

COORDINATE SYSTEMS IN BANACH SPACES AND LATTICES

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ABSTRACT. Using methods of descriptive set theory, in particular, the determinacy of infinite games of perfect information, we answer several questions from the literature, including [12, Question 2.5, Question 2.10], [8, Problem 1.3, Problem 5.2] and [6, Question 2], regarding different notions of bases in Banach spaces and lattices.

For the case of Banach lattices, our results follow from a general theorem stating that (under the assumption of projective determinacy), every order basis (e_n) for a Banach lattice $X = [e_n]$ is also a σ -order basis for X , every σ -order basis for X is a uniform basis for X , and every uniform basis is Schauder.

Regarding Banach spaces, we address two problems concerning filter Schauder bases for Banach spaces, i.e., in which the norm convergence of partial sums is replaced by norm convergence along some appropriate filter on \mathbb{N} . We first provide an example of a Banach space admitting such a filter Schauder basis, but no ordinary Schauder basis. Secondly, we show that every filter Schauder basis with respect to an analytic filter is also a filter Schauder basis with respect to a Borel filter.

1. INTRODUCTION

1.1. Order bases in Banach lattices. Suppose X is a Banach lattice. Then the lattice structure on X gives rise to three classical notions of sequential convergence, not available in a general Banach space. Namely,

- a sequence $(x_n)_{n=1}^{\infty}$ *converges uniformly* to x , denoted $x_n \xrightarrow{u} x$, if there is some $z \in X_+$ so that

$$\forall m \forall^{\infty} n |x_n - x| \leq \frac{z}{m},$$

- a sequence $(x_n)_{n=1}^{\infty}$ σ -*order converges* to x , denoted $x_n \xrightarrow{\sigma} x$, if there is some sequence $z_m \downarrow 0$ so that

$$\forall m \forall^{\infty} n |x_n - x| \leq z_m,$$

- a sequence $(x_n)_{n=1}^{\infty}$ *order converges* to x , denoted $x_n \xrightarrow{\circ} x$, if there is some net $z_{\mu} \downarrow 0$ so that

$$\forall \mu \forall^{\infty} n |x_n - x| \leq z_{\mu}.$$

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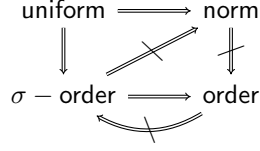


FIGURE 1. Implications between convergence types.

Here the notation $\forall^\infty n$ means for all but finitely many n , i.e., $\exists N \forall n \geq N$, while $z_m \downarrow 0$ and $z_\mu \downarrow 0$ mean that (z_m) and (z_μ) are decreasing and have infimum 0. It can be shown that, in all cases above, the limit is unique whenever it exists. Thus, if C is one of the above notions of convergence and $\sum_{n=1}^\infty x_n$ is a series in X , we can unambiguously write

$$x =^C \sum_{n=1}^\infty x_n$$

to denote that the sequence of partial sums $(\sum_{n=1}^m x_n)$ C -converges to x .

All three notions of convergence are evidently compatible with the algebraic structure of X , in the sense that if (x_n) and (y_n) converge to x and y respectively, then the sequence of sums $(x_n + y_n)$ converges to $x + y$ and similarly for scalar products. Nevertheless, neither uniform nor order convergence arise in general from Hausdorff topologies on X [13, Section 18].

As is evident from the definitions, uniform convergence implies norm convergence and σ -order convergence, whereas σ -order convergence implies order convergence. However, in absence of other hypotheses on X , no other implications hold, which is recorded in Figure 1.1.

Definition 1.1. *Let X be a Banach lattice and C one of the following convergence types: norm, uniform, order or σ -order. Then a sequence $(e_n)_{n=1}^\infty$ in X is said to be a C -basis for X provided that, for every $x \in X$, there is a unique sequence of scalars $(a_n) \in \mathbb{R}^\mathbb{N}$ so that*

$$x =^C \sum_{n=1}^\infty a_n e_n.$$

For example, it is easy to see that the standard unit vector sequence (e_n) forms an order basis for each of the spaces ℓ_p , $1 \leq p \leq \infty$, and also for c_0 . We are thus in the unfamiliar situation that the same sequence (e_n) is an order basis both for ℓ_∞ and its subspace (even sublattice) c_0 . In particular, we see that the norm-closed linear span $[e_n]$ of (e_n) may be strictly smaller than the lattice X for which it is an order basis.

When (e_n) is a C -basis for X , we may define functionals $X \xrightarrow{e_k^\sharp} \mathbb{R}$ by letting

$$e_k^\sharp(x) = a_k,$$

where (a_n) is the uniquely defined sequence referenced above. Similarly, we let $X \xrightarrow{P_m} [e_1, \dots, e_m]$ denote the corresponding sequence of basis projections,

$$P_m(x) = \sum_{n=1}^m e_n^\sharp(x) e_n.$$

Since the sequence constantly equal to e_n will C -converge to e_n , we find that $e_k^\sharp(e_n) = \delta_{k,n}$ for all k, n , that is, the functionals e_k^\sharp are *biorthogonal* to the sequence (e_n) . Observe however that a priori it is not clear that the functionals e_k^\sharp or the operators P_m are continuous (with respect to the norm topology on X ; Banach lattices often admit no order continuous linear functionals).

Norm bases are of course more commonly known as *Schauder bases* and we shall employ that terminology here. Moreover, it is a classical result [2, p. 111] that the biorthogonal functionals e_k^\sharp associated to a Schauder basis are always continuous.

Biorthogonal functionals associated with some sequence are typically denoted by e_k^* , but, since the very continuity of the functionals is at play here, we shall only use the notation e_k^* if we already know that they are continuous.

In our first main theorem, which settles the relationships between the different types of bases, for some of the implications we resort to additional set theoretical axioms, namely the determinacy of certain infinite games on \mathbb{N} [10, Definition 26.3]. Nevertheless, this usage should not be too disturbing as projective determinacy is arguably part of the right set theoretical foundations of mathematics.

Theorem 1.2 (Projective determinacy). *Suppose (e_n) is a sequence of vectors in a Banach lattice $X = [e_n]$ and (e_n^\sharp) is a sequence of (possibly discontinuous) biorthogonal functionals for (e_n) . Consider the following properties:*

- (1) (e_n) is an order basis for X with corresponding functionals (e_n^\sharp) ,
- (2) (e_n) is a σ -order basis for X with corresponding functionals (e_n^\sharp) ,
- (3) (e_n) is a uniform basis for X with corresponding functionals (e_n^\sharp) ,
- (4) (e_n) is a Schauder basis for X with corresponding functionals (e_n^\sharp) .

Then $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ and so the e_n^\sharp are continuous in all cases above.

