_n(x)| \right\| \leq C\|x\|.$$

Proof. It is clear that (i)⇒(ii)⇒(iii)⇒(iv).

(iii)⇒(v): Suppose not. Then using Lemma 3.2 we get a sequence (x_k) of elements of E such that for each k:

(a) $m_k := |supp(x_k)|$ is finite and $supp(x_k) \cap supp(x_i) = \emptyset$ for $i = 1, \dots, k-1$;
(b) $\|x_k\| \leq 2^{-k}$;
(c) $\|\mathcal{G}_{m_k}^{\vee}(x_k)\| \geq k$;
(d) $\max\{|e_n^*(x_k)| : n \in \mathbb{N}\} < \min\{|e_n^*(x_{k-1})| : n \in supp(x_{k-1})\}$.

Indeed, we can certainly do this for $k = 1$, so suppose that we have constructed $x_1, \dots, x_{k-1}$. Let $\mu = \min\{|e_n^*(x_{k-1})| : n \in supp(x_{k-1})\}$ and put $C_k = \max\{k2^k, 2Kk\mu^{-1}\}$ where $K = \sup_n \|e_n^*\|$. Using Lemma 3.2 there exists x_k of finite support disjoint from $\cup_{i=1}^{k-1} supp(x_i)$ such that $\|\mathcal{G}_{m_k}^{\vee}(x_k)\| > C_k\|x_k\|$. Scaling x_k we can take $\|x_k\| = kC_k^{-1} \leq 2^{-k}$, so that $\|\mathcal{G}_{m_k}^{\vee}(x_k)\| \geq k$. Then for every $n \in \mathbb{N}$ we have

$$|e_n^*(x_k)| \leq \|e_n^*\|\|x_k\| < \mu.$$

Now, the series $\sum_{k=1}^{\infty} x_k$ converges to some $x \in E$. Note by construction that $\sum_{k=1}^{j-1} x_k = \mathcal{G}_{l_j}(x)$ is a strictly greedy sum of x (here $l_j := \sum_{k=1}^{j-1} m_k$, $j \geq 2$, and $l_1 := 0$). Write x_j in its natural greedy ordering as $e^*_{\rho_j(1)}(x_j)e_{\rho_j(1)} + \cdots + e^*_{\rho_j(m_j)}(x_j)e_{\rho_j(m_j)}$ and notice that $\mathcal{G}_{l_j}(x) + e^*_{\rho_j(1)}(x_j)e_{\rho_j(1)} + \cdots + e^*_{\rho_j(r)}(x_j)e_{\rho_j(r)}$ is a natural greedy sum of x for each $1 \leq r \leq m_j$. Hence, by assumption, there is a u with $|\mathcal{G}_{l_j}(x) + e^*_{\rho_j(1)}(x_j)e_{\rho_j(1)} + \cdots + e^*_{\rho_j(r)}(x_j)e_{\rho_j(r)}| \leq u$. Note that u is uniform (independent of j and r) since we are building a natural greedy approximation of x, i.e., every term we are bounding is a greedy sum of the same approximation. In particular, $|\mathcal{G}_{l_j}(x)| \leq u$ for all j, hence $|e^*_{\rho_j(1)}(x_j)e_{\rho_j(1)} + \cdots + e^*_{\rho_j(r)}(x_j)e_{\rho_j(r)}| \leq 2u$. Taking sup we get $\mathcal{G}_{m_j}^{\vee}(x_j) \leq 2u$, so that $j \leq \|\mathcal{G}_{m_j}^{\vee}(x_j)\| \leq 2\|u\|$. This is a contradiction.

(v)⇒(i): By [5, Theorem 10.2.3, Lemma 10.2.5] we get that for each $y \in E$, $\mathcal{G}_m(y) \xrightarrow{\|\cdot\|} y$. Fix $x \in E$. Since $\mathcal{G}_m(x) \xrightarrow{\|\cdot\|} x$ there exists $m_1 < m_2 < \dots$ with $\mathcal{G}_{m_k}(x) \xrightarrow{u} x$. Passing to a further subsequence and using that $(\mathcal{G}_m(x))$ is Cauchy, we may assume that for all $i > m_k$, $\|\mathcal{G}_i(x) - \mathcal{G}_{m_k}(x)\| < \frac{1}{2^k}$. In particular, $\|x - \mathcal{G}_{m_k}(x)\| \leq \frac{1}{2^k}$.

Consider the element $y_k = x - \mathcal{G}_{m_k}(x)$. Then for each $i > m_k$, $\mathcal{G}_i(x) - \mathcal{G}_{m_k}(x)$ is just $\mathcal{G}_{i-m_k}(y_k)$. Hence, our assumption yields that

$$u_k = \bigvee_{i=m_k+1}^{i=m_{k+1}} |\mathcal{G}_i(x) - \mathcal{G}_{m_k}(x)|$$

has norm at most $C\|y_k\| \leq \frac{C}{2^k}$. Define $e := \sum_{k=1}^{\infty} ku_k$. Then for every $k \in \mathbb{N}$, $u_k \leq \frac{e}{k}$, so that $u_k \xrightarrow{u} 0$.

Since $|\mathcal{G}_{m_k}(x) - x| + u_k \xrightarrow{u} 0$ there exists a vector $f_1 > 0$ with the property that for any $\varepsilon > 0$ there exists k^*, for any $k \geq k^*$, $|\mathcal{G}_{m_k}(x) - x| + u_k \leq \varepsilon f_1$. Fix ε, and find the required k^*. Let $i \in \mathbb{N}$ with $i > m_{k^*}$. We can find $k \geq k^*$ with $m_k < i \leq m_{k+1}$ so that

$$|\mathcal{G}_i(x) - x| \leq |\mathcal{G}_{m_k}(x) - x| + |\mathcal{G}_i(x) - \mathcal{G}_{m_k}(x)| = |\mathcal{G}_{m_k}(x) - x| + u_k \leq \varepsilon f_1.$$

This shows that $\mathcal{G}_i(x) \xrightarrow{u} x$, and hence that $\mathcal{B}$ satisfies (i).

We have thus established that (i)⇔(ii)⇔(iii)⇔(v)⇒(iv). For (iv)⇒(i) assume that for each $x \in E$, $(\mathcal{G}_m^\vee(x))$ is norm bounded. Since $(\mathcal{G}_m^\vee(x))$ is an increasing sequence, it has a supremum in X^{**}, hence $(\mathcal{G}_m(x))$ is order bounded in X^{**}. Thus, (ii) holds in X^{**}, so (i) holds in X^{**}. However, uniform convergence of sequences passes freely from X^{**} to X by [65, Proposition 2.12]. Consequently, (i) holds in X. □

We will call a semi-normalized basic sequence (e_n) in a Banach lattice X *uniformly quasi-greedy* if it satisfies any, and hence all, of the conditions in Theorem 3.3. Note that the last condition in Theorem 3.3 says that $C_{qg}^\vee := \sup_m \|\mathcal{G}_m^\vee\| < \infty$. We will call $C_{qg}^\vee$ the *uniform quasi-greedy constant of* (e_n).

Example 3.4. *Wavelets:* By controlling the maximal function of the greedy approximations by the Hardy-Littlewood maximal function, Tao ([61], see also [40]) proved that all compactly supported wavelet bases (such as the Haar and Daubechies wavelets) are uniformly quasi-greedy in L_p for all $1 < p < \infty$. In particular, considering a rearrangement of the Haar basis that is not bibasic ([65, Example 6.2]), we see that uniformly quasi-greedy bases need not be bibases.

Example 3.5. *The trigonometric and Walsh bases:* In [38], Körner showed the existence of a real-valued function $f \in L_2(\mathbb{T})$ whose greedy algorithm with respect to the trigonometric system diverges almost everywhere. In particular, this provides an example of a basis that is both greedy and a bibasis but is not uniformly quasi-greedy. Similarly, in [39] it is shown that the greedy algorithm for the Walsh basis need not converge almost everywhere.

Remark 3.6. As mentioned in Section 1, due to the freedom in choosing the ambient Banach lattice X, the theory we will develop in this article for uniformly quasi-greedy bases will recover and extend the usual theory for quasi-greedy bases. To see this, let (e_n) be a semi-normalized Schauder basis of a Banach space E. As E is separable, we may consider E as a subspace of $C[0,1]$. In $C[0,1]$, it is easy to see that uniform convergence agrees with norm convergence and $\||x| \vee |y|\| = \|x\| \vee \|y\|$ for all $x, y \in C[0,1]$. Thus, choosing $X = C[0,1]$ in Theorem 3.3, we recover the classical characterization of quasi-greedy bases in Theorem 1.3. Similarly, when $X = C[0,1]$, our results in Section 3.3 will recover the classical structural properties of quasi-greedy bases. This leads to two different perspectives on Theorem 3.3:

First, suppose that the Banach lattice X is fixed. Then one would be interested in determining which quasi-greedy bases (or quasi-greedy basic sequences) are uniformly quasi-greedy in X, as these bases would exhibit the enhanced approximation properties in Theorem 3.3. On the other hand, suppose that the Banach space E – or even a basis (e_n) of E – is fixed, and consider an embedding of E into a Banach lattice X. If one chooses X to be $C[0,1]$, then several of the theorems we will prove in this paper will recover those from the theory of quasi-greedy bases. However, if one chooses X differently, then the sequence (e_n) may, or may not, be uniformly quasi-greedy with respect to X. Note that the property of being uniformly quasi-greedy is "stable" with respect to the choice of X, in the sense that if (e_n) is a basic sequence in X, and X is a closed sublattice of Y, then (e_n) will be uniformly quasi-greedy with respect to X iff it is uniformly quasi-greedy with respect to Y. In the next section, we will show that if one puts conditions on X (say, order continuity, or being a classical Banach lattice) or on how E sits in X (say, complementably, or even $E = X$) then one can prove theorems for bibasic and uniformly quasi-greedy bases that have no analogues for Schauder or quasi-greedy bases.

Remark 3.7. Theorem 3.3 is equally valid for Schauder decompositions. Since greedy decompositions do not seem to appear in the literature, we quickly sketch what we mean:

Let $\mathcal{B} = (E_n)$ be a sequence of closed non-zero subspaces of a Banach lattice X which forms a Schauder decomposition of $E := [E_n]$. Fix $x = \sum_{n=1}^{\infty} e_n \in E$, with $e_n \in E_n$ for all n. A *greedy ordering* of x is an injective map $\pi : \mathbb{N} \to \mathbb{N}$ such that $\{n : e_n \neq 0\} \subseteq \pi(\mathbb{N})$ and $(\|e_{\pi(n)}\|)$ is non-increasing. The *m-th greedy sum* of x associated to π is given by $G_{\pi,m}(x) := \sum_{n=1}^{m} e_{\pi(n)}$, and the sequence $(G_{\pi,m}(x))_{m=1}^{\infty}$ is called a *greedy approximation* of x. $G_{\pi,m}(x)$ is said to be a *strictly greedy sum* of x of order m if $G_{\pi,m}(x) = G_{\sigma,m}(x)$ for all greedy orderings σ. The *m-th natural greedy sum* of x is $\mathcal{G}_m(x) := \sum_{n=1}^{m} e_{\rho(n)}$ where $\rho : \mathbb{N} \to \mathbb{N}$ is such that $\{n : e_n \neq 0\} \subseteq \rho(\mathbb{N})$ and if $j < k$ then either $\|e_{\rho(j)}\| > \|e_{\rho(k)}\|$ or $\|e_{\rho(j)}\| = \|e_{\rho(k)}\|$ and $\rho(j) < \rho(k)$. A *quasi-greedy decomposition* is a decomposition for which $\mathcal{G}_m(x) \xrightarrow{\|\cdot\|} x$ for all $x \in E$. Defining $\mathcal{G}_m^{\vee}$ in the obvious way, the proof of the following theorem is almost identical to the semi-normalized basic sequence case.

Theorem 3.8. *Let $\mathcal{B} = (E_n)$ be a sequence of subspaces of a Banach lattice X which forms a Schauder decomposition of $[E_n]$. The following are equivalent.*

(i) For all $x \in E$, $\mathcal{G}_m(x) \xrightarrow{u} x$;
(ii) For all $x \in E$, $\mathcal{G}_m(x) \xrightarrow{o} x$;
(iii) For all $x \in E$, $|\mathcal{G}_m(x)| \leq u$ for some $u \in X$ and all m;
(iv) For all $x \in E$, $(\bigvee_{n=1}^{m} |\mathcal{G}_n(x)|)_m$ is norm bounded;
(v) There exists $C \geq 1$ such that for all $x \in E$ and $m \in \mathbb{N}$,

$$\left\| \bigvee_{n=1}^{m} |\mathcal{G}_n(x)| \right\| \leq C\|x\|.$$

In particular, arguing as in Remark 3.6, the "usual" characterization of quasi-greedy bases in Theorem 1.3 is valid for quasi-greedy decompositions – we will focus on the basis case for convenience of the reader; however, we will return to the subject of finite dimensional decompositions (FDDs) at various points in the paper.

We also mention that, although stated for the natural ordering, being uniformly quasi-greedy is independent of the greedy ordering. Indeed, noting that absolutely convergent series converge uniformly, the proof from [5, Lemma 10.2.5] can be used to show the following.

Proposition 3.9. *Let (e_n) be a basic sequence in a Banach lattice X. The following are equivalent.*

(i) $G_m(x) \xrightarrow{u} x$ for every $x \in E$ and every greedy approximation $(G_m(x))$;
(ii) (e_n) is uniformly quasi-greedy;
(iii) For every $x \in E$ there is a greedy approximation $(G_m(x))$ such that $G_m(x) \xrightarrow{u} x$;
(iv) For every $x \in E$ with infinite support its strictly greedy approximation converges uniformly to x.

Results analogous to Lemma 3.1 and Proposition 3.9 are also valid for conditions (ii), (iii) and (iv) of Theorem 3.3. For example, one can show that being uniformly quasi-greedy is equivalent to the fact that for every $x \in E$ and greedy ordering π, there is a $u^\pi \in X$ satisfying that for all m, $|G_{\pi,m}(x)| \leq u^\pi$. However, the next example shows that one cannot pick u^π independently of π:

Example 3.10. We will construct a uniformly quasi-greedy sequence (e_k) in the Banach lattice $X = \ell_2(L_2)$ such that there exists $x \in E$ for which there is no $u \in X$ such that for every $m \in \mathbb{N}$ and every greedy ordering π we have $|G_{\pi,m}(x)| \leq u$. In the n-th copy of L_2 we consider the first n^2 Rademacher vectors. Order these into a normalized basic sequence (e_k) in lexicographical order; by standard martingale inequalities the Rademacher's are bibasic in every ordering, from which it follows that (e_k) is uniformly quasi-greedy. Now consider the vector $x = \sum_{k=1}^{\infty} a_k e_k \in \ell_2(L_2)$ defined by having all of the a_k in the n-th copy of L_2 being $\frac{1}{n^2}$. Note that this series converges as (e_k) is an orthonormal sequence, and it is already in its natural greedy ordering. Next, recall the inequality $\sup_{\delta_n=0,1} |\sum_{n=1}^m \delta_n x_n| \geq \frac{1}{2}\sum_{n=1}^m |x_n|$ and suppose that $u \in X$ dominates $|G_{\pi,m}(x)|$ for each π and m. Then in the n-th block, u must dominate the partial sums of each permutation of the Rademacher's, hence it must dominate half of the sum of their absolute values. This means that the n-th block contributes at least $\frac{1}{2}$ to the norm of u. Since each of the blocks are disjoint, this forces $\|u\|$ to be arbitrarily large, a contradiction.