As hinted above, projective determinacy is only needed for parts of the theorem. Indeed, $(3) \Rightarrow (4)$ holds without any additional set theoretical assumptions, whereas $(2) \Rightarrow (3)$ is proved under the weaker assumption of analytic determinacy.

Coupled with [12, Theorem 2.1], we obtain the following characterisation of uniform bases for Banach lattices.

Corollary 1.3. *The following conditions are equivalent for a sequence (e_n) of non-zero vectors in a Banach lattice $X = [e_n]$:*

- (1) (e_n) is a uniform basis for X ,
- (2) (e_n) is a Schauder basis for X so that, for every $x \in X$, the sequence of partial sums

$$P_m x = \sum_{n=1}^m e_n^*(x) e_n$$

is order bounded,

- (3) there is a constant M so that, for all finite tuples of scalars $(a_n)_{n=1}^m$ one has

$$\left\| \bigvee_{k=1}^m \left| \sum_{n=1}^k a_n e_n \right| \right\| \leq M \left\| \sum_{n=1}^m a_n e_n \right\|.$$

Remark 1.4. It is a classical fact that, if X is a Banach lattice of measurable functions, we have $f_n \xrightarrow{o} f$ if and only if $f_n \xrightarrow{a.e.} f$ and, moreover, there exists a $g \in X_+$ satisfying $|f_n| \leq g$ for all n (i.e., the sequence (f_n) is order bounded).

Therefore, a uniform basis (e_n) of a Banach lattice X can be thought of as a coordinate system which guarantees both norm and dominated almost everywhere convergence of the basis expansions.

Statement (3) of Corollary 1.3 is simply the standard inequality for Schauder basic sequences with the supremum $\bigvee_{k=1}^m$ pulled inside the norm. The equivalence between (1) and (3) in Corollary 1.3 therefore shows that bounding the maximal function of a basic sequence is equivalent to establishing strong convergence properties of the series, even in the general setting of Banach lattices.

We remark that u -bases (and u -basic sequences; see Remark 2.6) occur frequently in applications. For example, it follows from Doob's inequality that martingale difference sequences in $L_p(\mu)$ ($1 < p < \infty$ and μ a probability measure) are u -basic. On the other hand, the combination of the Burkholder–Davis–Gundy and Khintchine inequalities yields that unconditional blocks of the Haar basis in $L_1[0, 1]$ are u -basic, and the Carleson–Hunt theorem [16] establishes the inequality in Corollary 1.3 (3) for the trigonometric basis. For several more examples and non-examples of u -basic sequences, the reader may consult [12, 13].

Analogously to Corollary 1.3, we may characterise order bases as follows, and note that there is a similar characterisation of σ -order bases.

Corollary 1.5 (Projective determinacy). *The following conditions are equivalent for a sequence (e_n) of non-zero vectors in a Banach lattice $X = [e_n]$:*

- (1) (e_n) is an order basis for X ,
- (2) (e_n) is a uniform basis for X such that $0 = {}^\circ \sum_{n=1}^\infty 0e_n$ is the unique order expansion of 0.

Theorem 1.2 immediately solves several problems listed in the literature regarding the relationships between these basis notions.

Problem 1.6. [12, Question 2.10] *If (e_n) is simultaneously a Schauder basis and a σ -order basis for a Banach lattice X , do the coefficients in the norm and in the order expansions of the same vector agree? That is, are the two associated sets of biorthogonal functionals equal?*

By the implications (2) \Rightarrow (3) \Rightarrow (4), proved under the assumption of analytic determinacy, the answer to Problem 1.6 is therefore positive.

For our next applications, we need to recall some facts about order continuous Banach lattices from [3].

Lemma 1.7. *Suppose X is an order continuous Banach lattice. Then, for all sequences (x_n) and vectors x ,*

$$x_n \xrightarrow{\circ} x \iff x_n \xrightarrow{\sigma\circ} x \iff x_n \xrightarrow{u} x.$$

Similarly, if X is σ -order continuous, then

$$x_n \xrightarrow{\sigma\circ} x \iff x_n \xrightarrow{u} x.$$

Thus, in an order continuous Banach lattice X , the notions of order, σ -order and uniform bases coincide and they will have the same associated biorthogonal functionals. Similarly, if X is σ -order continuous, the notions of σ -order and uniform bases coincide and have the same biorthogonal functionals. In particular, these types of bases will automatically also be Schauder bases with the same biorthogonal functionals.

Corollary 1.8. *Let (e_n) be an order basis for an order continuous Banach lattice. Then (e_n) is also a Schauder basis with the same biorthogonal functionals.*

This in turn provides a positive answer to the following questions.

Problem 1.9. [8, Problem 1.3] *Let (x_n) be an order basis of an order continuous Banach lattice E . Is then (x_n) a Schauder basis of E ? What about $E = L_p$ with $1 \leq p < \infty$?*

In the same paper, the authors consider the specific example of L_1 and conjecture a negative answer to the following question.

Problem 1.10. [8, Problem 5.2] *Does L_1 have a σ -order basis?*

Regarding this, they show that L_1 does not admit a sequence (e_n) that is simultaneously a Schauder and an order basis for L_1 [8, Theorem 5.1]. However, given that L_1 is order continuous, every order basis for L_1 is also a Schauder basis, which gives a negative answer to Problem 1.10.

Corollary 1.11. *The Banach lattice L_1 admits no σ -order basis.*

A few cautionary remarks on the terminology are in order. Namely, our notion of σ -order convergence for sequences is simply called *order convergence* for sequences in [8] and therefore our notion of σ -order basis is similarly designated *order basis* in [8]. In the same paper, a sequence (e_n) in a Banach lattice X which is simultaneously a Schauder basis and a σ -order basis for X is denoted a *bibasis*. By Theorem 1.2, under analytic determinacy, bibases are thus simply σ -order bases. However, the authors exclusively work in σ -order continuous Banach lattices, where of course the notions of σ -order and uniform bases coincide. On the other hand, in [12], a sequence (e_n) in a Banach lattice X is called a *bibasis* for X provided that it is both a Schauder basis and a uniform basis. Because of Theorem 1.2, these two competing notions of bibases are superfluous as they just correspond to σ -order and uniform bases respectively. To avoid any confusion, we shall exclusively employ the terminology of Definition 1.1 and eschew the, in hindsight, unnecessary notion of bibasis.

1.2. Filter bases. The second topic of our study concerns a generalisation of Schauder bases in the context of general Banach spaces, not lattices. Assume that F is a filter of subsets of \mathbb{N} , that is, $F \subseteq \mathcal{P}(\mathbb{N})$ is closed under taking intersections and supersets,

- $a, b \in F \Rightarrow a \cap b \in F$,
- $a \subseteq b \ \& \ a \in F \Rightarrow b \in F$.

For reasons that will become apparent later, we shall also assume that all filters are *proper*, i.e., $\emptyset \notin F$, and contain the *Fréchet filter* consisting of all cofinite subsets of \mathbb{N} . Recall that a sequence (x_n) is said to *converge along F* to x , denoted $x_n \xrightarrow[n \rightarrow F]{} x$, if

$$\{m \in \mathbb{N} \mid \|x_n - x\| < \epsilon\} \in F$$

for all $\epsilon > 0$. In complete analogy with Definition 1.1, we have the following definition due to M. Ganichev and V. Kadets [7].

Definition 1.12. A sequence (e_n) in a Banach space X is said to be an F -basis for X provided that, for all $x \in X$, there is a unique sequence $(a_n) \in \mathbb{R}^{\mathbb{N}}$ so that $\sum_{n=1}^m a_n e_n \xrightarrow{m \rightarrow F} x$, which we denote by

$$x =^F \sum_{n=1}^{\infty} a_n e_n.$$

More generally, (e_n) is said to be a filter basis for X if it is an F -basis for X for some filter F , in which case F is said to be compatible with (e_n) .

Let us note that, if F is just the Fréchet filter itself, then an F -basis (e_n) is nothing but a Schauder basis. Although [4, Example 1] provides a basis for ℓ_2 with respect to the ideal of sets of density 1, which however is not a Schauder basis, T. Kania asked whether there is an example of a Banach space without a Schauder basis that nevertheless has an F -basis for some appropriate filter F . We answer this by the following simple example.

Example 1.13 (A Banach space with a filter basis, but no Schauder basis). Let X be a Banach space with a finite-dimensional decomposition $(X_n)_{n=1}^{\infty}$, but without a Schauder basis. That such spaces exist follows for example from [11, Theorem 1.1]. Choose now sequences $(e_i)_{i \in \mathbb{N}}$ and $0 = k_0 < k_1 < k_2 < \dots$ so that

$$\{e_i \mid k_{n-1} < i \leq k_n\}$$

is a basis for X_n for all n . Let also

$$F = \{a \subseteq \mathbb{N} \mid k_n \in a \text{ for all but finitely many } n\}.$$

Since (X_n) is an F.D.D. for X , we have that, for every $x \in X$, there are unique vectors $x_n \in X_n$ so that $x = \sum_{n=1}^{\infty} x_n$. Writing $x_n = \sum_{i=k_{n-1}+1}^{k_n} a_i e_i$ for appropriate scalars $a_i \in \mathbb{R}$, we see that, for all $\epsilon > 0$,

$$k_n \in \left\{ m \in \mathbb{N} \mid \left\| x - \sum_{i=1}^m a_i e_i \right\| < \epsilon \right\}$$

for all but finitely many n and hence that the latter set belongs to F . Furthermore, by the uniqueness of the x_n , (a_i) is the only such sequence, which shows that (e_i) is an F -basis for X .

Note that, if F is a filter and (e_n) is an F -basis for X , we may define the associated biorthogonal functionals e_n^\sharp just as for order bases etc. However, it might be possible that (e_n) is simultaneously an F' -basis for X with respect to some other filter F' , in which case it is unclear whether the biorthogonal functionals associated with F' are the same as those associated with F . Furthermore, even under additional set theoretical axioms, it is no longer clear whether any of these functionals are continuous. To discuss these issues, we must introduce a more refined concept.

Definition 1.14. Let (e_n, e_n^\sharp) be a biorthogonal system in a Banach space X , i.e., (e_n) is a sequence of vectors in X and $e_n^\sharp: X \rightarrow \mathbb{R}$ are (possibly discontinuous) functionals biorthogonal to the e_n . We say that (e_n, e_n^\sharp) is a filter basis system for X provided that there is a filter F so that (e_n) is an F -basis for X with associated biorthogonal functionals e_n^\sharp . Such a filter F is said to be compatible with (e_n, e_n^\sharp) .

Let us note that, to every filter basis system (e_n, e_n^\sharp) , there is a smallest compatible filter F , which we shall return to later on. Recall also that a sequence (e_n) in a Banach space is said to be *minimal* if, for all k , we have that $e_k \notin [e_n]_{n \neq k}$.

Lemma 1.15. *Let (e_n, e_n^\sharp) be a filter basis system for X . Then the sequence (e_n) is minimal if and only if the functionals e_n^\sharp are continuous.*

Thus, the continuity of the associated biorthogonal functionals can be detected directly on the filter basis (e_n) itself without even involving the functionals.

Although Ganichev and Kadets [7] operate with a slightly more general notion of filter basis, the following problem remains open even in our setting.

Problem 1.16. [7] *Suppose (e_n) is a filter basis for a Banach space X . Is (e_n) necessarily a minimal sequence?*

The papers [6,9] address Problem 1.16 and show that a filter basis (e_n) is minimal if and only if it admits a compatible filter F that is analytic when viewed as a subset of $\mathcal{P}(\mathbb{N}) = \{0, 1\}^{\mathbb{N}}$ [6, Theorem A, Theorem B]. In connection with this, [6, Question 2] asks whether it is possible to improve this so as to get the filter F to be Borel. We resolve this even while keeping the associated biorthogonal functionals e_n^\sharp fixed.

Theorem 1.17. *Let (e_n, e_n^\sharp) be a filter basis system for X . Then the following are equivalent.*

- (1) *The functionals e_n^\sharp are continuous,*
- (2) *the sequence (e_n) is minimal,*
- (3) *the smallest compatible filter is analytic,*
- (4) *there is a compatible analytic filter,*
- (5) *there is a compatible Borel filter.*

1.3. A higher order Fatou property. It is of course natural to ask whether the use of projective and analytical determinacy in Theorem 1.2 is really necessary, i.e., if there is not some other more insightful proof bypassing these issues. We do not know the answer to this question, but note that the problem resides in the fact that the very notions of order and σ -order convergence are a priori of too high descriptive complexity. Indeed, in the proofs of Theorems 2.3 and 2.4, we are only able to show that, in a general separable Banach lattice X , the sets

$$\left\{ ((x_n), x) \in X^{\mathbb{N}} \times X \mid x_n \xrightarrow{\circ} x \right\} \quad \text{and} \quad \left\{ ((x_n), x) \in X^{\mathbb{N}} \times X \mid x_n \xrightarrow{\sigma \circ} x \right\}$$

are respectively $\mathbf{\Pi}_2^1$ and $\mathbf{\Sigma}_2^1$. On the other hand, if we can show them to be $\mathbf{\Sigma}_1^1$ (i.e., analytic) instead, then our proofs no longer necessitate the additional set theoretical assumptions. This happens, for example, if the Banach lattice X admits a countable π -basis, i.e., a countable set $P \subseteq X$ of positive elements so that, for every $x > 0$, there is some $p \in P$ with $0 < p \leq x$.