Remark 3.11. The motivation for Example 3.10 stems from the various characterizations of absolute sequences given in [65]. Suppose that (x_k) is an unconditional basic sequence in a Banach lattice X. Then (x_k) is said to be *permutable* if it is bibasic in every ordering; that is, if for each permutation σ and $x \in [x_k]$ there exists u^σ such that $|P_n^\sigma x| \leq u^\sigma$ for all n, where P_n^σ is the n-th partial sum associated to the permutation σ. It turns out that one can choose u^σ independently of σ if and only if (x_k) is absolute – see [65, Proposition 7.5]. As shown in Example 3.10, the property that the order bound u^π in statement (iii) of Theorem 3.3 can be chosen independently of the greedy ordering π can be viewed as an intermediate between being uniformly quasi-greedy and absolute (see [65, Theorem 7.2] for further characterizations of absolute sequences).

Remark 3.12. Apart from Theorem 1.3, the other main characterization in the theory of greedy bases is the equivalence between being greedy and being unconditional and democratic. This characterization can also be generalized to the Banach lattice setting as follows: Let (M, Σ, μ) be a measure space and let X be a Banach function space over (M, Σ, μ). For each $x \in X$ we let $G_x \in \Sigma$ denote any finite measure set such that

$$\text{ess sup}_{s \in G_x^c} |x(s)| \leq \text{ess inf}_{t \in G_x} |x(t)| < \infty.$$

For each $A \in \Sigma$, we let P_A be the restriction operator to A. We say that X is *greedy* if there exists some constant $C > 0$ such that for all $x \in X$ and all sets of the form G_x we have

$$\|x - P_{G_x} x\| \leq C \inf_{\mu(A) \leq \mu(G_x); y \in P_A X} \|x - y\|.$$

We say that X is *democratic* if there exists $D > 0$ such that $\|1_A\| \leq D\|1_B\|$ for all $A, B \in \Sigma$ with $\mu(A) \leq \mu(B) < \infty$. Routine arguments then show that X is greedy if and only if it is democratic, which extends the characterization of (weight)-greedy bases from the discrete setting where μ is atomic to the continuous setting of Banach function spaces.

3.2. **The bibasis theorem fails for frames.** One of the surprising features of Theorem 1.1 is that although order convergence and order boundedness generally do not pass between sublattices, statement (iv) is stable under passing between sublattices, in the sense that if $(x_k) \subseteq X \subseteq Y$ then (iv) holds in X iff it holds in Y. In statement (ii) of Theorem 1.1 one actually has a simultaneously norm and order convergent sequence, so one may wonder if these sequences pass freely between closed sublattices. The next result says that the answer is no; we will later expand on this example to show that the bibasis theorem fails for frames.

Example 3.13. Let φ be an Orlicz function and for each sequence $x = (x_k)$ of real numbers, define $I(x) = \sum_{k=1}^{\infty} \varphi(|x_k|)$. Then $\ell_\varphi := \{x : \exists\lambda > 0,\ I(\lambda x) \leq 1\}$ is a Banach lattice under the Luxemburg norm, and $h_\varphi := \{x : \forall\lambda > 0,\ I(\lambda x) < \infty\}$ is a closed order dense ideal of ℓ_φ. It is well-known that $\ell_\varphi = h_\varphi$ iff φ satisfies the Δ_2-condition near zero, so choose any φ that fails the Δ_2-condition near zero. Fix some positive vector $x = (x_k)$ in $\ell_\varphi \setminus h_\varphi$ and define $y^k = x_k e^k \in h_\varphi$, where e^k is the k-th standard unit vector. Next, define z^k by $z^k_n = x_n$ if $n \geq k$, and 0 otherwise. Then $0 \leq y^k \leq z^k \downarrow 0$ in ℓ_φ, so that $y^k \xrightarrow{o} 0$ in ℓ_φ. Note that since $x \in \ell_\varphi$, (y^k) is norm null, but (y^k) is not even order bounded in h_φ, since any upper bound for (y^k) must coordinate-wise dominate $x \notin h_\varphi$.

We are now ready to show that Theorem 1.1 fails for frames. Recall that, given a Banach space E, a sequence $(x_k, f_k) \in E \times E^*$ is a *Schauder frame* for E if for all $x \in E$ we have $x = \sum_{k=1}^{\infty} f_k(x)x_k$. Note that each of the statements in Theorem 1.1 has an obvious "frame" analogue. Specifically, for a frame (x_k, f_k) with (x_k) contained in a Banach lattice, we can consider the following properties:

(i) For all $x \in [x_k]$, $\sum_{k=1}^{n} f_k(x)x_k \xrightarrow{u} x$;
(ii) For all $x \in [x_k]$, $\sum_{k=1}^{n} f_k(x)x_k \xrightarrow{o} x$;
(iii) For all $x \in [x_k]$, $\left(\sum_{k=1}^{n} f_k(x)x_k\right)_n$ is order bounded;
(iv) For all $x \in [x_k]$, $\left(\bigvee_{n=1}^{m} \left|\sum_{k=1}^{n} f_k(x)x_k\right|\right)_m$ is norm bounded;
(v) There exists $M \geq 1$ such that for all $x \in [x_k]$ and $m \in \mathbb{N}$,

$$\left\| \bigvee_{n=1}^{m} \left| \sum_{k=1}^{n} f_k(x)x_k \right| \right\| \leq M\|x\|.$$

It is easy to see that (i)⇒(ii)⇒(iii)⇒(iv)⇔(v). However, it turns out that the reverse implications do not hold, in general. To see that (iv)⇏(iii), we will modify Example 3.13. However, before doing so, we give an easier example which shows that (iii)⇏(ii).

Example 3.14. Let $X = L_p[0,1]$, $1 < p < \infty$. Since X is reflexive, order convergence agrees with uniform convergence (see [8]) and increasing norm bounded sequences have supremum. Hence, for any frame of X, properties (i) and (ii) coincide, as do (iii), (iv) and (v). We will show that (iii)⇏(ii). For this, we let (t_n) be the "typewriter" sequence, $t_n = \chi_{[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}]}$, where $k \geq 0$ is such that $2^k \leq n < 2^{k+1}$. Let (h_n, h_n^*) denote the Haar basis with its coordinate functionals and choose $f \in X^*$ with $f(\mathbb{1}) = 1$. We define our sequence (x_k, f_k) by weaving the typewriter sequence through the Haar basis as follows:

$$(x_k) = (h_1, t_1, -t_1, h_2, t_2, -t_2, h_3, t_3, -t_3, \dots),$$
$$(f_k) = (h_1^*, f, f, h_2^*, f, f, h_3^*, f, f, \dots).$$

It is easy to see that (x_k, f_k) is a frame. Moreover, since (h_k) is a bibasis, it is easy to deduce that for every $x \in X$, $P_n x := \sum_{k=1}^{n} f_k(x)x_k$ defines an order bounded sequence. However, $P_n(\mathbb{1}) \overset{a.e.}{\not\to} \mathbb{1}$.

Remark 3.15. A slight modification of the above construction (by instead weaving through a weakly null normalized sequence) allows one to give examples of reproducing pairs that are not Schauder frames. This solves an open problem from [27].

As we have already remarked, the implication (iv)⇒(iii) above also fails for frames; however, there is a significant strengthening of this fact. Indeed, recall that a frame (x_k, f_k) for a closed subspace E of a Banach lattice X is *absolute* if for each $x \in E$,

$$\sup_n \left\| \sum_{k=1}^n |f_k(x)x_k| \right\| < \infty.$$

Baire Category arguments then give the existence of some $A \geq 1$ with

$$\left\| \sum_{k=1}^n |f_k(x)x_k| \right\| \leq A\|x\|$$

for each $x \in E$. Clearly, absolute frames satisfy property (v). However, as we will now show, they need not satisfy property (iii).

Example 3.16. Consider an Orlicz sequence space ℓ_φ failing the Δ_2-condition near zero. The unit vectors (e_k) are disjoint and hence absolute. We will view $(e_k) \subseteq h_\varphi \subseteq \ell_\varphi$. As in Example 3.13, we choose $0 \leq x = (x^k)$ in $\ell_\varphi \setminus h_\varphi$ and define $y_k = x^k e_k$. Let f be a non-zero functional. We consider the sequence

$$(e_1, y_1, -y_1, e_2, y_2, -y_2, e_3, \dots)$$

equipped with the functionals

$$(e_1^*, f, f, e_2^*, f, f, e_3^*, \dots).$$

For $u \in h_\varphi$, the partial sums of this sequence will look like $\sum_{k=1}^n e_k^*(u)e_k$ or $\sum_{k=1}^n e_k^*(u)e_k + f(u)x^n e_n$, and so this sequence will be a frame. Also, if we sum moduli we can bound the norm by

$$\|\sum_{k=1}^\infty |e_k^*(u)e_k|\| + 2|f(u)|\|\sum_{n=1}^\infty x^k e_k\| < \infty.$$

Hence, we have constructed an absolute frame.

Now, let u be such that $f(u) \neq 0$. If the partial sums for u were order bounded in h_φ, we would deduce that the sequence $(x^n e_n)$ is order bounded in h_φ, which we showed in Example 3.13 is not true.

The main reason why the above example works is that the space $X = h_\varphi$ is not σ-monotonically complete. Recall that a Banach lattice X is *σ-monotonically complete* if all positive increasing norm bounded sequences in X have supremum. In this case, one can deduce order convergence of the partial sums for absolute frames.

Proposition 3.17. *If X is σ-monotonically complete and (x_k, f_k) is an absolute frame for $E \subseteq X$ then for each $x \in E$, $\sum_{k=1}^\infty f_k(x)x_k \xrightarrow{o} x$. In particular, each absolute frame has order convergent expansions, when the order convergence in X^{**} is used.*

Proof. Fix $x \in E$. Since (x_k, f_k) is absolute, the sequence $(\sum_{k=1}^n |f_k(x)x_k|)$ is increasing and norm bounded, so has a supremum u. It is then easy to see that $|x - \sum_{k=1}^n f_k(x)x_k| \leq u - \sum_{k=1}^n |f_k(x)x_k| \downarrow 0$. To finish the proof, recall the standard fact that if Y is a Banach lattice then Y^* is monotonically complete. □

Proposition 3.17 shows that absolute frames admit a reconstruction formula with respect to an appropriate order convergence. However, being an absolute frame is a very strong property. Indeed, by modifying the proof of [34, Theorem 3], one obtains the following result.

Proposition 3.18. *Let (x_k, f_k) be an absolute frame in $L_p[0,1]$, $1 \leq p < \infty$. Then $[x_k]$ embeds isomorphically into ℓ_p.*

On the other hand, although property (i) is the strongest of the five properties listed above, we will now show that frames satisfying this condition are in complete abundance. To set the stage, let E be a closed subspace of a Banach lattice X and recall that a sequence (x_k, f_k) in $E \times E^*$ is a *u-frame* if for each $x \in E$ we have $\sum_{k=1}^{n} f_k(x)x_k \xrightarrow{u} x$. The following result shows that if E admits a u-frame (which happens, in particular, if E has a bibasis) then one can construct a u-frame for E with the (x_k) lying in any subset of E with dense span.

Proposition 3.19. *Let E be a closed subspace of a Banach lattice X and let $M \subseteq E$ have dense span. If E admits a u-frame then there is a u-frame (x_k, f_k) for E with each $x_k \in M$.*

Proof. This result was proven in [21, Theorem 4.4] under the assumption that $E = X$ and (x_k, f_k) is a bibasis. Therefore, it suffices to extend the result by only assuming that (x_k, f_k) is a u-frame for a subspace of X. The key to doing this is to prove a perturbation result; namely, that if (x_k, f_k) is a u-frame for a closed subspace E of a Banach lattice X and $0 < \varepsilon < 1$ then any sequence (y_k) in E for which

$$\|x_k - y_k\| \leq \frac{\varepsilon}{2^{2k+1}\|f_k\|}$$

can be paired with a sequence $g_k \in E^*$ so that (y_k, g_k) is a u-frame for E. Such a perturbation result allows one to construct, for any dense set M of E, a u-frame for E consisting of elements of M. After this, the proof of [21, Theorem 4.4] can be used to replace the assumption that M is dense with the assumption that M has dense span. To see that the above perturbation result holds, let $x \in E$ and define $S(x) = \sum_{k=1}^{\infty} f_k(x)(x_k - y_k)$. It is shown in [52, Lemma 2.3] that S is well-defined and $\|S\| < 1$. In fact, it is easy to see that the sum is uniformly Cauchy and hence uniformly converges. One then defines $T = I - S$, so that $T(x) = u - \sum_{k=1}^{\infty} f_k(x)y_k$. Replacing x with $T^{-1}x$, we see that $x = u - \sum_{k=1}^{\infty} f_k(T^{-1}x)y_k$, so that $(y_k, (T^{-1})^* f_k)$ is a u-frame for E. ☐

Remark 3.20. Interesting examples of Gabor frames exhibiting enhanced convergence properties can be found in [25]; see also [70] for nonharmonic Fourier series.

3.3. **Properties of uniformly quasi-greedy bases.** As explained in Remark 3.6, any result about quasi-greedy bases can be viewed as a result about uniformly quasi-greedy bases with ambient Banach lattice $C[0,1]$. In this subsection, we record some extensions of the fundamental results on quasi-greedy bases to other ambient Banach lattices X.

We begin by generalizing the well-known fact that quasi-greedy bases are unconditional for constant coefficients.

Proposition 3.21. *Suppose that (e_n) is a uniformly quasi-greedy basic sequence. Then the following hold.*

(i) For any distinct indices $n_1, \dots, n_m$ we have

$$\left\| \bigvee_{k=1}^{m} \left| \sum_{i=1}^{k} e_{n_i} \right| \right\| \leq C_{qg}^{\vee} \left\| \sum_{i=1}^{m} e_{n_i} \right\|.$$

(ii) (e_n) *is permutable for constant coefficients, i.e., for any distinct indices* $n_1, \dots, n_m$ *and any choices of signs* ε_i *we have*

$$(2C_{qg}^{\vee})^{-1}\Big\|\bigvee_{k=1}^m\Big|\sum_{i=1}^k e_{n_i}\Big|\Big\| \le \Big\|\sum_{i=1}^m \varepsilon_i e_{n_i}\Big\| \le \Big\|\bigvee_{k=1}^m\Big|\sum_{i=1}^k \varepsilon_i e_{n_i}\Big|\Big\| \le 2C_{qg}^{\vee}\Big\|\sum_{i=1}^m e_{n_i}\Big\|.$$

Proof. (i) follows since $(\sum_{i=1}^k e_{n_i})$ is a greedy approximation of $x = \sum_{i=1}^m e_{n_i}$. For the right inequality in (ii), notice that

$$\Big|\sum_{i=1}^k \varepsilon_i e_{n_i}\Big| \le \Big|\sum_{\{i\le k\,:\,\varepsilon_i=1\}} \varepsilon_i e_{n_i}\Big| + \Big|\sum_{\{i\le k\,:\,\varepsilon_i=-1\}} \varepsilon_i e_{n_i}\Big|,$$

where if we happen to sum over the empty set, we declare the sum to be 0. It follows that

$$\Big\|\bigvee_{k=1}^m\Big|\sum_{i=1}^k \varepsilon_i e_{n_i}\Big|\Big\| \le \Big\|\bigvee_{k=1}^m\Big|\sum_{\{i\le k\,:\,\varepsilon_i=1\}} \varepsilon_i e_{n_i}\Big|\Big\| + \Big\|\bigvee_{k=1}^m\Big|\sum_{\{i\le k\,:\,\varepsilon_i=-1\}} \varepsilon_i e_{n_i}\Big|\Big\|.$$

In this form, we can ignore the ε_i and interpret the above terms as greedy sums of x, which gives us our desired estimate.

To obtain the first inequality in (ii), let $A = \{i : \varepsilon_i = 1\}$ and $B = \{1, \dots, m\} \setminus A$. Then a generic term $\sum_{i=1}^k e_{n_i}$ can be decomposed as follows:

$$\Big|\sum_{i=1}^k e_{n_i}\Big| \le \Big|\sum_{A\cap\{1,\dots,k\}} e_{n_i}\Big| + \Big|\sum_{B\cap\{1,\dots,k\}} (-1)e_{n_i}\Big|.$$

Taking sup over the above inequality and using that both the first and second terms on the right-hand side form part of a greedy approximation of $\sum_{i=1}^m \varepsilon_i e_{n_i}$, we obtain our desired estimate with constant $(2C_{qg}^{\vee})^{-1}$.

□

Corollary 3.22. *Suppose that* (e_n) *is a uniformly quasi-greedy basic sequence. Then for any distinct indices* $n_1, \dots, n_m$ *and any real numbers* $a_1, \dots, a_m$*, we have*

$$\Big\|\bigvee_{k=1}^m\Big|\sum_{i=1}^k a_i e_{n_i}\Big|\Big\| \le 2\max|a_i| C_{qg}^{\vee}\Big\|\sum_{i=1}^m e_{n_i}\Big\|.$$

Proof. Without loss of generality, we may assume that $\max|a_i| = 1$. By [28, Theorem 3.13], there exists $c_l \ge 0$ and signs ε_l^i, $l = 1, \dots, m+1$, $i = 1, \dots, m$ such that $\sum_{l=1}^{m+1} c_l = 1$ and $\sum_{l=1}^{m+1} \varepsilon_l^i c_l = a_i$ for $i = 1, \dots, m$. It follows that

$$\begin{aligned}\Big\|\bigvee_{k=1}^m\Big|\sum_{i=1}^k a_i e_{n_i}\Big|\Big\| &\le \sum_{l=1}^{m+1} c_l \Big\|\bigvee_{k=1}^m\Big|\sum_{i=1}^k \varepsilon_l^i e_{n_i}\Big|\Big\| \le 2C_{qg}^{\vee}\sum_{l=1}^{m+1} c_l\Big\|\sum_{i=1}^m e_{n_i}\Big\| \\ &= 2C_{qg}^{\vee}\Big\|\sum_{i=1}^m e_{n_i}\Big\|.\end{aligned}$$

□

Corollary 3.23. *Suppose that (e_n) is a uniformly quasi-greedy basic sequence. Then for every $x \in E$, every greedy ordering π of x and every $m \in \mathbb{N}$,*

$$|e^*_{\pi(m)}(x)| \left\| \bigvee_{k=1}^m \left| \sum_{i=1}^k e_{n_i} \right| \right\| \leq 4C_{qg}C^{\vee}_{qg}\|x\|,$$

where $\{n_1, \dots, n_m\} = \{\pi(1), \dots, \pi(m)\}$. In particular, for any distinct indices $n_1, \dots, n_m$, any real numbers $a_1, \dots, a_m$, and any permutation σ of $\{1, \dots, m\}$

$$\min_{j \in \{1,\dots,m\}} |a_j| \left\| \bigvee_{k=1}^m \left| \sum_{i=1}^k e_{n_{\sigma(i)}} \right| \right\| \leq 4C_{qg}C^{\vee}_{qg} \left\| \sum_{i=1}^m a_i e_{n_i} \right\|.$$

Proof. The proof is the same as [5, Theorem 10.2.12] except that we replace the first inequality by $\| \bigvee_{k=1}^m \left| \sum_{i=1}^k e_{n_i} \right| \| \leq 2C^{\vee}_{qg} \| \sum_{i=1}^m \varepsilon_i e_{\pi(i)} \|$, which is allowed by Proposition 3.21.

□

We next introduce some notation to measure non-permutability. Let (e_n) be a basic sequence in a Banach lattice X and let $A = \{n_1, \dots, n_m\}$ be an ordered subset of $\mathbb{N}$. Define a continuous, sublinear, absolutely homogeneous map $P^{\vee}_A$ on E by $P^{\vee}_A(x) := \bigvee_{i=1}^m \left| \sum_{k=1}^i e^*_{n_k}(x) e_{n_k} \right|$, where we take into account that A is ordered via the order in which we sup up. Define $\|P^{\vee}_A\| = \sup_{x \in S_E} \left\| P^{\vee}_A(x) \right\|$ and note that the following are equivalent:

- (e_n) is bibasic;
- For each $x \in E$ the sequence $(P^{\vee}_{\{1,\dots,m\}}(x))_m$ is norm bounded;
- The sequence $(\|P^{\vee}_{\{1,\dots,m\}}\|)_m$ is norm bounded.

Hence, noting the trivial bound $\|P^{\vee}_A(x)\| \leq \mathbf{k}|A|\|x\|$ where $\mathbf{k} = \sup_n \|e_n\|\|e^*_n\|$, the maps $P^{\vee}_A$ can be used as a measure of how far (e_n) is from being bibasic via the asymptotic growth of their norms. Taking into account more sets A, one can measure non-permutability. For uniformly quasi-greedy bases, we get the following bound.

Corollary 3.24. *Suppose that (e_n) is a uniformly quasi-greedy basic sequence. Then for every $x \in E$ and every ordered set $A = \{n_1, \dots, n_m\} \subseteq supp(x)$ we have*

$$\|P^{\vee}_A(x)\| \leq 8C^2_{qg}C^{\vee}_{qg} \frac{\max\{|e^*_n(x)| : n \in A\}}{\min\{|e^*_n(x)| : n \in A\}} \|x\|.$$

Proof. Take $A = \{n_1, \dots, n_m\} \subseteq \text{supp}(x)$ and let

$$B = \{n \in \mathbb{N} : \alpha \leq |e^*_n(x)| \leq \beta\}$$

with $\alpha = \min_A |e^*_n(x)|$ and $\beta = \max_A |e^*_n(x)|$. We can extend the order on A to the set B by writing $B = \{n_1, ..., n_m, n_{m+1}, ..., n_l\}$. Hence,

$$\left\| \bigvee_{k=1}^m \left| \sum_{i=1}^k e^*_{n_i}(x) e_{n_i} \right| \right\| \leq \left\| \bigvee_{k=1}^l \left| \sum_{i=1}^k e^*_{n_i}(x) e_{n_i} \right| \right\|. \tag{3.3}$$

Applying Corollary 3.22, we obtain the estimate

$$\begin{aligned}\left\|\bigvee_{k=1}^{l}\left|\sum_{i=1}^{k} e_{n_i}^*(x)e_{n_i}\right|\right\| &\leq 2C_{gq}^{\vee}\max_A |e_n^*(x)|\|\sum_{i\in B}\varepsilon_i e_{n_i}\| \\ &= 2C_{qg}^{\vee}\frac{\max_A |e_n^*(x)|}{\min_A |e_n^*(x)|}\min_B |e_n^*(x)|\|\sum_{i=1}^{l}\varepsilon_i e_{n_i}\|.\end{aligned}$$

Using [9, Lemma 2.3], we see that

$$\min_B |e_n^*(x)|\|\sum_{i\in B}\varepsilon_i e_{n_i}\| \leq 2C_{qg}\|\sum_{i=1}^{l} e_{n_i}^*(x)e_{n_i}\|. \tag{3.4}$$

We next notice that for $\varepsilon > 0$ small enough we have

$$\sum_{i=1}^{l} e_{n_i}^*(x)e_{n_i} = \sum_{\{n_i:|e_{n_i}^*(x)|>\alpha-\varepsilon\}} e_{n_i}^*(x)e_{n_i} - \sum_{\{n_i:|e_{n_i}^*(x)|>\beta\}} e_{n_i}^*(x)e_{n_i}.$$

Hence, $\| \sum_{i=1}^{l} e_{n_i}^*(x)e_{n_i}\| \leq 2C_{qg}\|x\|$. Combining this with (3.4) we obtain the estimate

$$\min_B |e_n^*(x)|\|\sum_{i=1}^{l}\varepsilon_i e_{n_i}\| \leq 4C_{qg}^2\|x\|. \tag{3.5}$$

Combining everything, we conclude that

$$\left\|\bigvee_{k=1}^{m}\left|\sum_{i=1}^{k} e_{n_i}^*(x)e_{n_i}\right|\right\| \leq 8C_{qg}^{\vee}C_{qg}^2\frac{\max_A |e_n^*(x)|}{\min_A |e_n^*(x)|}\|x\|,$$

as desired

□

Remark 3.25. To measure conditionality of (e_n) it is standard to use the growth of the sequence $\mathbf{k}_m := \sup_{|A|\leq m}\|P_A\|$. As mentioned, the sequence $\mathbf{k}_m^{\vee} := \sup_{|A|\leq m}\|P_A^{\vee}\|$ is a measure of non-permutability, where the sup is over all ordered subsets of $\mathbb{N}$ of cardinality at most m. Clearly, $\mathbf{k}_m \leq \mathbf{k}_m^{\vee}$. By applying Corollary 3.24 to $P_A(x)$ and using the identity $P_A^{\vee}(P_A(x)) = P_A^{\vee}(x)$ we see that, for certain $x \in E$, we can bound $\|P_A^{\vee}(x)\|$ in terms of $\|P_A(x)\|$. However, in general, $\|P_A^{\vee}(x)\|$ and $\|P_A(x)\|$ behave qualitatively different. Indeed, the Haar in $L_p[0,1]$, $p > 1$, is a uniformly quasi-greedy bibasis, it is unconditional so $\mathbf{k}_m = \mathcal{O}(1)$, but it is not permutable so $\mathbf{k}_m^{\vee} \neq \mathcal{O}(1)$.

Proposition 3.26. *Let (e_n) be a uniformly quasi-greedy basic sequence. Then $\boldsymbol{k}_m^{\vee} = \mathcal{O}(\log_2(m))$.*

Proof. Consider an integer $m \geq 2$ and find p such that $2^p \leq m < 2^{p+1}$. Let $x \in S_E$ and note that we have $|e_n^*(x)| \leq \mathbf{K}$ for all $n \in \mathbb{N}$, where $\mathbf{K} := \sup_n \|e_n^*\|$. Construct a partition $(B_j)_{j=0}^p$ of $\mathbb{N}$ as in the proof of [5, Theorem 10.2.14].

Let $A = \{n_1, \dots, n_m\}$ be an ordered subset of $\mathbb{N}$. Then for each $1 \leq l \leq m$ we have

$$\left|\sum_{k=1}^{l} e_{n_k}^*(x)e_{n_k}\right| \leq \left|\sum_{k\in\{1,\dots,l\}\cap B_0} e_{n_k}^*(x)e_{n_k}\right| + \cdots + \left|\sum_{k\in\{1,\dots,l\}\cap B_p} e_{n_k}^*(x)e_{n_k}\right|.$$

Hence,

$$P_A^{\vee}(x) \leq \bigvee_{l=1}^{m} \left| \sum_{k\in\{1,\dots,l\}\cap B_0} e_{n_k}^*(x)e_{n_k} \right| + \cdots + \bigvee_{l=1}^{m} \left| \sum_{k\in\{1,\dots,l\}\cap B_p} e_{n_k}^*(x)e_{n_k} \right|.$$

For $j = 1, \dots, p$, Corollary 3.24 gives

$$\left\| \bigvee_{l=1}^{m} \left| \sum_{k\in\{1,\dots,l\}\cap B_j} e_{n_k}^*(x)e_{n_k} \right| \right\| \leq 16C_{qg}^2 C_{qg}^{\vee},$$

while the trivial estimate gives

$$\left\| \bigvee_{l=1}^{m} \left| \sum_{k\in\{1,\dots,l\}\cap B_0} e_{n_k}^*(x)e_{n_k} \right| \right\| \leq m\mathbf{K}2^{-p}c \leq 2c\mathbf{K},$$

with $c = \sup_n \|e_n\|$. Combining the above estimates yields the bound $\|P_A^{\vee}(x)\| \leq 16pC_{qg}^2 C_{qg}^{\vee} + 2c\mathbf{K}$, as required. □

Remark 3.27. One cannot hope for a better bound in Proposition 3.26, in general, as one can consider embeddings into $C[0,1]$ (the estimate $\mathbf{k}_m = \mathcal{O}(\log_2(m))$ for quasi-greedy bases is sharp for general Banach spaces). One may hope for better estimates if one only considers bases, or if one works in nicer spaces, e.g., $L_p(\mu)$. In this direction, martingale theory gives that the bibasis constant of the standard ordering of the Haar in $L_p[0,1]$ is q, where $\frac{1}{p} + \frac{1}{q} = 1$. Knowing this exact bound, it is natural to inquire what the value – or at least the asymptotics – of $\mathbf{k}_m^{\vee}$, $C_{qg}^{\vee}$ and the bibasis constant are for other bases. In particular, which bases of $L_p[0,1]$ minimize these quantities? Although we do not have a complete answer to this question, in Proposition 4.28 we will show that $\mathbf{k}_m^{\vee} \neq \mathcal{O}(1)$ for any basis of $L_p[0,1]$.

4. The role of the ambient space

In [65] and Section 3.3 it is shown that many of the major results about Schauder and quasi-greedy bases generalize to bibases and uniformly quasi-greedy bases. In this section, we delve deeper into the interplay between coordinate systems and lattice structures and prove results that have no direct analogues in the classical theories.

4.1. **Embedding bases into Banach lattices.** Given any basis $(e_i)_i$ of a Banach space E, we may always embed E into a Banach lattice X so that $(e_i)_i$ becomes bibasic, and hence has good order properties. In this subsection, we aim to do the following:

- Take a basis $(e_i)_i$ of a Banach space E possessing certain good properties.
- Embed E into a Banach lattice X so that the "uniform" analogues of these properties fail.

More specifically, we deal with:

(i) The Lindenstrauss basis in ℓ_1, or rather its version described in [19]; this is an example of a conditional quasi-greedy basis.
(ii) The canonical basis in ℓ_p, $1 < p < \infty$.