In Section 4 we introduce a hierarchy, indexed by countable ordinal numbers α , of properties similar to the well-known Fatou property of separable Banach lattices (see Definition 4.8). Our main result in this context is the following.

Theorem 1.18. *The following conditions are equivalent for a separable Banach lattice X .*

- (1) *The set*

$$\left\{ (x_n)_{n=1}^{\infty} \in X^{\mathbb{N}} \mid x_1 \geq x_2 \geq \dots \geq 0 = \inf_n x_n \right\}$$

is Borel,

(2) the set

$$\left\{ \left((x_n)_{n=1}^\infty, x \right) \in X^\mathbb{N} \times X \mid x_n \xrightarrow[n]{\sigma_0} x \right\}$$

is analytic,

(3) X is α -Fatou for some $\alpha < \omega_1$.

Furthermore, we show that, if a separable Banach lattice X satisfies these three equivalent conditions, then every σ -order basis for X is also a uniform basis for X without any additional set theoretical assumptions. As noted above, this happens, for example, if X has a countable π -basis.

In Section 5 we construct a sequence $(X_\alpha)_{\alpha < \omega_1}$ of separable Banach lattices with countable π -bases so that X_α fails to be β -Fatou for all $\beta \leq \alpha$. However, the question of whether one can find a fixed separable Banach lattice failing to be α -Fatou for all $\alpha < \omega_1$ remains open:

Problem 1.19. *Is every separable Banach lattice necessarily α -Fatou for some $\alpha < \omega_1$?*

An intriguing case is when X is assumed to be σ -complete, that is, every countable set $\{z_n\}_{n \in \mathbb{N}} \subseteq X$ that is bounded above has a least upper bound, $\bigvee_{n \in \mathbb{N}} z_n \in X$. In this case, we have that

$$x_n \xrightarrow[n]{\circ} x \iff x_n \xrightarrow[n]{\sigma_0} x$$

for all (x_n) and x , which means that both conditions are simultaneously $\mathbf{\Pi}_2^1$ and $\mathbf{\Sigma}_2^1$, that is, $\mathbf{\Delta}_2^1$. However, using the fact that separable σ -complete Banach lattices are order continuous [5, Proposition 1.a.7], it is easy to see that these conditions are in fact $\mathbf{\Sigma}_1^1$, which by Theorem 1.18 ensures that

$$z = \bigvee_{n \in \mathbb{N}} z_n$$

is Borel in the pair $(z, (z_n))$. Therefore, any counterexample to Problem 1.19 cannot be σ -complete.

Although Problem 1.19 remains open in general, our construction in Section 5 implies that there is no global Borel definition of when a sequence in a separable Banach lattice has infimum zero. In order to make such a statement precise, one needs to consider a Borel structure on the collection of all separable Banach lattices, as has been done by M. A. Tursi [14] following the ideas initiated by Bossard in Banach space theory. Using this language, it can be derived from the results of this paper that the collection of pairs $(X, (x_n))$ such that X is a separable Banach lattice and $(x_n)_{n=1}^\infty \in X^\mathbb{N}$ satisfy $x_1 \geq x_2 \geq \dots \geq 0 = \inf_n x_n$ in X constitutes a true coanalytic set.

2. PROOFS FOR ORDER BASES

As is well-known from the case of Banach spaces, it is often useful to operate with basic sequences as opposed to bases. So let us introduce this notion in our context. Recall first that a sequence (e_n) in a Banach space X is said to be *Schauder basic* in case it is a Schauder basis for its closed linear span $[e_n]$. When dealing with uniform, σ -order and order convergence, extra caution is required since the closed linear span $[e_n]$ of a sequence in X need not be a sublattice. Furthermore, even the notions

of σ -order and order convergence are not absolute, but depend on the ambient lattice. On the other hand, uniform convergence is absolute [12, Proposition 2.12]. In fact, as shown in Lemma 2.1 below, uniform convergence can be equivalently reformulated so as to avoid any reference to the ambient lattice.

Lemma 2.1. *For a sequence (x_n) and vector x in a Banach lattice X , we have that $x_n \xrightarrow{u} x$ if and only if*

$$\forall \epsilon > 0 \exists k \forall m \geq k \left\| \bigvee_{n=k}^m |x_n - x| \right\| < \epsilon.$$

In particular, the set

$$\left\{ ((x_n)_{n=1}^\infty, x) \in X^\mathbb{N} \times X \mid x_n \xrightarrow{u} x \right\}$$

is Borel.

Proof. Suppose first that $x_n \xrightarrow{u} x$ and find some $z > 0$ so that

$$\forall l \forall^\infty n |x_n - x| \leq \frac{z}{l}.$$

Thus, if $\epsilon > 0$ is given, choose l large enough that $\|\frac{z}{l}\| < \epsilon$ and find k so that $|x_n - x| \leq \frac{z}{l}$ for all $n \geq k$. We then see that

$$\left\| \bigvee_{n=k}^m |x_n - x| \right\| \leq \|\frac{z}{l}\| < \epsilon$$

for all $m \geq k$.

Conversely, suppose that

$$\forall \epsilon > 0 \exists k \forall m \geq k \left\| \bigvee_{n=k}^m |x_n - x| \right\| < \epsilon.$$

We choose $0 = k_0 < k_1 < k_2 < \dots$ so that

$$\left\| \bigvee_{n=k_l}^m |x_n - x| \right\| < \frac{1}{4^l}$$

whenever $l \geq 1$ and $m \geq k_l$. This implies that the series

$$\sum_{l=0}^{\infty} 2^l \bigvee_{n=k_l}^{k_{l+1}-1} |x_n - x|$$

converges in norm to some element $z \in X_+$. Moreover, for any $l \geq 1$ and all $n \geq k_l$, we have that $2^l |x_n - x| \leq z$ and hence $|x_n - x| \leq \frac{z}{2^l}$. Therefore, $x_n \xrightarrow{u} x$. \square

We may thus define a sequence (e_n) in a Banach lattice X to be *u-basic* in case, for every $x \in [e_n]$, there is a unique sequence $(a_n) \in \mathbb{R}^\mathbb{N}$ so that $x =^u \sum_{n=1}^\infty a_n e_n$. By Lemma 2.1, this notion is intrinsically defined and independent of the choice of the ambient Banach lattice, which we may therefore always assume to be the separable Banach lattice $[e_n]_\wedge \subseteq X$ generated by (e_n) . Note however that $[e_n]$ itself will in general only be a Banach space, not a lattice. The corresponding biorthogonal functionals $e_k^\sharp: [e_n] \rightarrow \mathbb{R}$ are defined as before.

The following establishes the implication (3) \Rightarrow (4) of Theorem 1.2.

Theorem 2.2. *Suppose that (e_n) is a \mathbf{u} -basic sequence in a Banach lattice X . Then the biorthogonal functionals e_n^\sharp are continuous and hence (e_n) is Schauder basic.*

To simplify notation, it is slightly easier to work with the operator $[e_n] \xrightarrow{E} \mathbb{R}^{\mathbb{N}}$ defined by $Ex = (e_n^\sharp(x))_n$ in place of the biorthogonal functionals themselves. Observe that E is continuous if and only if all the e_n^\sharp are continuous.

Proof. We recall that, since both $[e_n]$ and $\mathbb{R}^{\mathbb{N}}$ are separable Fréchet spaces and hence Polish spaces, the operator E is continuous if and only if the graph $\mathcal{G}E \subseteq [e_n] \times \mathbb{R}^{\mathbb{N}}$ is Borel. Indeed, if $\mathcal{G}E$ is Borel or even analytic, then E is Borel measurable [10, Theorem 14.12] and therefore continuous [10, Theorem 9.10]. Now,

$$(x, (a_n)) \in \mathcal{G}E \Leftrightarrow \sum_{n=1}^m a_n e_n \xrightarrow{\mathbf{u}}_m x,$$

which is Borel by Lemma 2.1. Thus, the biorthogonal functionals e_n^\sharp are all continuous.

To see that (e_n) is Schauder basic, i.e., a Schauder basis for $[e_n]$, note that, for every $x \in [e_n]$, we have that $x = \mathbf{u} \sum_{n=1}^{\infty} e_n^\sharp(x) e_n$ and hence also $x = \|\cdot\| \sum_{n=1}^{\infty} e_n^\sharp(x) e_n$. Also, if $(a_n) \in \mathbb{R}^{\mathbb{N}}$ is any sequence so that $x = \|\cdot\| \sum_{n=1}^{\infty} a_n e_n$, we see that

$$a_k = \lim_m e_k^\sharp \left(\sum_{n=1}^m a_n e_n \right) = e_k^\sharp \left(\lim_m \sum_{n=1}^m a_n e_n \right) = e_k^\sharp(x),$$

so the norm-expansion $x = \|\cdot\| \sum_{n=1}^{\infty} e_n^\sharp(x) e_n$ is unique. \square

In order to obtain a similar result for either σ -order or order convergence, we are forced to rely on additional set theoretical assumptions, namely, the determinacy of increasingly complicated sets.

Theorem 2.3 (Σ_1^1 -determinacy or $\text{MA} + \neg\text{CH}$). *Suppose that (e_n) is a σ -order basis for a separable Banach lattice X . Then the biorthogonal functionals e_n^\sharp are continuous and hence (e_n) is Schauder basic.*

Proof. Observe that, for $x \in X$ and $(a_n) \in \mathbb{R}^{\mathbb{N}}$, we have

$$x = \sigma \sum_{n=1}^{\infty} a_n e_n \Leftrightarrow \exists (z_m) \in X_+^{\mathbb{N}} \left(z_m \downarrow 0 \quad \& \quad \forall m \forall^{\infty} n \left| x - \sum_{k=1}^n a_k e_k \right| \leq z_m \right).$$

Clearly, the condition $\forall m \forall^{\infty} n \left| x - \sum_{k=1}^n a_k e_k \right| \leq z_m$ is Borel in the tuple $(x, (a_n), (z_m)) \in X \times \mathbb{R}^{\mathbb{N}} \times X_+^{\mathbb{N}}$, but unfortunately the condition $z_m \downarrow 0$ appears only to be Π_1^1 in $(z_m) \in X_+^{\mathbb{N}}$,

$$z_m \downarrow 0 \Leftrightarrow \forall m z_m \geq z_{m+1} \quad \& \quad \forall y > 0 \exists m y \not\leq z_m.$$

Thus, a priori, the graph $\mathcal{G}E$ of $X \xrightarrow{E} \mathbb{R}^{\mathbb{N}}$ is only Σ_2^1 , which means that the inverse image $E^{-1}(U)$ of an open set $U \subseteq \mathbb{R}^{\mathbb{N}}$ is Σ_2^1 and therefore has the property of Baire if we assume either Σ_1^1 -determinacy [10, Theorem 36.20] or $\text{MA} + \neg\text{CH}$ [10, Exercise 38.8] and [15, Theorem 19.23]. Therefore, E is continuous by [10, Theorem 14.12] and hence so are the associated partial sum projections $X \xrightarrow{P_m} [e_1, \dots, e_m]$.

We claim that the sequence of operators $(P_m)_{m=1}^{\infty}$ is uniformly bounded, which by Grunblum's criterion [1, Proposition 1.1.9] implies that (e_n) is Schauder basic. To see this, it suffices by the principle of uniform boundedness to show that

$(P_m x)_{m=1}^\infty$ is bounded in norm for each $x \in X$. However, given x , observe that, as $P_m x \xrightarrow[m \rightarrow \infty]{\sigma} x$, there is $z \geq 0$ for which $|x - P_m x| \leq z$ for all but finitely many m , which shows that the sequence $(\|P_m x\|)_{m=1}^\infty$ is bounded. \square

Theorem 2.4 (Projective determinacy). *Suppose that (e_n) is an order basis for a separable Banach lattice X . Then the biorthogonal functionals e_n^\sharp are continuous and hence (e_n) is Schauder basic.*

Proof. We claim that, for any sequence (x_n) and vector x in X , we have

$$x_n \xrightarrow{\circ} x \Leftrightarrow \forall y > 0 \exists z (y \not\leq z \ \& \ \forall^\infty n |x_n - x| \leq z).$$

As the right-hand side is clearly a $\mathbf{\Pi}_2^1$ -condition on $((x_n), x) \in X^\mathbb{N} \times X$, this will imply that the biorthogonal operator $X \xrightarrow{E} \mathbb{R}^\mathbb{N}$ has $\mathbf{\Pi}_2^1$ -graph and hence is continuous under the assumption of projective determinacy [10, Theorem 38.17]. As in the proof of Theorem 2.3, this in turn ensures that (e_n) is Schauder basic.

To establish our claim, suppose first that $x_n \xrightarrow{\circ} x$. This means that there is a decreasing net (z_μ) with $\inf_\mu z_\mu = 0$ so that $\forall \mu \forall^\infty n |x_n - x| \leq z_\mu$. In particular, if $y > 0$, then $y \not\leq z_\mu$ for some μ and hence the sequence (x_n) satisfies

$$(1) \quad \forall y > 0 \exists z (y \not\leq z \ \& \ \forall^\infty n |x_n - x| \leq z).$$

Conversely, suppose that (x_n) satisfies (1). Then the set

$$A = \{z \in X \mid \forall^\infty n |x_n - x| \leq z\}$$

becomes directed under the ordering $z \prec z' \Leftrightarrow z' \leq z$. For if $z, z' \in A$, then also $z, z' \prec z \wedge z' \in A$. It follows that (A, \prec) can be viewed as a decreasing net with infimum 0 witnessing that $x_n \xrightarrow{\circ} x$. \square

The proofs of Theorem 1.2 and Corollaries 1.3 and 1.5 heavily rely on the relationship between the different types of convergence of partial sums established in [12, Theorem 3.1] (see also [8, Theorem 2.3] for a related earlier result).