4.1.1. *A copy of the Lindenstrauss basis that is neither bibasic nor uniformly quasi-greedy.* A fundamental example of a conditional quasi-greedy basis in ℓ_1 is given in [19]. We begin by outlining their construction.

For $n \in \mathbb{N}$ and $j \in \{1,2\}$, define $\phi_j(n) = 2n + j$. Denote by (e_i) the canonical basis of ℓ_1, and let $x_n = e_n - \big(e_{\phi_1(n)} + e_{\phi_2(n)}\big)/2$. Denote by F_n the span of $x_1, \ldots, x_n$ in ℓ_1^{2n+2}. By [19], each F_n is C-isomorphic to ℓ_1^n, where C is a uniform constant. Denote by $x_1^{(n)}, \ldots, x_n^{(n)}$ the copies of $x_1, \ldots, x_n$ inside of F_n. By [19], the vectors $\big(x_i^{(n)}\big)_{1\le i\le n, n\in\mathbb{N}}$ form a quasi-greedy basis inside of $\big(\sum_n F_n\big)_1 \sim \ell_1$. Now equip $\operatorname{span}\big[\big(x_i^{(n)}\big)_{1\le i\le n}\big]$ with the lattice structure of $\big(\sum_n \ell_1^{2n+2}\big)_1$, which contains $\big(\sum_n F_n\big)_1$ in the natural way.

Proposition 4.1. *In the above notation, the sequence $\big(x_i^{(n)}\big)_{1\le i\le n, n\in\mathbb{N}}$ is neither uniformly quasi-greedy nor bibasic.*

Proof. For any $C > 1$, find $m \in \mathbb{N}$ with $m + 1 > C/2$. For $n \ge 3 \cdot 2^m$, we construct $x \in \operatorname{span}\big[\big(x_i^{(n)}\big)_{1\le i\le n}\big]$ which witnesses the fact that the uniformly quasi-greedy and bibasis constants of the sequence $\big(x_i^{(n)}\big)_{1\le i\le n}$ in ℓ_1^{2n+2} are at least C. Since n is fixed, we shall use x_i instead of $x_i^{(n)}$.

For $\overline{j} = (j_1, \ldots, j_k) \in \{1,2\}^k$ define $\phi_{\overline{j}}(t) = \phi_{j_k}\big(\ldots\phi_{j_2}(\phi_{j_1}(t))\ldots\big)$; then $\phi_{\overline{j}}(t) = 2^k t + 2^{k-1} j_1 + \ldots + 2j_{k-1} + j_k$. For convenience, let $\phi_\emptyset(t) = t$. Note that, if $\overline{j} = (j_1, \ldots, j_k)$ and $\overline{i} = (i_1, \ldots, i_\ell)$, then $\phi_{\overline{j}}(1) = \phi_{\overline{i}}(1)$ holds iff $\ell = k$ and $j_1 = i_1, \ldots, j_k = i_k$. Indeed, for $k = \ell$ this follows from the uniqueness of the binary decomposition. On the other hand, if $\ell > k$ then

$$\phi_{\overline{j}}(1) \le 2^k + 2\big(2^{k-1} + \ldots + 1\big) < 2^k + \big(2^k + 2^{k-1} + \ldots + 1\big) \le \phi_{\overline{i}}(1).$$

The case of $\ell < k$ is handled in a similar way.

For $m \in \mathbb{N}$, let

$$y_m = \sum_{k=0}^{m-1} 2^{-k} \sum_{j_1,\ldots,j_k \in\{1,2\}} x_{\phi_{j_k,\ldots,j_1}(1)},$$

where, by convention, the term for $k = 0$ corresponds to $x_{\phi_\emptyset(1)} = x_1$. A direct computation shows that

$$y_m = e_1 - 2^{-m} \sum_{j_1,\ldots,j_m\in\{1,2\}} e_{\phi(j_m,\ldots,j_1)(1)}. \tag{4.1}$$

By the definition of ϕ,

$$\sum_{j_1,\ldots,j_m\in\{1,2\}} e_{\phi(j_m,\ldots,j_1)(1)} = \sum_{i\in I_m} e_i, \text{ where } I_m = \{2^{m+1} - 1, \ldots, 2^{m+1} + 2^m - 2\}, \tag{4.2}$$

hence $\|y_m\| = 2$.

Next, we show that, for any $N \in \mathbb{N}$, $\| \vee_{m=0}^{N-1} |y_m| \| = N + 1$. To this end, observe that, in (4.2), $|I_m| = 2^m$, and $I_m \cap I_k = \emptyset$ if $m \ne k$. Therefore, $\vee_{m=0}^{N-1} |y_m| = e_1 + \sum_{m=1}^N 2^{-m} \sum_{i\in I_m} e_i$, yielding the desired estimate.

Note that, if the Lindenstrauss basis is bibasic (uniformly quasi-greedy), then there exists $\mathbf{B} \in (0,\infty)$ so that $(x_i)_{i=1}^n$ is bibasic (resp. uniformly quasi-greedy) with constant $\mathbf{B}$, no matter how large n is. Fix N, and pick $n \ge \phi_{2,\ldots,2}(1)$ (2 is repeated N times). If $m \le N$, then y_m is both a greedy sum and a partial sum of y_N, hence $\| \vee_{m=0}^{N-1} |y_m| \| \leq \mathbf{B} \|y_N\|$. This gives $\mathbf{B} \geq (N+1)/2$ (here $\mathbf{B}$ can stand for either the bibasic or the uniformly quasi-greedy constant of $(x_i)_{i=1}^n$), leading to the desired contradiction. $\square$

4.1.2. *A copy of the canonical ℓ_p-basis that is not bibasic nor uniformly quasi-greedy.* Here, we shall denote by (e_i) the canonical ℓ_p-basis. We establish the following proposition, which partially resolves a question in [50, Remark 7.6].

Proposition 4.2. *For $1 < p < \infty$ the space ℓ_p contains a basis (u_i), equivalent to (e_i), which is neither bibasic nor uniformly quasi-greedy.*

Proof. Our reasoning is similar to [50, Section 7.1] and relies on the investigation of the main triangular truncation carried out in [41].

For each n, consider the $n \times n$ matrix $T^{(n)} = \left[T_{ij}^{(n)}\right]$ with $T_{ij}^{(n)} = 1/(i-j)$ if $i \neq j$ and $T_{ij}^{(n)} = 0$ if $i = j$. As noted in [41, (1.7)] (where a "mirror image" of this matrix is considered), $\|T^{(n)}\| \leq K(p)$ with $K(p)$ a uniform constant and $T^{(n)}$ viewed as an operator on ℓ_p^n. Let $S^{(n)} = \alpha T^{(n)}$ where $\alpha = 1/(2K(p))$. When the value of n is not in doubt, we shall write S and T instead of $S^{(n)}$ and $T^{(n)}$.

Identify ℓ_p with $\left(\sum_n (\ell_p^n \oplus_p \ell_p^n)\right)_p$. Denote by $(f_i)_{i=1}^n$ and $(g_i)_{i=1}^n$ the canonical bases of the first and the second copies of ℓ_p^n, respectively. Inside of $\ell_p^n \oplus_p \ell_p^n$ consider the basis consisting of elements v_i, w_i $(1 \leq i \leq n)$ with $v_i = f_i \oplus S^{(n)} g_i$ and $w_i = -S^{(n)} f_i \oplus g_i$. We claim that the basis (v_i, w_i) is equivalent to $(e_i)_{i=1}^{2n}$, with the constant of equivalence independent of n. Indeed, we can write $\sum_{i=1}^n (\alpha_i u_i + \beta_i v_i) = A \sum_i (\alpha_i f_i + \beta_i g_i)$, where

$$A = \begin{pmatrix} I & S \\ -S & I \end{pmatrix} = I + \begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix}.$$

Clearly, $\left\| \begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix} \right\| \leq \frac{1}{2}$. Hence, $\|A\| \leq \frac{3}{2}$. A Neumann series expansion gives us the identity

$$A^{-1} = I + \sum_{k=1}^{\infty} (-1)^k \begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix}^k,$$

hence $\|A^{-1}\| \leq 2$.

Concatenate the bases (v_i, w_i) into (u_i). As shown above, (u_i) is equivalent to the ℓ_p basis. Now we show that (v_i, w_i) is neither bibasic in $\ell_p^n \oplus_p \ell_p^n$ nor uniformly quasi-greedy there. To this end, consider $x = \sum_{i=1}^n v_i$. Clearly, $\|x\| \leq 3n^{1/p}/2$. We shall show that

$$\Big\| \bigvee_{k=1}^n \Big| \sum_{i=1}^k v_i \Big| \Big\| \succ n^{1/p} \log n.$$

To estimate the left-hand side, we focus on the second copy of ℓ_p^n. For $2 \leq j \leq n$, the j-th term of $\sum_{i=1}^j v_i$ (that is, the coefficient in front of g_j) equals $\alpha\left(\frac{1}{j-1} + \ldots + 1\right)$. Consequently,

$$\bigvee_{k=1}^n \Big| \sum_{i=1}^k v_i \Big| \geq \alpha \sum_{j=2}^n \Big(\frac{1}{j-1} + \ldots + 1 \Big) g_j.$$

The term in front of g_j is $\succ \log n$ for $j \geq n/2$, hence the norm of the left-hand side $\succ n^{1/p} \log n$. This finishes the proof. $\square$

In the specific case of $p = 2$, one can do somewhat better.

Proposition 4.3. *The space ℓ_2 contains a basis (u_i), isometrically equivalent to (e_i), which is neither bibasic nor uniformly quasi-greedy.*

This is a consequence of the following lemma.

Lemma 4.4. *There is a constant c such that for every n there is an orthonormal basis $h_1, \ldots, h_n$ of ℓ_2^n so that*

$$\Big\| \bigvee_{k=1}^{n} |h_1 + \ldots + h_k| \Big\| \geq c \log n \sqrt{n}.$$

Sketch of a proof of Proposition 4.3. We closely follow the proof of Proposition 4.2. The only difference is that now (in the same notation as before) we take $v_i = 12f_i/13 \oplus 5h_i/13$ and $w_i = -5h_i/13 \oplus 12g_i/13$, with $h_1, \ldots, h_n$ coming from Lemma 4.4. □

Proof of Lemma 4.4. Note that there exists a bijection between orthonormal bases $h_1, \ldots, h_n$ and $n \times n$ unitary matrices U, implemented by $U \mapsto (Ue_1, \ldots, Ue_n)$. By the Russo-Dye Theorem (for the real version, see [48]), the unit ball of $B(\ell_2^n)$ is the closed convex hull of unitaries, hence it suffices to show that the operator

$$\Phi : B(\ell_2^n) \to \ell_2^n(\ell_\infty^n) : T \mapsto \big(Te_1, Te_1 + Te_2, \ldots, Te_1 + \ldots + Te_n\big)$$

has norm $\succ \log n \sqrt{n}$. Here, we define

$$\big\|(\eta_1, \ldots, \eta_n)\big\|_{\ell_2^n(\ell_\infty^n)} = \big\| \vee_k |\eta_k| \big\|_{\ell_2^n},$$

or in other words, for $\eta_k = \sum_{\ell=1}^n \eta_{k\ell} e_\ell$,

$$\big\|(\eta_1, \ldots, \eta_n)\big\|_{\ell_2^n(\ell_\infty^n)} = \big\| \sum_\ell \big(\vee_k |\eta_{k\ell}| \big) e_\ell \big\|_{\ell_2^n} = \Big(\sum_\ell \big(\vee_k |\eta_{k\ell}| \big)^2 \Big)^{1/2}.$$

We shall actually establish the corresponding estimate for the norm of Φ^*, which takes $\ell_2^n(\ell_1^n)$ to $B(\ell_2^n)^*$ (the space of trace class $n \times n$ matrices). Here, we consider the "trace duality" on $B(\ell_2^n)$: for $n \times n$ matrices A and B, $\langle A, B \rangle = \mathrm{tr}(AB^*)$. One can observe that Φ^* maps $\overline{\xi} = (\xi_1, \ldots, \xi_n)$ to the operator

$$\Phi^*\big(\overline{\xi}\big) : e_k \mapsto \xi_k + \ldots + \xi_n \ (1 \leq k \leq n).$$

In particular, taking $\xi_k = e_k$ $(1 \leq k \leq n)$, we obtain

$$\big\|\overline{\xi}\big\|_{\ell_2^n(\ell_1^n)} = \big\| \sum_k |e_k| \big\|_{\ell_2^n} = \sqrt{n}.$$

On the other hand, $\Phi^*(\overline{\xi})$ is represented by the lower triangular matrix $\tau = \big[\tau_{k\ell}\big]_{k,\ell=1}^n$, with $\tau_{k\ell} = 1$ if $k \geq \ell$, $\tau_{k\ell} = 0$ otherwise. It remains to show that $\big\|\tau\big\|_{B(\ell_2^n)^*} \succ n \log n$.

To obtain this estimate, consider the $n \times n$ Toeplitz matrix $A = \big[A_{k\ell}\big]$, with $A_{k\ell} = 1/(k - \ell)$ if $k \neq \ell$ and $A_{k\ell} = 0$ if $k = \ell$. By [6] (or [41]), $\|A\|_{B(\ell_2^n)} \leq \pi$. By parallel duality,

$$\|A\|_{B(\ell_2)} \big\|\tau\big\|_{B(\ell_2^n)^*} \geq \langle A, \tau \rangle = \sum_{k,\ell} A_{k\ell} \tau_{k\ell} = \sum_{m=1}^n \sum_{j=1}^m \frac{1}{j} \sim n \log n,$$

which yields the desired estimate for $\big\|\tau\big\|_{B(\ell_2^n)^*}$. □

4.1.3. *ℓ_1 is the only basis that is absolute in every Banach lattice.* Here we give an alternative proof of one of the results of [50].

Proposition 4.5. *Suppose that (e_i) is a semi-normalized basis in a Banach space E, not equivalent to the ℓ_1 basis. Then there exists a Banach lattice X containing a basic sequence (x_i) equivalent to (e_i) so that (x_i) is not an absolute basic sequence.*

Remark 4.6. We do not know whether this proposition can be strengthened in one of the following ways:

- Can we replace "absolute" by a weaker condition such as "permutable bibasic?"
- Can (x_i) be a basis and not just a basic sequence? (here we are implicitly assuming that (x_i) is unconditional so that $E = [x_i]$ admits lattice structures).

Proof. Any absolute basis is unconditional, so if (e_i) is conditional an embedding of E into a suitable $C(K)$ will generate the desired (x_i). We henceforth assume that (e_i) is an unconditional basis of E. Due to the uniqueness of unconditional bases in ℓ_1 [44, Theorem 2.b.10], E is not isomorphic to ℓ_1. By renorming, we can assume that (e_i) is 1-unconditional. We shall view E as a Banach lattice with the order determined by (e_i).

Fix $p \in [1, \infty)$. By [67, Theorem 2.3], for any $N \in \mathbb{N}$ and $C > 0$ there exists a finite rank contraction $T : \mathrm{span}[e_i : i > N] \to \ell_p$ of regular norm greater than C. That is, $\|\|T\|\| > C$. Note that ℓ_p is positively contractively complemented in its second dual, and hence the regular norm is equal to the order bounded norm. Approximating, we can find a norm one $x = \sum_i \alpha_i e_i$ (finite sum, with $\alpha_i \geq 0$) so that

$$\Big\| \vee_{\varepsilon_i = \pm 1} \big| \sum_i \varepsilon_i \alpha_i T e_i \big| \Big\| > C.$$

Note that

$$\vee_{\varepsilon_i = \pm 1} \big| \sum_i \varepsilon_i \alpha_i T e_i \big| = \sum_i \big| \alpha_i T e_i \big|,$$

so we have, in fact,

$$\Big\| \sum_i \big| \alpha_i T e_i \big| \Big\| > C.$$

This allows us to find $1 = N_0 < N_1 < N_2 < \ldots$, contractions $T_j : \mathrm{span}[e_i : N_{j-1} \leq i < N_j] \to \ell_p$, and $\alpha_i \geq 0$ so that

$$\big\| \sum_{i=N_{j-1}}^{N_j - 1} \alpha_i e_i \big\| = 1, \quad \Big\| \sum_i \alpha_i |T_j e_i| \Big\| > j.$$

In the space $X = E \oplus_\infty c_0(\ell_p)$, let $x_i = e_i \oplus 0 \oplus \ldots \oplus 0 \oplus T_j e_i \oplus 0 \oplus \ldots$ (the term $T_j e_i$ is located in the j-th copy of ℓ_p). The sequence (x_i) is equivalent to (e_i) as each T_j is a contraction, but (x_i) is not absolute in X. ☐

4.2. **Blocking bases.** In this subsection, we show that the absolute and bibasis properties behave very well under blocking, whereas the greedy property does not.

4.2.1. *Blocking to gain the absolute property.* In Section 4.1 we have given several methods to construct basic sequences with poor lattice properties. We now present some techniques to produce good basic sequences (or more specifically FDDs, from which one can extract good basic sequences). The main insight is that most bases are built in disjoint blocks, so if we lump the blocks together, we can get better lattice behavior.

Proposition 4.7. *Let X be a Banach lattice and (C_n) an absolute FDD of $[C_n] \subseteq X$. Then every unconditional and shrinking FDD (B_n) with $[B_n] \subseteq [C_n]$ can be blocked to be absolute.*

Proof. Let (B_n) be an unconditional and shrinking FDD with $[B_n] \subseteq [C_n]$. Let $\varepsilon_i = \frac{1}{2^i}$ and apply [44, Proposition 1.g.4] to get a blocking (B_i') of (B_n) and a blocking (C_i') of (C_n) such that for every $x \in B_i'$ there is a $y \in C_{i-1}' \oplus C_i'$ with $\|x - y\| \leq \varepsilon_i \|x\|$.

Take normalized $x_i \in B_i' \setminus \{0\}$; it suffices to show that (x_i) is absolute. For this, we let $x \in [x_i]$, write $x = \sum_{i=1}^{\infty} a_i x_i$, and find $y_i \in C_{i-1}' \oplus C_i'$ with $\|a_i x_i - y_i\| \leq \varepsilon_i |a_i|$. By unconditionality, $x^1 = \sum_{i \text{ odd}} a_i x_i$ and $x^2 = \sum_{i \text{ even}} a_i x_i$ exist. Moreover, we have

$$\| \sum_{i \text{ odd}} |a_i x_i| \| \leq \| \sum_{i \text{ odd}} |a_i x_i - y_i| \| + \| \sum_{i \text{ odd}} |y_i| \| \leq \sum_{i \text{ odd}} \varepsilon_i |a_i| + \| \sum_{i \text{ odd}} |y_i| \| < \infty.$$

The reason the above is finite is because the sequence (x_i) is normalized, $a_i \to 0$, and since we are only summing over odd terms the y_i do not overlap. This means that $\| \sum_{i \text{ odd}} |y_i| \| \leq A \| \sum_{i \text{ odd}} y_i \|$ and $\| \sum_{i \text{ odd}} y_i \| < \infty$ since $\sum_{i \text{ odd}} (a_i x_i - y_i)$ exists. Similarly, $\| \sum_{i \text{ even}} |a_i x_i| \| < \infty$, so $\| \sum_{i=1}^{\infty} |a_i x_i| \| < \infty$ and hence (x_i) is absolute. □

Remark 4.8. By [32], every FDD of ℓ_p, $p > 1$, can be blocked to be unconditional. As the canonical basis of ℓ_p is absolute, Proposition 4.7 shows that every FDD of ℓ_p, $p > 1$, can be blocked to be an absolute FDD. By contrast, the "dual summing basis" in ℓ_1 cannot even be blocked as a bi-FDD.

Next, we present two examples to illustrate the sharpness of the assumptions in Proposition 4.7.

Example 4.9. Based on the reasoning from [31, Example 2.13], we show that $X = \ell_p \oplus \ell_q$ $(1 < q < p < \infty)$ has a shrinking FDD which cannot be blocked to be either unconditional or a bi-FDD. Denote by (δ_i) and (e_i) the canonical bases of ℓ_q and ℓ_p, respectively. Let $E_1 = [0 \oplus \delta_1]$ and for $n \geq 2$ let $E_n = [e_{n-1} \oplus \delta_{n-1}, 0 \oplus \delta_n]$. For a blocking $F_n = [E_i : k(n) < i \leq k(n+1)]$, take $f_1 = 0 \oplus e_{k(2)} \in F_1$, and, for $2 \leq n \leq m$, $f_n = e_{k(n)} \oplus (\delta_{k(n)} + \delta_{k(n+1)}) \in F_n$. It is easy to see that $\| \sum_{n=1}^{m} (-1)^n f_n \| \sim m^{1/p}$, while $\| \sum_{n=1}^{m} f_n \| \sim m^{1/q}$, and also, $\| \vee_{k=1}^{m} | \sum_{n=1}^{k} (-1)^n f_n | \| \sim m^{1/q}$.

Example 4.10. A *Haar system* is a family of functions $f_{i,n}$ $(n \geq 0, 1 \leq i \leq 2^n)$ so that there exist sets $A_{i.n} \subset (0,1)$ of measure 2^{-n} so that $A_{i,n} = A_{2i-1,n+1} \cup A_{2i,n+1}$, $A_{i,n} \cap A_{j,n} = \emptyset$ if $i \neq j$, $f_{i,n} = 1$ on $A_{2i-1,n+1}$, and $f_{i,n} = -1$ on $A_{2i,n+1}$. [33] gives an example of a weakly null normalized sequence $(g_k) \subset L_1(0,1)$ so that, for every subsequence g_k', and every sequence of positive numbers $\varepsilon_{i,n}$, there exists a Haar sequence $h_{i,n}$ and a block sequence $g_{i,n}''$ of g_k' so that $\|g_{i,n}'' - h_{i,n}\| < \varepsilon_{i,n}$. By [26, Theorem 4.6], "branches" of any Haar system fail to be bibasic. Hence, no subsequence of (g_k) is bibasic. In particular, although every norm convergent sequence has a uniformly convergent subsequence, it is not true that every basic sequence admits a bibasic subsequence. Actually, much stronger results can be shown if one allows the ambient Banach lattice to be non-classical – in [50, Theorem 7.5] it is shown that there are subspaces with no bibasic sequences at all.

4.2.2. *Blocking to lose the greedy property.* In contrast to the situation in Section 4.2.1, we now show that greediness is very unstable under blockings. For this purpose, it is instructive to look at an example. Recall that we are starting with a greedy basis (e_i), which we want to block into an FDD which fails the democracy condition. To begin, let us take (e_i) to be the standard basis of the Lorentz space $\ell_{p,q}$. One can find a sequence of mutually disjoint elements which is equivalent to ℓ_q. Thus, one can block the canonical basis for $\ell_{p,q}$ into an FDD (E_i) so that

- For i odd, the space E_i is 1-dimensional, and spanned by a canonical basis element.
- For i even, E_i contains a unit vector $u_{i/2}$ such that $u_1, u_2, \dots$ span a copy of ℓ_q.

The resulting FDD is not democratic. Indeed, the fundamental function evaluated from E_i's with i odd gives $\sim n^{\frac{1}{p}}$; on the other hand, if we look at $u_j \in E_{2j}$, we obtain the fundamental function $\sim n^{\frac{1}{q}}$.

In a similar fashion, one can use the fact that the space L_p with $p \in (1, \infty) \setminus \{2\}$ contains a Hilbert space to show that the Haar basis, although greedy, can be blocked into a non-greedy FDD. In general, we have the following.

Proposition 4.11. *For a subsymmetric basis (e_i) of a Banach space X, the following are equivalent.*

(i) (e_i) is equivalent to the canonical ℓ_p $(1 \le p < \infty)$ or c_0 basis;
(ii) Every (not necessarily consecutive) blocking of a permutation of (e_i) produces a greedy FDD.

Proof. The implication (i)⇒(ii) is immediate, so we focus on (ii)⇒(i).

We assume that every blocking of a permutation of (e_i) is a greedy FDD. By renorming, we can and do assume that (e_i) is normalized, and 1-unconditional. Thus, any FDD we produce from (e_i) will be 1-unconditional as well. We let $\phi(n) = \left\| \sum_{i=1}^n e_i \right\|$ and claim that there exists a constant C so that, whenever $(x_k)_{k=1}^n$ is a disjoint collection of norm one vectors, then

$$C^{-1}\phi(n) \le \left\| \sum_{k=1}^n x_k \right\| \le C\phi(n). \tag{4.3}$$

Suppose for the sake of contradiction that (4.3) fails. Then for any $M \in \mathbb{N}$ and $C > 1$ there exists $n \in \mathbb{N}$ and $x_1, \dots, x_n$ disjointly supported on $[M, \infty)$ for which (4.3) fails. Then, concatenating, we can find a sequence (N_k) so that $N_0 = 1$ and $N_k > 2N_{k-1}$ for any k, together with norm one vectors (x_{ik}) $(k \in \mathbb{N}, 1 \le i \le n_k)$, so that $(x_{ik})_{i=1}^{n_k}$ is disjointly supported on $[2N_{k-1}, N_k - 1]$ and

$$\left\| \sum_{i=1}^{n_k} x_{ik} \right\| \notin [k^{-1}\phi(n_k), k\phi(n_k)].$$

We then consider an FDD so that

(i) Some blocks contain the vectors x_{ik} described above;
(ii) If $N_k \le j < 2N_k$ for some k, then $\mathrm{span}[e_j]$ is a one-dimensional member of this FDD.

Recall that, as (e_i) is itself greedy, there exists a constant c so that

$$c^{-1}\phi(n) \le \left\| \sum_{i \in A} e_i \right\| \le c\phi(n)$$

whenever $A \subseteq \mathbb{N}$ has cardinality n. Therefore, the FDD constructed above fails the democracy condition, so cannot be greedy. This is the desired contradiction.

Reasoning as in the proof of [44, Theorem 2.a.9], we show that there exists $\gamma \in [0,1]$ so that $C^{-2}n^\gamma \leq \phi(n) \leq C^2 n^\gamma$ for every n, where C is as before. If $\gamma = 0$, then clearly (e_i) is equivalent to the c_0 basis. We show that, if $\gamma > 0$, then (e_i) is equivalent to the ℓ_p basis, for $p = \frac{1}{\gamma}$.

Consider a finitely supported norm one vector $a = \sum_{i \in S} \alpha_i e_i$. We have to show that $\sum_i |\alpha_i|^p \sim 1$. Due to unconditionality, we can and do assume that $\alpha_i > 0$ for all $i \in S$. Clearly, $\sup_i \alpha_i \leq 1$. For $j \in \mathbb{N}$, let $S_j = \{i \in S : 2^{-j/p} < \alpha_i \leq 2^{(1-j)/p}\}$. Let J be the largest index j for which S_j is non-empty. Let $y = \sum_j 2^{-j/p} \mathbb{1}_{S_j}$. Then $2^{-1/p} \leq \|y\| \leq 1$. It therefore suffices to show that $\sum_j 2^{-j}|S_j| \sim 1$.

Fix $M > \max S$, divisible by 2^J. For $0 \leq k \leq M-1$, let $y_k = \sum_j 2^{-j/p} \mathbb{1}_{S_j + kM}$, and

$$y' = \sum_{k=0}^{M-1} 2^{-j/p} \mathbb{1}_{S'_j}, \text{ where } S'_j = \cup_{k=0}^{M-1}(S_j + (k-1)M).$$

Note that $\|y_k\| \sim 1$ by subsymmetry, and hence $\|y'\| \sim \phi(M) \sim M^{1/p}$.

For every $j \leq J$, $2^{-j}M|S_j| = L_j$ is an integer. S'_j can be written as a union of L_j disjoint sets $F_{j1}, \ldots, F_{jL_j}$, of cardinality 2^j. Write

$$y' = \sum_j \sum_{\ell=1}^{L_j} 2^{-j/p} \mathbb{1}_{F_{j\ell}}.$$

Each of the vectors $2^{-j/p}\mathbb{1}_{F_{j\ell}}$ has norm ~ 1, hence $M \sim \|y'\|^p \sim \sum_j L_j \sim \sum_j 2^{-j}M|S_j|$, which yields the desired result; namely, $\sum_j 2^{-j}|S_j| \sim 1$. □

Remark 4.12. We do not know to what extent the subsymmetry assumption in Proposition 4.11 is redundant. It is not required to establish that $\phi(n) \sim n^{\frac{1}{p}}$. If (e_i) is not subsymmetric, our proof (with minor modifications) shows that any spreading basis (e'_i) arising from a subsequence of (e_i) is equivalent to the ℓ_p basis. In fact, in the terminology of [16], X must be strongly asymptotic ℓ_p. In general, being strongly asymptotic c_0 does not imply being greedy, as the example of the Tsirelson space shows (see [18, Remark 5.8]).

Example 4.13. It is shown in [17] that certain spaces of the form $(\oplus_{n=1}^\infty \ell_p^n)_{\ell_q}$ – for appropriate p, q – do not have greedy bases. However, one can block the canonical basis in the evident way to get greedy FDD for these spaces. There are also more exotic spaces which have greedy FDD but no greedy bases; for example, [60] produces a space with an ℓ_2-FDD but no basis. On the other hand, there are several spaces for which it is unclear whether a greedy FDD can be produced; for example, $\ell_p \oplus \ell_q$, $\ell_q(\ell_p)$ (for appropriate p, q) and the Schatten classes. For $\ell_p \oplus \ell_q$ it was shown in [53, 68] that there is a unique unconditional basis up to permutation – which is, of course, not greedy when $p \neq q$. See [55] for a proof that there are no greedy bases for matrix spaces with mixed ℓ_p and ℓ_q norms. For the same reason as with $(\oplus_{n=1}^\infty \ell_p^n)_{\ell_q}$, it is clear that $\ell_q(\ell_p)$ has a greedy Schauder decomposition.