Theorem 2.5. [12, Theorem 3.1] *The following statements are equivalent for a Schauder basic sequence (e_n) in a Banach lattice X with associated basis projections $[e_n] \xrightarrow{P_m} [e_n]$.*

- (i) For all $x \in [e_n]$, $P_m x \xrightarrow{u} x$,
- (ii) For all $x \in [e_n]$, $P_m x \xrightarrow{\sigma\sigma} x$,
- (iii) For all $x \in [e_n]$, $P_m x \xrightarrow{\circ} x$,
- (iv) For all $x \in [e_n]$, $(P_m x)$ is order bounded in X ,
- (v) For all $x \in [e_n]$, $(\sum_{n=1}^m |P_n x|)$ is norm bounded,
- (vi) There is $M \geq 1$ so that, for all $n \in \mathbb{N}$ and scalars a_1, \dots, a_n , one has

$$\left\| \bigvee_{m=1}^n \left| \sum_{k=1}^m a_k e_k \right| \right\| \leq M \left\| \sum_{k=1}^n a_k e_k \right\|.$$

Proof of Theorem 1.2. (1) \Rightarrow (2): Assume that (e_n) is an order basis for X with corresponding functionals (e_n^\sharp) . Then, by Theorem 2.4, the biorthogonal functionals (e_n^\sharp) are continuous and (e_n) is a Schauder basis for $X = [e_n]$. Furthermore, for all $x \in X$, we have $P_m x \xrightarrow{\circ} x$, which means that (e_n) satisfies condition (iii) of [12, Theorem 3.1] and hence must also satisfy condition (ii) of the same theorem,

namely that, for every $x \in X$, $P_m x \xrightarrow{\sigma^\circ} x$, i.e., $x =^{\sigma^\circ} \sum_{n=1}^{\infty} e_n^\#(x) e_n$. On the other hand, if (a_n) is any sequence so that $x =^{\sigma^\circ} \sum_{n=1}^{\infty} a_n e_n$, then also $x =^\circ \sum_{n=1}^{\infty} a_n e_n$, whereby $a_n = e_n^\#(x)$ as (e_n) is an order basis. This shows uniqueness of the σ -order expansion and hence implies that (e_n) is a σ -order basis for X .

The implication (2) \Rightarrow (3) is entirely analogous using conditions (ii) and (i) of [12, Theorem 2.1] in place of conditions (iii) and (ii), together with Theorem 2.3. Finally, the implication (3) \Rightarrow (4) follows directly from Theorem 2.2. \square

Proof of Corollary 1.3. Fix a sequence (e_n) in a Banach lattice X so that $X = [e_n]$. Assume first that condition (3) holds, i.e., that, for some constant M and all finite tuples of scalars $(a_n)_{n=1}^m$ one has

$$\left\| \bigvee_{k=1}^m \left| \sum_{n=1}^k a_n e_n \right| \right\| \leq M \left\| \sum_{n=1}^m a_n e_n \right\|,$$

whereby also

$$\left\| \sum_{n=1}^k a_n e_n \right\| \leq M \left\| \sum_{n=1}^m a_n e_n \right\|,$$

for all $k \leq m$. Thus, by Grunblum's criterion [1, Proposition 1.1.9], we see that (e_n) is a Schauder basis for X . We let P_m denote the corresponding basis projections and e_n^* the biorthogonal functionals. By the implication (vi) \Rightarrow (i) of [12, Theorem 3.1], we find that, for all $x \in X$, $P_m x \xrightarrow{u} x$, i.e., that $x =^u \sum_{n=1}^{\infty} e_n^*(x) e_n$. On the other hand, to see that this expansion is unique, note that, if $x =^u \sum_{n=1}^{\infty} a_n e_n$ for some sequence (a_n) , then also $x =^{\|\cdot\|} \sum_{n=1}^{\infty} a_n e_n$, which in turn implies that $a_n = e_n^*(x)$ for all n . This shows that (e_n) is also a uniform basis for X and hence verifies the implication (3) \Rightarrow (1).

Now, assume instead that (e_n) is a uniform basis for X . Then, by Theorem 2.2, (e_n) is also a Schauder basis for X . Let again P_m denote the corresponding basis projections. Then, for all $x \in X$, $P_m x \xrightarrow{u} x$, which implies that the sequence $(P_m x)$ is order bounded. Thus (1) \Rightarrow (2).

Finally, the implication (2) \Rightarrow (3) is a direct consequence of [12, Theorem 3.1]. \square

Proof of Corollary 1.5. By Theorem 1.2, if (e_n) is an order basis for X , it is also a uniform basis for X . Note also, that because (e_n) is an order basis, $0 =^\circ \sum_{n=1}^{\infty} 0 e_n$ must be the unique order expansion of 0. This shows that (1) \Rightarrow (2).

Conversely, if (2) holds, then, by Theorem 1.2, (e_n) is also a Schauder basis and must satisfy condition (3) of Corollary 1.3. So, if e_n^* denote the biorthogonal functionals associated to the Schauder basis (e_n) , then by [12, Theorem 2.1] we have that $x =^\circ \sum_{n=1}^{\infty} e_n^*(x) e_n$ for all $x \in X$. To see that this order expansion of x is unique, note that, if $x =^\circ \sum_{n=1}^{\infty} a_n e_n$ for some sequence (a_n) , then $0 =^\circ \sum_{n=1}^{\infty} (e_n^*(x) - a_n) e_n$ and so $a_n = e_n^*(x)$ by the uniqueness of the order expansion for 0. Thus, (e_n) is an order basis for X . \square

Remark 2.6. As a consequence of the above discussion, it follows that a sequence (e_n) of non-zero vectors in a Banach lattice X is u-basic if and only if the inequality in Theorem 2.5 (vi) holds. This shows that the assumption that $X = [e_n]$ is not required for the equivalence of the statements in Corollary 1.3. Moreover, it yields a significant generalization of Grunblum's criterion [1, Proposition 1.1.9] for Schauder

basic sequences. Indeed, if (e_n) is a sequence of non-zero vectors in a Banach space E , then we may always view E as contained in the Banach lattice $X = C(B_{E^*})$. In $C(K)$ -spaces, it is clear that uniform convergence agrees with norm convergence – hence the notions of u-basic and Schauder basic coincide – and the supremum in Theorem 2.5 (vi) commutes with the norm. Therefore, we recover the standard Grunblum criterion [1, Proposition 1.1.9] in the particular case $X = C(B_{E^*})$.