Remark 4.14. Results similar to Proposition 4.11 (providing criteria for "tensor-stable" bases to be equivalent to the canonical basis of either c_0 or ℓ_p) can be found in [13].

4.3. **Absolute sequences characterize AM-spaces.** In AM-spaces (that is, closed sublattices of $C(K)$-spaces), every basic sequence is bibasic and every unconditional sequence is absolute. In [65] it was asked whether the converse holds: If every basic (resp. unconditional) sequence in X is bibasic (resp. absolute), must X be lattice isomorphic to an AM-space? Here we make some progress on these conjectures.

Recall that a Banach lattice X has an *upper p-estimate* if there exists a constant C so that the inequality $\|\sum_i x_i\| \leq C\big(\sum_i \|x_i\|^p\big)^{1/p}$ holds for any disjoint $x_1, \ldots, x_n$. Clearly, an upper p-estimate implies upper p'-estimates for $p' < p$. Denote by $s(X)$ the supremum of all p's for which X has an upper p-estimate. By reversing inequalities, one obtains the definition of a lower p-estimate; denote by $\sigma(X)$ the infimum of all p's for which X has a lower p-estimate. We refer the reader to [23, 24] for a comprehensive study of upper p-estimates. Note that, if X is finite dimensional, then $s(X) = \infty$ and $\sigma(X) = 1$. Otherwise, $1 \leq s(X) \leq \sigma(X) \leq \infty$. Furthermore, if $\sigma(X) < \infty$, then X is order continuous.

Proposition 4.15. *Suppose X is an infinite dimensional Banach lattice, and $\{s(X), \sigma(X)\} \cap (1, \infty) \neq \emptyset$. Then X contains an unconditional sequence which is not bibasic.*

Note that the hypothesis of the proposition fails if one of the three holds: (i) $s(X) = \sigma(X) = \infty$, (ii) $s(X) = \sigma(X) = 1$, or (iii) $s(X) = 1$, $\sigma(X) = \infty$. Note that $s(x) = \sigma(X) = \infty$ does not imply that X is an AM-space. Similarly, $s(x) = \sigma(X) = 1$ does not imply that X is an AL-space.

For the proof, we need the following proposition, which strengthens Krivine's Theorem for lattices.

Proposition 4.16. *For a Banach lattice X, $c > 1$, and $n \in \mathbb{N}$, there exist disjoint normalized $x_i \in X_+$ ($1 \leq i \leq n$) so that $(x_1, \ldots, x_n)$ is c-equivalent to the canonical basis of $\ell^n_{s(X)}$ (of $\ell^n_{\sigma(X)}$), and an infinite dimensional Banach lattice $Y \subset X$, disjoint from $x_1, \ldots, x_n$, so that $s(Y) = s(X)$ (resp. $\sigma(Y) = \sigma(X)$).*

Here, we say that the bases $(e_i)_{i=1}^n$ and $(f_i)_{i=1}^n$ are *c-equivalent* if, for any sequence of scalars $(\alpha_i)_{i=1}^n$,

$$c^{-1}\Big\|\sum_i \alpha_i e_i\Big\| \leq \Big\|\sum_i \alpha_i f_i\Big\| \leq c\Big\|\sum_i \alpha_i e_i\Big\|.$$

We postpone the proof of this proposition and instead show how it implies Proposition 4.15.

Proof of Proposition 4.15. We use Proposition 4.16 to find disjoint unit vectors $\big(e_{in}\big)_{n\in\mathbb{N}, 1\leq i\leq 2n}$ so that, for every n, $\big(e_{in}\big)_{1\leq i\leq 2n}$ is 2-equivalent to the ℓ_r^{2n} basis, where $r \in \{s(X), \sigma(X)\} \cap (1, \infty)$. By the construction from Proposition 4.2, span$[e_{in} : 1 \leq i \leq 2n]$ has a basis $\big(u_{in}\big)_{i=1}^{2n}$ which is c-equivalent to the ℓ_r basis on the Banach space level, but has bibasis constant $\sim \log n$. Concatenating the u_{in}'s, one obtains an unconditional sequence in X which is not bibasic. □

In the proof of Proposition 4.16, we rely on some ideas from [1] and [2]. We also need two lemmas, the first of which is fairly straightforward.

Lemma 4.17. *Any Banach lattice X contains a separable Banach lattice X' so that $s(X') = s(X)$ and $\sigma(X') = \sigma(X)$.*

Henceforth, we assume that X is separable. We find a weak unit $u \in X_+$, and let X_u be the corresponding principal ideal, with the norm $\|x\|_\infty = \inf\{\lambda > 0 : |x| \leq \lambda u\}$. This ideal can be identified with $C(K)$, for some compact Hausdorff space K, with u corresponding to 1. Adjusting

u, we can assume that $\|\cdot\| \leq \|\cdot\|_\infty$. As X_u is $\|\cdot\|$-dense in X, we have $s(X) = s(X_u, \|\cdot\|)$ and $\sigma(X) = \sigma(X_u, \|\cdot\|)$.

Suppose now that Ω is an open subset of K. We shall denote by X^Ω the set of all $x \in X_u$ which vanish outside of Ω.

Lemma 4.18. *There exists $t \in K$ so that, for every neighborhood $\Omega \ni t$, $s(X^\Omega) = s(X)$ ($\sigma(X^\Omega) = \sigma(X)$).*

The spaces listed in this lemma are equipped with the norm $\|\cdot\|$.

Proof. We deal with $s(\cdot)$, as $\sigma(\cdot)$ is handled similarly. Suppose, for the sake of contradiction, that any $t \in K$ has a neighborhood Ω_t so that $s(X^{\Omega_t}) > s(X)$. By the compactness of K, we can find $t_1, \ldots, t_n \in K$ so that $K = \cup_{i=1}^n \Omega_{t_i}$. We shall achieve a contradiction by showing that $s(X) = s := \min_i s(X^{\Omega_{t_i}})$.

We shall show that, for any $r < s$, there exists a constant C so that the inequality

$$\Big\| \sum_{j=1}^m y_j \Big\| \leq C \Big(\sum_j \|y_j\|^r \Big)^{1/r} \tag{4.4}$$

holds for any disjoint $y_1, \ldots, y_m \in C(K)_+$.

Indeed, for any $i \in \{1, \ldots, n\}$ there exists C_i so that, for any disjoint $z_1, \ldots, z_k \in C(K)_+$, vanishing outside of Ω_{t_i}, we have

$$\Big\| \sum_{j=1}^k z_j \Big\| \leq C_i \Big(\sum_j \|z_j\|^r \Big)^{1/r}.$$

Find $x_1, \ldots, x_n \in C(K)_+$ so that $\sum_i x_i = 1$, and x_i vanishes outside of Ω_{t_i}. Let $y_{ij} = x_i \wedge y_j$. Then

$$\begin{aligned} \Big\| \sum_{j=1}^m y_j \Big\| &\leq \sum_{i=1}^n \Big\| \sum_{j=1}^m y_{ij} \Big\| \leq \sum_i C_i \Big(\sum_j \|y_{ij}\|^r \Big)^{1/r} \\ &\leq \max_i C_i n^{1-1/r} \Big(\sum_j \sum_i \|y_{ij}\|^r \Big)^{1/r} \leq \max_i C_i n \Big(\sum_j \|y_j\|^r \Big)^{1/r}, \end{aligned}$$

the last inequality being due to the fact that

$$\|y_j\| \geq \max_i \|y_{ij}\| \geq n^{-1/r} \Big(\sum_i \|y_{ij}\|^r \Big)^{1/r}.$$

This establishes (4.4) with $C = n \cdot \max_i C_i$.

The case of $\sigma(\cdot)$ is handled similarly, except that one has to use the inequality $\big\| \sum_{j=1}^m y_j \big\| \geq \vee_{i=1}^n \big\| \sum_{j=1}^m y_{ij} \big\|$ instead of $\big\| \sum_{j=1}^m y_j \big\| \leq \sum_{i=1}^n \big\| \sum_{j=1}^m y_{ij} \big\|$. ☐

Proof of Proposition 4.16. Fix $\varepsilon > 0$ and $c' \in (1, c)$ so that

$$c > c' + n\varepsilon \text{ and } \frac{1}{c} < \frac{1}{c'} - n\varepsilon.$$

By Krivine's Theorem [57], X contains disjoint normalized vectors $y_1, \ldots, y_{n+1}$, c'-equivalent to the ℓ_r-basis, where r is either $s(X)$ or $\sigma(X)$, depending on the case. Perturbing these vectors slightly, we can assume that they belong to X_u (in the notation of Lemma 4.18); the latter space is identified with $C(K)$, again as in Lemma 4.18.

Let $t_0 \in K$ be the special point whose existence is guaranteed by Lemma 4.18. At most one of our disjoint vectors $y_1, \ldots, y_{n+1}$ does not vanish at t_0; up to relabeling, we can assume that $y_1, \ldots, y_n$ do vanish there. Find vectors $y_i' \in C(K)_+$ $(1 \leq i \leq n)$ so that, for every i, y_i' vanishes on some neighborhood of t_0, $y_i' \leq y_i$, and $\|y_i - y_i'\|_\infty < \varepsilon$. Then $\|y_i - y_i'\| \leq \|y_i - y_i'\|_\infty < \varepsilon$.

We claim that the vectors $x_i = y_i'/\|y_i'\|$ are disjoint, and c-equivalent to the ℓ_r basis. Indeed, for any $(\alpha_i)_{i=1}^n$,

$$\Big\| \sum_i \alpha_i x_i \Big\| \geq \Big\| \sum_i \alpha_i y_i' \Big\| \geq \Big\| \sum_i \alpha_i y_i \Big\| - \sum_i |\alpha_i| \|y_i - y_i'\| \\ \geq \frac{1}{c'} \Big(\sum_i |\alpha_i|^r \Big)^{1/r} - \varepsilon \sum_i |\alpha_i| \geq \Big(\frac{1}{c'} - n\varepsilon \Big) \Big(\sum_i |\alpha_i|^r \Big)^{1/r}.$$

On the other hand, for any i,

$$\big\| x_i - y_i' \big\| = \|y_i'\| \Big(\frac{1}{\|y_i\|} - 1 \Big) = 1 - \|y_i'\| < \varepsilon,$$

hence

$$\Big\| \sum_i \alpha_i x_i \Big\| \leq \Big\| \sum_i \alpha_i y_i \Big\| + \varepsilon \sum_i |\alpha_i| \leq \big(c' + n\varepsilon \big) \Big(\sum_i |\alpha_i|^r \Big)^{1/r}.$$

Moreover, there exists an open neighborhood $\Omega \ni t_0$ disjoint from the x_i's; then X^Ω has the required upper or lower estimate. □

In a manner similar to Proposition 4.15 (but using Proposition 4.1 instead of Proposition 4.2), we establish the following.

Proposition 4.19. *Suppose an infinite dimensional Banach lattice X satisfies $s(X) = 1$. Then X has a (conditional) basic sequence which is not bibasic.*

Taken together, Proposition 4.15 and Proposition 4.19 immediately imply the following.

Corollary 4.20. *Suppose an infinite dimensional Banach lattice X has $s(X) < \infty$. Then X contains a basic sequence which is not bibasic.*

By Corollary 4.20, if every basic sequence in X is bibasic then $s(X) = \infty$. However, as mentioned above, $s(X) = \infty$ does not imply that X is lattice isomorphic to an AM-space. We now show that if we instead assume that every unconditional sequence in X is absolute, we can reach this stronger conclusion.

Theorem 4.21. *If every unconditional basic sequence in a Banach lattice X is absolute, then X is lattice isomorphic to an AM-space.*

For future use, we recall [47, Theorem 2.1.12]: X is lattice isomorphic to an AM-space iff there exists $C \geq 1$ so that the inequality $\| \vee_j x_j \| \leq C \vee_j \|x_j\|$ holds for any $x_1, \ldots, x_n \in X_+$ (actually it suffices to verify the preceding inequality for disjoint n-tuples only). Consequently, X is lattice isomorphic to an AM-space iff the same is true for any separable sublattice of X.

In the proof, we re-use the notation and facts introduced earlier in this section. We begin by establishing a version of Proposition 4.16.

Lemma 4.22. *Suppose a separable Banach lattice X is not lattice isomorphic to an AM-space and u is a weak unit in X. There exists $t \in K$ so that, for every neighborhood of $\Omega \ni t$, the $\|\cdot\|$-completion of X^Ω is not lattice isomorphic to an AM-space.*

Sketch of a proof. Suppose, for the sake of contradiction, that every $t \in K$ possesses an open neighborhood $\Omega_t \ni t$ so that the completion of X^{Ω_t} is lattice isomorphic to an AM-space. Use compactness to find $t_1, \dots, t_n \in K$ so that $\cup_{i=1}^n \Omega_{t_i} = K$. For every i there exists a constant C_i so that, whenever $z_1, \dots, z_m \in X^{\Omega_{t_i}}$, we have $\| \vee_j |z_j| \| \le C_i \vee_j \|z_j\|$.

To achieve a contradiction, we show that there exists a constant C so that, for any $y_1, \dots, y_m \in C(K)_+$ (we identify $C(K)$ with X_u), $\| \vee_j y_j \| \le C \vee_j \|y_j\|$. As in the proof of Proposition 4.16, let (x_i) be a partition of unity subordinate to (Ω_{t_i}), and let $y_{ij} = x_i \wedge y_j$. Then $y_j \le \sum_i y_{ij}$, hence

$$\| \vee_j y_j \| \le \sum_i \| \vee_j y_{ij} \| \le \sum_i C_i \vee_j \|y_{ij}\| \le C \vee_j \|y_j\|,$$

where $C = \sum_{i=1}^n C_i$. □

Lemma 4.23. *Suppose a separable Banach lattice X is not lattice isomorphic to an AM-space, while $s(X) = \infty$. Then there exist disjoint vectors $e_{kn}, f_{kn} \in X_+$ ($n \in \mathbb{N}$, $1 \le k \le M_n$) so that, for every n,*

(i) $(e_{kn})_{k=1}^{M_n}$ is 2-equivalent to the $\ell_\infty^{M_n}$ basis.
(ii) $\| \sum_k f_{kn} \| > n$, while $\| \sum_k \alpha_k f_{kn} \| \le (\sum_k \alpha_k^2)^{1/2}$ for any (α_k).

For the proof of this lemma, we shall use a characterization of AM-spaces from [12, Lemma 4]. The completion of a normed lattice X is not lattice isomorphic to an AM-space iff for any K we can find disjoint $x_1, \dots, x_n \in X_+$ so that $\| \sum_k x_k \| > K$, while $\sum_k |x^*(x_k)|^2 \le 1$ for any $x^* \in X^*$ with $\|x^*\| \le 1$. Equivalently, the operator $T : X^* \to \ell_2^n : x^* \mapsto (x^*(x_k))_k$ is contractive. By duality, this is equivalent to $T^* : \ell_2^n \to X \subseteq X^{**} : e_i \mapsto x_i$ being contractive. In other words, we require that $\| \sum_k \alpha_k x_k \| \le 1$ whenever $\sum_k \alpha_k^2 \le 1$.

Proof. Since the vectors in question can be produced recursively, it suffices to prove the following: Suppose X is separable and is not lattice isomorphic to an AM-space. Then, for any n, there exists $M \in \mathbb{N}$ and mutually disjoint positive norm one vectors $e_1, \dots, e_M, f_1, \dots, f_M$, so that:

- $(e_k)_{k=1}^M$ is 2-equivalent to the ℓ_∞^M basis.
- $\| \sum_{k=1}^M f_k \| > n$, while $\| \sum_k \alpha_k f_k \| \le (\sum_k \alpha_k^2)^{1/2}$ for any (α_k).
- There exists a closed sublattice $X' \subset X$, disjoint from $e_1, \dots, e_M, f_1, \dots, f_M$ and not lattice isomorphic to an AM-space.

By Lemma 4.22, there exists $t \in K$ so that, for any open $\Omega \ni t$, the completion of X^Ω is not lattice isomorphic to an AM-space. Find norm one positive disjoint $f_1, \dots, f_{M+1} \in X$ so that $\| \sum_{k=1}^{M+1} f_k \| > n + 1$, while $\| \sum_k \alpha_k f_k \| \le (\sum_k \alpha_k^2)^{1/2}$ for any (α_k). By removing one of the vectors (say f_{M+1}) and perturbing the rest, we may assume that $f_1, \dots, f_M$ are supported outside of some open $\Omega \ni t$, where t is a special point whose existence is guaranteed by Lemma 4.22. Let X_0 be the closure of X^Ω. Repeat the same reasoning to construct suitable $e_1, \dots, e_M \in X_0$, disjoint from suitable X'. □

Proof of Theorem 4.21. As noted before, we can assume that X is separable. If $p = s(X) < \infty$, use Krivine's Theorem to find disjoint lattice copies of $\ell_p^{2^n}$ in X. Each of these contains "independent discrete Rademachers" r_{kn} ($1 \le k \le n$). We know that $\| \sum_k \alpha_k r_{kn} \| \sim \left(\sum_k \alpha_k^2 \right)^{1/2}$ while

$\|\sum_k |\alpha_k r_{kn}|\| = \sum_k |\alpha_k|$, hence the sequence obtained by concatenating the r_{kn}'s is unconditional but not absolute. Actually, this sequence is also bibasic, due to [65].

Now suppose that $s(X) = \infty$ but X is not an AM-space. Use Lemma 4.23 to find the sequences (e_{kn}) and (f_{kn}) in X_+ ($n \in \mathbb{N}$, $1 \leq k \leq 2^n$) so that the vectors involved are disjoint, (e_{kn}) is 2-equivalent to the $\ell_\infty^{2^n}$ basis, $\|e_{kn}\| = 1$, and the vectors (f_{kn}) are such that $\|\sum_k f_{kn}\| \nearrow \infty$, while $\|\sum_k \alpha_k f_{kn}\| \leq 1$ whenever $\sum_k \alpha_k^2 \leq 1$. Let $H_n = (h_{ijn})_{i,j=1}^{2^n}$ be the Hadamard matrix of size $2^n \times 2^n$ (see [65, Section 8]). The entries of this matrix are equal to ± 1, and the rows $\hat{h}_{kn} = (h_{kjn})_j$ are mutually orthogonal.

Taking inspiration from [3] we let, for $1 \leq k \leq 2^n$,

$$u_{kn} = e_{kn} + 2^{-n} \sum_{j=1}^{2^n} h_{kjn} f_{jn}.$$

Then (u_{kn}) is a (double-indexed) unconditional basic sequence. Indeed, by disjointness, it suffices to establish the unconditionality of (u_{kn}) for a fixed n. In fact, we shall show that (u_{kn}) is 3-equivalent to the $\ell_\infty^{2^n}$ basis. For any (α_k) in c_{00} we have

$$\begin{aligned} \vee_k |\alpha_k| &\leq \Big\| \sum_k \alpha_k e_{kn} \Big\| \leq \Big\| \sum_k \alpha_k u_{kn} \Big\| \\ &\leq \Big\| \sum_k \alpha_k e_{kn} \Big\| + 2^{-n} \Big\| \sum_k \alpha_k \sum_j h_{kjn} f_{kn} \Big\|. \end{aligned}$$

Moreover, $\big\| \sum_k \alpha_k e_{kn} \big\| \leq 2 \vee_k |\alpha_k|$. Therefore, it suffices to show that

$$\Big\| \sum_k \alpha_k \sum_j h_{kjn} f_{jn} \Big\| \leq 2^n \text{ whenever } \vee_k |\alpha_k| \leq 1.$$

By the properties of the vectors f_{jn},

$$\Big\| \sum_k \alpha_k \sum_j h_{kjn} f_{jn} \Big\|^2 = \Big\| \sum_j \Big(\sum_k \alpha_k h_{kjn} \Big) f_{jn} \Big\|^2 \leq \sum_j \Big| \sum_k \alpha_k h_{kjn} \Big|^2.$$

However,

$$\sum_j \Big| \sum_k \alpha_k h_{kjn} \Big|^2 = \Big\| \sum_k \alpha_k \hat{h}_{kn} \Big\|^2 = \sum_k |\alpha_k|^2 \|\hat{h}_{kn}\|^2 = 2^{2n},$$

which is the desired inequality.

Finally, we note that the sequence (u_{kn}) is not absolute. Indeed, for each n, $\|\sum_k \pm u_{kn}\| \leq 3$, yet $\|\sum_k |u_{kn}|\| \geq \|\sum_k f_{kn}\| \nearrow \infty$. □

4.4. **Complemented absolute sequences.** In this subsection, we prove some additional results that require conditions on the ambient space X, or how $[x_k]$ sits inside of X.

Recall that a sequence (x_k) is disjoint if and only if $\sum_{k=1}^n |a_k x_k| = \bigvee_{k=1}^n |a_k x_k|$ for all scalars $a_1, \ldots, a_n$. Our next result shows that *complemented* absolute sequences behave very much like disjoint sequences.

Proposition 4.24. *Let (x_k) be an absolute sequence in a Banach lattice X and suppose that $[x_k]$ is complemented in X. Then*

$$\Big\|\sum_{k=1}^n |a_k x_k|\Big\| \sim \Big\|\bigvee_{k=1}^n |a_k x_k|\Big\|.$$

Proof. By the remark after [45, Theorem 1.d.6] we have

$$\Big\|\sum_{k=1}^n a_k x_k\Big\| \sim \Big\|\Big(\sum_{k=1}^n |a_k x_k|^2\Big)^{\frac{1}{2}}\Big\|.$$

Take $\theta=\frac{1}{2}$, then $\frac{1}{2}=\frac{\theta}{1}+\frac{1-\theta}{\infty}$; applying [45, Proposition 1.d.2 (i) and (ii)] we get that

$$\Big\|\sum_{k=1}^n a_k x_k\Big\| \leq C\Big\|\Big(\sum_{k=1}^n |a_k x_k|^2\Big)^{\frac{1}{2}}\Big\| \leq C\Big\|\sum_{k=1}^n |a_k x_k|\Big\|^{\theta}\Big\|\bigvee_{k=1}^n |a_k x_k|\Big\|^{1-\theta},$$

where C is the constant of equivalence.

Due to the absoluteness of (x_k), there exists a constant M^* so that, for any (a_k), $\|\sum_{k=1}^n |a_k x_k|\| \leq M^*\|\sum_{k=1}^n a_k x_k\|$. Thus,

$$\Big\|\sum_{k=1}^n |a_k x_k|\Big\| \leq CM^*\Big\|\sum_{k=1}^n |a_k x_k|\Big\|^{1/2}\Big\|\bigvee_{k=1}^n |a_k x_k|\Big\|^{1/2},$$

which leads to

$$\Big\|\sum_{k=1}^n |a_k x_k|\Big\| \leq (CM^*)^2\Big\|\bigvee_{k=1}^n |a_k x_k|\Big\|.$$

Consequently,

$$\Big\|\sum_{k=1}^n a_k x_k\Big\| \leq \Big\|\sum_{k=1}^n |a_k x_k|\Big\| \leq (CM^*)^2\Big\|\bigvee_{k=1}^n |a_k x_k|\Big\|,$$

and on the other hand,

$$\Big\|\bigvee_{k=1}^n |a_k x_k|\Big\| \leq \Big\|\sum_{k=1}^n |a_k x_k|\Big\| \leq M^*\Big\|\sum_{k=1}^n a_k x_k\Big\|.$$

This completes the proof. □

Proposition 4.24 allows us to give a new Banach lattice proof of the well-known characterization of complemented unconditional sequences in $C[0,1]$.

Corollary 4.25. *The only complemented semi-normalized unconditional basic sequences in AM-spaces are those equivalent to the unit vector basis of c_0.*

Proof. Suppose that (x_k) is such a sequence. In AM-spaces, unconditional is the same as absolute, so (x_k) is absolute. Now, using the AM-property and Proposition 4.24 we see that

$$\begin{aligned}\max_k |a_k| &\lesssim \|\sum_{k=1}^n a_k x_k\| \lesssim \|\sum_{k=1}^n |a_k x_k|\| \sim \|\bigvee_{k=1}^n |a_k x_k|\| \\ &= \bigvee_{k=1}^n \|a_k x_k\| \sim \max_k |a_k|.\end{aligned}$$

□

Remark 4.26. The proof of Corollary 4.25 shows us that the complementability assumption is critical in Proposition 4.24. Since $C[0,1]$ is universal, it contains copies of every normalized unconditional basis, and the conclusion of Proposition 4.24 must fail for all of them except c_0.

Remark 4.27. Of course, Corollary 4.25 is well-known; it is actually known ([15, p. 74]) that the only complemented semi-normalized unconditional basic sequences in $\mathcal{L}_\infty$-spaces are those equivalent to the unit vector basis of c_0.

4.5. **Bibasic sequences in non-atomic Banach lattices.** We now consider the case when the ambient lattice is L_p. It was shown in [65] that L_1 does not reasonably embed into the span of a bibasic sequence, so it would be interesting to know if L_1 admits a uniformly quasi-greedy basis. Although we do not know the answer to this question, we will prove an L_p-version of it. Specifically, the next proposition proves that L_p cannot admit a permutable bibasis, which is in contrast to the fact that uniformly quasi-greedy bases are "almost" permutable (in the same sense that quasi-greedy bases are "almost" unconditional) and that L_p *does* admit uniformly quasi-greedy bases when $p > 1$.

In the next proposition, we use the concept of unbounded convergence. Given a convergence $\xrightarrow{\tau}$ on a vector lattice X, a net (x_α) is said to *unbounded τ-converge* to $x \in X$ if $|x_\alpha - x| \wedge u \xrightarrow{\tau} 0$ for all $u \in X_+$, see [64]. When τ is the convergence of a locally solid topology, the convergence $\xrightarrow{u\tau}$ on X defines the weakest locally solid topology on X agreeing with τ on the order intervals. On the other hand, unbounded order convergence acts as the natural generalization of almost everywhere convergence to vector lattices. For a comprehensive study of such convergences, see [14, 36, 62]. Although we do not pursue it here, we note that the unbounded convergences could provide a systematic solution to [4, Problem 12.3] as they encompass convergence in measure, convergence almost everywhere, as well as various convergences that are weaker than the norm.

Proposition 4.28. *Suppose that $1 \le p < \infty$ and (E_n) is a sequence of subspaces of a Banach lattice X which forms a permutable bi-FDD of $[E_n]$. Then there is no isomorphic embedding $T : L_p \to [E_n]$ with the property that $T^{-1} : T(L_p) \subseteq X \to L_p$ maps uniformly null sequences in $T(L_p)$ to uo-null sequences in L_p.*

Proof. By [37, Corollary 9 and Remark 10, p. 102] no Haar type system in L_p $(1 \le p < \infty)$ is a permutable *uo*-bibasic sequence in L_p. Now proceed as in [26, Theorem 5.1], using the stability results proven in [65]. □

We next present a result of a similar spirit for absolute FDD's. Recall that the sequentially u-to-u-continuous isomorphisms (see [65]) are the natural morphisms which preserve bibasic and uniformly quasi-greedy basic sequences. In [50, Proposition 7.10] it is shown that a linear map is sequentially u-to-u-continuous if and only if it is multibounded, in the sense that there exists $M \ge 1$ such that for any m and $x_1, \ldots, x_m$ we have $\| \bigvee_{k=1}^m |Tx_k|\| \leq M \| \bigvee_{k=1}^m |x_k|\|$. Such maps are sometimes called (∞, ∞)-regular, and have been studied by many authors. We show that such maps take absolute sequences to absolute sequences. More precisely, we have the following proposition.

Proposition 4.29. *Suppose X, Y are Banach lattices, E is a subspace of X, and $T : E \to Y$ is multibounded, with a bounded inverse. If a sequence $(x_k) \subset E$ is absolute, then the same is true for (Tx_k).*

Proof. The absoluteness of (x_k) means the existence of a constant C_0 with the property that, for every $(a_i)_{i=1}^n$, $\| \sum_i |a_i x_i| \| \leq C_0 \| \sum_i a_i x_i \|$. T being multibounded gives us the existence of a constant C_1 so that the inequality $\big\| \vee_{k=1}^m |Te_k| \big\| \leq C_1 \big\| \vee_{k=1}^m |e_k| \big\|$ holds. If, in addition, for any k there exists an ℓ so that $e_\ell = -e_k$ (the sequence (e_k) is "symmetric" – it contains the opposite of any of its elements), then we have $\big\| \vee_{k=1}^m Te_k \big\| \leq C_1 \big\| \vee_{k=1}^m e_k \big\|$. For $a_1, \ldots, a_n \in \mathbb{R}$, $\sum_i |a_i x_i| = \bigvee_{\varepsilon_i = \pm 1} \sum_i \varepsilon_i a_i x_i$, and the family $\big(\sum_i \varepsilon_i a_i x_i \big)_{\varepsilon_i = \pm 1}$ is symmetric in the above sense, hence

$$\Big\| \sum_i |a_i T x_i| \Big\| = \Big\| \bigvee_{\varepsilon_i = \pm 1} T\big(\sum_i \varepsilon_i a_i x_i \big) \Big\| \leq C_1 \Big\| \bigvee_{\varepsilon_i = \pm 1} \big(\sum_i \varepsilon_i a_i x_i \big) \Big\| = C_1 \Big\| \sum_i |a_i x_i| \Big\|.$$

However, by the boundedness of T^{-1}, there exists $C_2 > 0$ so that, for any (a_i), $\| \sum_i a_i x_i \| \leq C_2 \| \sum_i a_i T x_i \|$. Then

$$\Big\| \sum_i |a_i T x_i| \Big\| \leq C_1 \Big\| \sum_i |a_i x_i| \Big\| \leq C_1 C_0 \Big\| \sum_i a_i x_i \Big\| \leq C_2 C_1 C_0 \Big\| \sum_i a_i T x_i \Big\|,$$

which shows the absoluteness of (Tx_i). □

Proposition 4.30. *If a σ-order complete Banach lattice embeds with multibounded inverse into the span of an absolute FDD then it is purely atomic.*

Proof. Let X be a σ-order complete Banach lattice, E a Banach lattice, (x_k) an absolute basic sequence in E (which can be weakened to FDD, but we use basic for ease of reference), and $T : X \to [x_k] \subseteq E$ an isomorphic embedding with multibounded inverse. We begin with a few reductions.