3. PROOFS FOR FILTER BASES

In the following, we shall identify the powerset $\mathcal{P}(\mathbb{N})$ with the Cantor space $\{0, 1\}^{\mathbb{N}}$. If (e_n) is any sequence in a Banach space X , we may define a Borel measurable function

$$X \times \mathbb{R}^{\mathbb{N}} \times \mathbb{R}_+ \xrightarrow{\theta} \mathcal{P}(\mathbb{N})$$

by letting, for all $x \in X$, $(a_n) \in \mathbb{R}^{\mathbb{N}}$ and $\epsilon > 0$,

$$(2) \quad \theta(x, (a_n), \epsilon) = \left\{ m \in \mathbb{N} \mid \left\| x - \sum_{n=1}^m a_n e_n \right\| < \epsilon \right\}.$$

Assume now that (e_n, e_n^\sharp) is a fixed biorthogonal system in X . Then a filter F on \mathbb{N} (always assumed to be proper and containing the Fréchet filter of all cofinite sets) is compatible with (e_n, e_n^\sharp) if, for all $x \in X$,

$$\sum_{n=1}^m e_n^\sharp(x) e_n \xrightarrow{m \rightarrow F} x$$

and, for all sequences $(a_n) \in \mathbb{R}^{\mathbb{N}}$ other than $\vec{0} = (0, 0, \dots)$, we have

$$\sum_{n=1}^m a_n e_n \not\xrightarrow{m \rightarrow F} 0.$$

Indeed, these two conditions taken together ensure that $\sum_{n=1}^{\infty} e_n^\sharp(x) e_n$ is the unique F -expansion of an element $x \in X$. Rewriting these conditions in terms of θ , we find that the filter F is compatible with (e_n, e_n^\sharp) if and only if

$$(3) \quad \theta(x, (e_n^\sharp(x)), \epsilon) \in F$$

for all $x \in X$ and $\epsilon > 0$ and, moreover, F satisfies the property Φ defined by

$$(4) \quad \Phi(F) \Leftrightarrow \forall (a_n) \in \mathbb{R}^{\mathbb{N}} \setminus \{\vec{0}\} \exists k \in \mathbb{N} \quad \theta(0, (a_n), \frac{1}{k}) \notin F.$$

To simplify notation, if (e_n, e_n^\sharp) is a biorthogonal system, we let $X \xrightarrow{E} \mathbb{R}^{\mathbb{N}}$ denote the biorthogonal operator $Ex = (e_n^\sharp(x))$. In particular, E is continuous if and only if all the e_n^\sharp are continuous.

Lemma 3.1. *Every filter basis system (e_n, e_n^\sharp) for a Banach space X has a smallest compatible filter.*

Proof. Observe that

$$(5) \quad A = \left\{ a \in \mathcal{P}(\mathbb{N}) \mid \bigcap_{i=1}^m \theta(x_i, Ex_i, \epsilon) \subseteq a \text{ for some } x_i \in X \text{ and } \epsilon > 0 \right\}$$

is the smallest filter on \mathbb{N} containing all images $\theta(x, Ex, \epsilon)$ for $x \in X$ and $\epsilon > 0$. In particular, A is contained in every compatible filter. On the other hand, if F is a compatible filter and $(a_n) \neq \vec{0}$, then there is some k so that

$$\theta(0, (a_n), \frac{1}{k}) \notin F \supseteq A,$$

which shows that $\Phi(A)$ holds and hence that A is a compatible filter for (e_n, e_n^\sharp) . \square

Note that, if E is continuous, then the smallest compatible filter A (see Equation (5)) is analytic when viewed as a subset of $\mathcal{P}(\mathbb{N})$. This is [6, Theorem B]. Observe also that, if F is a compatible analytic filter, then E has analytic graph,

$$(x, (a_n)) \in \mathcal{G}E \Leftrightarrow \forall k \in \mathbb{N} \quad \theta(x, (a_n), \frac{1}{k}) \in F,$$

and thus is Borel measurable [10, Theorem 14.12] and therefore continuous [10, Theorem 9.10]. This is [6, Theorem A].

Proof of Theorem 1.17. Let (e_n, e_n^\sharp) be a fixed filter basis system for $X = [e_n]$. We remark that the implications (4) \Rightarrow (1) \Rightarrow (3) have been noted above. Also, the implications (5) \Rightarrow (4) and (1) \Rightarrow (2) are trivial, so it suffices to show (2) \Rightarrow (1) and (3) \Rightarrow (5).

(2) \Rightarrow (1): Assume that (e_n) is minimal, i.e., that $e_k \notin [e_n]_{n \neq k}$ for all k . This means that there is a set (e_k^*) of continuous functionals e_k^* on $X = [e_n]$ biorthogonal to the vectors e_n . Now, for each $x \in X$, the series $\sum_{n=1}^{\infty} e_n^\sharp(x)e_n$ converges in norm along a filter to x and so, for some increasing sequence (m_i) , we have that

$$x = \lim_{i \rightarrow \infty} \sum_{n=1}^{m_i} e_n^\sharp(x)e_n,$$

whereby

$$\begin{aligned} e_k^*(x) &= e_k^* \left(\lim_{i \rightarrow \infty} \sum_{n=1}^{m_i} e_n^\sharp(x)e_n \right) \\ &= \lim_{i \rightarrow \infty} e_k^* \left(\sum_{n=1}^{m_i} e_n^\sharp(x)e_n \right) \\ &= \lim_{i \rightarrow \infty} \sum_{n=1}^{m_i} e_n^\sharp(x)e_k^*(e_n) \\ &= e_k^\sharp(x) \end{aligned}$$

for all k . So, $e_k^\sharp = e_k^*$ is continuous for each k .

(3) \Rightarrow (5): Assume that the minimal compatible filter A is analytic. We define a binary predicate Ψ on subsets of $\mathcal{P}(\mathbb{N})$ by letting for $B, C \subseteq \mathcal{P}(\mathbb{N})$

$$\begin{aligned} \Psi(B, C) &\Leftrightarrow \forall x \subseteq y \subseteq \mathbb{N} (x \in B \rightarrow y \notin C) \ \& \\ &\quad \forall x, y \subseteq \mathbb{N} (x, y \in B \rightarrow x \cap y \notin C) \ \& \\ &\quad \forall x \subseteq \mathbb{N} (x \text{ is cofinite} \rightarrow x \notin C) \ \& \\ &\quad \emptyset \notin B. \end{aligned}$$

Observe that, if $\sim F$ denotes the complement of a set $F \subseteq \mathcal{P}(\mathbb{N})$, then $\Psi(F, \sim F)$ holds if and only if F is a proper filter on \mathbb{N} containing all cofinite sets.