Clearly, X must be separable; let us assume that it does not have atoms. Since every separable σ-order complete nonatomic Banach lattice X can be represented as a Köthe function space on $[0,1]$ with $L_\infty \subseteq X \subseteq L_1$, we can assume that X is Köthe.

The point of assuming that T is an embedding with multibounded inverse is that the inverse map sends absolute sequences to absolute sequences by Proposition 4.29. Suppressing T, we view $X \subseteq [x_k] \subseteq E$. The combination of X being separable and σ-order complete yields that X is order continuous. By [43, Corollary 3.1.25], the Rademacher's form a weakly null sequence in X. Hence, by passing to a subsequence and using the Bessaga-Pełczyński's selection principle, we may find a block sequence (y_k) of (x_k) such that $\|y_k - r_k\| \to 0$. Here, (r_k) is a subsequence of the Rademacher's. Passing to further subsequences, we may assume that $\|y_k - r_k\| \to 0$ sufficiently fast so that (r_k) is a small perturbation of (y_k), and hence is absolute, by the stability results proved in [65] (more specifically, unsuppressing T we get that (Tr_k) is absolute, hence (r_k) is as well by the multibounded inverse assumption). Therefore,

$$\|\mathbb{1}\|_X \sum_{k=1}^{n} |a_k| = \| \sum_{k=1}^{n} |a_k r_k| \|_X \leq A \| \sum_{k=1}^{n} a_k r_k \|_X,$$

so that a subsequence of the Rademacher's is equivalent to the unit vector basis of ℓ_1. This means that the unit vector basis of ℓ_1 is weakly null, a contradiction.

Until this point, we assumed that X was atomless, and we now reduce to the case that X is not purely atomic. Since X is σ-order complete and separable, it is order continuous, and hence has the projection property. Let B be the band generated by the atoms, so that $B \oplus B^d = X$. B^d is atomless and σ-order complete, so X cannot be nicely embedded into the span of an absolute FDD without B^d being as well. This concludes the proof. □

Remark 4.31. In [42] (see also [46]) a nonatomic AM-space X is constructed that is linearly isomorphic to c_0, hence has an unconditional basis, which is absolute since X is AM. Hence, σ-order completeness cannot be dropped in Proposition 4.30.

Remark 4.32. The reader may check that a bi-FDD version of Proposition 4.7 is valid, which when combined with Proposition 4.28 leads to an interesting phenomena: Start with any unconditional FDD of L_p, $p > 1$. Then one can block it so that it is a bi-FDD. After that, using Proposition 4.28, one can rearrange the blocked FDD so that it fails to be a bi-FDD. However, one can then find a further blocking of this blocked and rearranged FDD to regain the bi-property, and so on ad infinitum.

References

[1] Y. A. Abramovich. Some new characterizations of AM-spaces. *An. Univ. Craiova Mat. Fiz.-Chim.*, 6:15–26, 1978.

[2] Y. A. Abramovich and G. Ja. Lozanovskii. Some numerical characteristics of KN-lineals. *Mat. Zametki*, 14:723–732, 1973.

[3] Y. A. Abramovich, E. D. Positselskii, and L. P. Yanovskii. On some parameters associated with normed lattices and on series characterisation of M-spaces. *Studia Math.*, 63(1):1–8, 1978.

[4] Fernando Albiac, José L. Ansorena, Pablo M. Berná, and Przemysław Wojtaszczyk. Greedy approximation for biorthogonal systems in quasi-Banach spaces. *Dissertationes Math.*, 560:88, 2021.

[5] Fernando Albiac and Nigel J. Kalton. *Topics in Banach space theory*, volume 233 of *Graduate Texts in Mathematics*. Springer, second edition, 2016. With a foreword by Gilles Godefroy.

[6] James R. Angelos, Carl C. Cowen, and Sivaram K. Narayan. Triangular truncation and finding the norm of a Hadamard multiplier. *Linear Algebra Appl.*, 170:117–135, 1992.

[7] A. Avilés, C. Rosendal, M. A. Taylor, and P. Tradacete. Coordinate systems in Banach spaces and lattices. *arXiv preprint, arXiv:2406.11223v1*, 2024.

[8] Susan E. Bedingfield and Andrew Wirth. Norm and order properties of Banach lattices. *J. Austral. Math. Soc. Ser. A*, 29(3):331–336, 1980.

[9] Pablo M. Berná, Óscar Blasco, and Gustavo Garrigós. Lebesgue inequalities for the greedy algorithm in general bases. *Rev. Mat. Complut.*, 30(2):369–392, 2017.

[10] J. Bourgain. On Kolmogorov's rearrangement problem for orthogonal systems and Garsia's conjecture. In *Geometric aspects of functional analysis (1987–88)*, volume 1376 of *Lecture Notes in Math.*, pages 209–250. Springer, Berlin, 1989.

[11] D. L. Burkholder. Maximal inequalities as necessary conditions for almost everywhere convergence. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 3:75–88, 1964.

[12] Donald I. Cartwright and Heinrich P. Lotz. Some characterizations of AM- and AL-spaces. *Math. Z.*, 142:97–103, 1975.

[13] Peter G. Casazza, Stephen J. Dilworth, Denka Kutzarova, and Pavlos Motakis. New characterizations of the unit vector basis of c_0 or ℓ_p. *Canad. Math. Bull.*, 66(4):1073–1083, 2023.

[14] Y. Deng, M. O'Brien, and V. G. Troitsky. Unbounded norm convergence in Banach lattices. *Positivity*, 21(3):963–974, 2017.

[15] Joe Diestel, Hans Jarchow, and Andrew Tonge. *Absolutely summing operators*, volume 43 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.

[16] S. J. Dilworth, V. Ferenczi, Denka Kutzarova, and E. Odell. On strongly asymptotic l_p spaces and minimality. *J. Lond. Math. Soc. (2)*, 75(2):409–419, 2007.

[17] S. J. Dilworth, D. Freeman, E. Odell, and T. Schlumprecht. Greedy bases for Besov spaces. *Constr. Approx.*, 34(2):281–296, 2011.

[18] S. J. Dilworth, N. J. Kalton, and Denka Kutzarova. On the existence of almost greedy bases in Banach spaces. *Studia Math.*, 159(1):67–101, 2003.

[19] S. J. Dilworth and David Mitra. A conditional quasi-greedy basis of l_1. *Studia Math.*, 144(1):95–100, 2001.

[20] Xiumin Du and Ruixiang Zhang. Sharp L^2 estimates of the Schrödinger maximal function in higher dimensions. *Ann. of Math. (2)*, 189(3):837–861, 2019.

[21] Daniel Freeman, Alexander M. Powell, and Mitchell A. Taylor. A Schauder basis for L_2 consisting of non-negative functions. *arXiv:2003.09576v1*, 2020.

[22] Daniel Freeman, Alexander M. Powell, and Mitchell A. Taylor. A Schauder basis for L_2 consisting of non-negative functions. *Math. Ann.*, 381(1-2):181–208, 2021.

[23] Enrique García-Sánchez, Denny H. Leung, Mitchell A. Taylor, and Pedro Tradacete. Banach lattices with upper p-estimates: free and injective objects. *Math. Ann.*, 391(3):3363–3398, 2025.

[24] Enrique García-Sánchez, Denny H. Leung, Mitchell A. Taylor, and Pedro Tradacete. Banach lattices with upper p-estimates: renorming and factorization. *preprint*, 2026.

[25] Loukas Grafakos and Chris Lennard. Characterization of $L^p(\mathbf{R}^n)$ using Gabor frames. *J. Fourier Anal. Appl.*, 7(2):101–126, 2001.

[26] Anna Gumenchuk, Olena Karlova, and Mikhail Popov. Order Schauder bases in Banach lattices. *J. Funct. Anal.*, 269(2):536–550, 2015.

[27] Logan Hart, Christopher Heil, Ian Katz, and Michael Northington V. Overcomplete Reproducing Pairs. *arXiv preprint arXiv:2311.04421*, 2023.

[28] Christopher Heil. *A basis theory primer*. Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, New York, expanded edition, 2011.

[29] Guixiang Hong, Samya Kumar Ray, and Simeng Wang. Maximal ergodic inequalities for some positive operators on noncommutative L_p-spaces. *J. Lond. Math. Soc. (2)*, 108(1):362–408, 2023.

[30] Richard A. Hunt. On the convergence of Fourier series. In *Orthogonal Expansions and their Continuous Analogues (Proc. Conf., Edwardsville, Ill., 1967)*, pages 235–255. Southern Illinois Univ. Press, Carbondale, IL, 1968.

[31] W. B. Johnson and E. Odell. Subspaces and quotients of $l_p \oplus l_2$ and X_p. *Acta Math.*, 147(1-2):117–147, 1981.

[32] W. B. Johnson and M. Zippin. On subspaces of quotients of $(\sum G_n)_{lp}$ and $(\sum G_n)_{c_0}$. *Israel J. Math.*, 13:311–316, 1972.

[33] William B. Johnson, Bernard Maurey, and Gideon Schechtman. Weakly null sequences in L_1. *J. Amer. Math. Soc.*, 20(1):25–36, 2007.

[34] William B. Johnson and Gideon Schechtman. A Schauder basis for $L_1(0,\infty)$ consisting of non-negative functions. *Illinois J. Math.*, 59(2):337–344, 2015.

[35] Marius Junge and Quanhua Xu. Noncommutative maximal ergodic theorems. *J. Amer. Math. Soc.*, 20(2):385–439, 2007.

[36] M. Kandić and M. A. Taylor. Metrizability of minimal and unbounded topologies. *J. Math. Anal. Appl.*, 466(1):144–159, 2018.

[37] B. S. Kashin and A. A. Saakyan. *Orthogonal series*, volume 75 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1989. Translated from the Russian by Ralph P. Boas.

[38] T. W. Körner. Divergence of decreasing rearranged Fourier series. *Ann. of Math. (2)*, 144(1):167–180, 1996.

[39] T. W. Körner. Hard summation, Olevskiĭ, Tao and Walsh. *Bull. London Math. Soc.*, 38(5):705–729, 2006.

[40] S. Kostyukovsky and A. Olevskii. Note on decreasing rearrangement of Fourier series. *J. Appl. Anal.*, 3(1):137–142, 1997.

[41] S. Kwapień and A. Pełczyński. The main triangle projection in matrix spaces and its applications. *Studia Math.*, 34:43–68, 1970.

[42] E. Lacey and P. Wojtaszczyk. Nonatomic Banach lattices can have l_1 as a dual space. *Proc. Amer. Math. Soc.*, 57(1):79–84, 1976.

[43] Pei-Kee Lin. *Köthe-Bochner function spaces.* Birkhäuser Boston, Inc., Boston, MA, 2004.

[44] Joram Lindenstrauss and Lior Tzafriri. *Classical Banach spaces. I.* Springer-Verlag, Berlin-New York, 1977.

[45] Joram Lindenstrauss and Lior Tzafriri. *Classical Banach spaces. II*, volume 97. Springer-Verlag, Berlin-New York, 1979.

[46] Lea McClaran. Banach spaces with an unconditional basis that are isomorphic to a nonatomic Banach lattice. *Proc. Amer. Math. Soc.*, 123(2):471–476, 1995.

[47] Peter Meyer-Nieberg. *Banach lattices.* Universitext. Springer-Verlag, Berlin, 1991.

[48] J. C. Navarro-Pascual and M. A. Navarro. Unitary operators in real von Neumann algebras. *J. Math. Anal. Appl.*, 386(2):933–938, 2012.

[49] Amos Nevo and Elias M. Stein. A generalization of Birkhoff's pointwise ergodic theorem. *Acta Math.*, 173(1):135–154, 1994.

[50] Timur Oikhberg, Mitchell A. Taylor, Pedro Tradacete, and Vladimir G. Troitsky. Free Banach lattices. *J. Eur. Math. Soc. (JEMS), to appear*, 2024.

[51] Gilles Pisier. Regular operators between non-commutative L_p-spaces. *Bull. Sci. Math.*, 119(2):95–118, 1995.

[52] Alexander M. Powell and Anneliese H. Spaeth. Nonnegativity constraints for structured complete systems. *Trans. Amer. Math. Soc.*, 368(8):5783–5806, 2016.

[53] Edelstein I. S. and P. Wojtaszczyk. On projections and unconditional bases in direct sums of Banach spaces. *Studia Math.*, 56(3):263–276, 1976.

[54] S. Sawyer. Maximal inequalities of weak type. *Ann. of Math. (2)*, 84:157–174, 1966.

[55] Gideon Schechtman. No greedy bases for matrix spaces with mixed ℓ_p and ℓ_q norms. *J. Approx. Theory*, 184:100–110, 2014.

[56] Thomas Scheckter. *Martingale Convergence Techniques in Noncommutative Integration.* PhD thesis, UNSW Sydney, 2020.

[57] Anton R. Schep. Krivine's theorem and the indices of a Banach lattice. *Acta Appl. Math.*, 27(1-2):111–121, 1992.

[58] Per Sjölin. An inequality of Paley and convergence a.e. of Walsh-Fourier series. *Ark. Mat.*, 7:551–570, 1969.

[59] E. M. Stein. On limits of seqences of operators. *Ann. of Math. (2)*, 74:140–170, 1961.

[60] Stanisław J. Szarek. A Banach space without a basis which has the bounded approximation property. *Acta Math.*, 159(1-2):81–98, 1987.

[61] Terence Tao. On the almost everywhere convergence of wavelet summation methods. *Appl. Comput. Harmon. Anal.*, 3(4):384–387, 1996.

[62] Mitchell A. Taylor. Completeness of unbounded convergences. *Proc. Amer. Math. Soc.*, 146(8):3413–3423, 2018.

[63] Mitchell A. Taylor. Unbounded convergences in vector lattices. *Master thesis, University of Alberta*, 2019.

[64] Mitchell A. Taylor. Unbounded topologies and *uo*-convergence in locally solid vector lattices. *J. Math. Anal. Appl.*, 472(1):981–1000, 2019.

[65] Mitchell A. Taylor and Vladimir G. Troitsky. Bibasic sequences in Banach lattices. *J. Funct. Anal.*, 278(10):108448, 33, 2020.

[66] Vladimir Temlyakov. *Greedy approximation*, volume 20 of *Cambridge Monographs on Applied and Computational Mathematics.* Cambridge University Press, Cambridge, 2011.

[67] A. W. Wickstead. Regular operators between Banach lattices. *Positivity*, pages 255–279, 2007.

[68] P. Wojtaszczyk. On projections and unconditional bases in direct sums of Banach spaces. II. *Studia Math.*, 62(2):193–201, 1978.

[69] P. Wojtaszczyk. Greedy algorithm for general biorthogonal systems. *J. Approx. Theory*, 107(2):293–314, 2000.

[70] Robert M. Young. *An introduction to nonharmonic Fourier series.* Academic Press, Inc., San Diego, CA, first edition, 2001.

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