Consider now the conjunction

$$\Gamma(B, C) \Leftrightarrow \Phi(B) \ \& \ \Psi(B, C),$$

where Φ is defined as in (4), and observe that Γ is a *hereditary* predicate in both variables, i.e., passes to subsets, and is *continuous upwards in the second variable*, i.e., if $C_1 \subseteq C_2 \subseteq \dots$ and $\Gamma(B, C_n)$ hold for all n , then also $\Gamma(B, \bigcup_n C_n)$. Furthermore, Γ is Π_1^1 on Σ_1^1 . That is, if Y is a Polish space and $B, C \subseteq Y \times \mathcal{P}(\mathbb{N})$ are Σ_1^1 , then

$$\{y \in Y \mid \Gamma(B_y, C_y) \text{ holds}\}$$

is Π_1^1 .

By the discussion above, we see that, if $A \subseteq F \subseteq \mathcal{P}(\mathbb{N})$, then F is a compatible filter if and only if $\Gamma(F, \sim F)$. In particular, $\Gamma(A, \sim A)$ and hence, by the Second Reflection Theorem [10, Theorem 35.16], there is some Borel set $F \subseteq \mathcal{P}(\mathbb{N})$ so that $A \subseteq F$ and $\Gamma(F, \sim F)$. Thus, F is a compatible Borel filter. \square

Remark 3.2. Observe that, if the biorthogonal operator $X \xrightarrow{E} \mathbb{R}^{\mathbb{N}}$ associated with some F -filter basis (e_n) for X is continuous, then the operator range $E[X]$ is the continuous injective image of a separable Banach space and is therefore a Borel linear subspace of $\mathbb{R}^{\mathbb{N}}$ [10, Theorem 15.1]. However, if F is actually Borel, we have explicit bounds on the Borel complexity of $E[X]$ in terms of the Borel complexity of F . Indeed,

$$(a_n) \in E[X] \Leftrightarrow \forall l \exists k \left\{ m \in \mathbb{N} \mid \left\| \sum_{n=1}^m a_n e_n - \sum_{n=1}^k a_n e_n \right\| < \frac{1}{l} \right\} \in F.$$

To see this, note that the implication from left to right is immediate. For the implication from right to left, note that, if k_l are such that

$$\left\{ m \in \mathbb{N} \mid \left\| \sum_{n=1}^m a_n e_n - \sum_{n=1}^{k_l} a_n e_n \right\| < \frac{1}{l} \right\} \in F$$

for all l , then $(\sum_{i=1}^{k_l} a_n e_n)_l$ is Cauchy and converges to some x so that $Ex = (a_n)$.

4. HIGHER ORDER FATOU PROPERTIES

We now return to the setting of Banach lattices and investigate conditions under which the assumptions of projective or analytic determinacy may be eliminated from Theorem 1.2. Recall that a π -basis for a Banach lattice X is a subset $B \subseteq X$ for which $b > 0$ for all $b \in B$ and so that, for all $x > 0$, there is $b \in B$ with $b < x$. Observe that, for example, $C([0, 1])$ and the sequence spaces c_0 and ℓ_p , $1 \leq p \leq \infty$, all have countable π -bases, while $L_1[0, 1]$ fails to have a countable π -basis.

Lemma 4.1. *Suppose X is a separable Banach lattice with a countable π -basis B . Then, for all sequences (x_n) and vectors x , we have*

$$\begin{aligned} x_n \xrightarrow{\circ} x &\Leftrightarrow x_n \xrightarrow{\sigma \circ} x \\ &\Leftrightarrow \forall b \in B \exists z (\forall^\infty n |x_n - x| \leq z \ \& \ b \not\leq z). \end{aligned}$$

Thus, order and σ -order convergence of sequences coincide and define an analytic relation on (x_n) and x .

Proof. Assume first that $x_n \xrightarrow[n]{\circ} x$. Then there is a decreasing net (z_μ) with infimum 0 so that every z_μ bounds all but finitely many of the expressions $|x_n - x|$. In particular, if $b \in B$, then we have that $b \not\leq z_\mu$ for some μ , which shows that

$$\forall b \in B \exists z (\forall^\infty n |x_n - x| \leq z \ \& \ b \not\leq z).$$

Assume now, in turn, that $\forall b \in B \exists z (\forall^\infty n |x_n - x| \leq z \ \& \ b \not\leq z)$. Enumerate B as $B = \{b_1, b_2, \dots\}$ and, for each k , choose some z_k so that $\forall^\infty n |x_n - x| \leq z_k$, whereas $b_k \not\leq z_k$. Let also $y_m = \bigwedge_{k=1}^m z_k$. Then $y_m \downarrow 0$ and, for every m , we have $|x_n - x| \leq y_m$ for all but finitely many n , i.e., $x_n \xrightarrow[n]{\sigma\circ} x$. As σ -order convergence implies order convergence, this finishes the proof. \square

Corollary 4.2. *Suppose X is a separable Banach lattice with a countable π -basis. Then Theorem 1.2 holds without the additional assumption of projective determinacy.*

We now turn to other weaker conditions than the existence of a countable π -basis. So, in the following, let X denote a fixed separable Banach lattice. Let us begin by recalling the following simple calculation.

Lemma 4.3. *For all $x, y \in X$, we have*

$$(y - x)^+ = y - (x \wedge y).$$

Proof. Recall that $+$ distributes over the lattice operation \vee , so

$$y - (x \wedge y) = y + ((-x) \vee (-y)) = (y - x) \vee 0 = (y - x)^+$$

for all $x, y \in X$. \square

If Σ is any set, we let $\Sigma^{<\mathbb{N}}$ denote the set of all finite strings of elements of Σ . Recall that a subset $T \subseteq \Sigma^{<\mathbb{N}}$ is a *tree* provided that T contains the empty string \emptyset and is closed under taking initial segments. Recall also that a tree T is said to be *ill-founded* provided that it has an *infinite branch*, i.e., if there is an infinite sequence $(y_n)_{n=1}^\infty$ in Σ so that

$$(y_1, \dots, y_n) \in T$$

for all n . Otherwise, T is *well-founded*. For a well-founded tree T , we may define an ordinal valued rank function $\rho_T: T \rightarrow \text{Ord}$ by letting

$$\rho_T(s) = 0 \Leftrightarrow s \text{ has no proper extensions in } T$$

and otherwise

$$\rho_T(s) = \sup \{ \rho_T(t) + 1 \mid t \in T \ \& \ s \subsetneq t \}.$$

We also define the *rank* of T itself by

$$\rho(T) = \sup \{ \rho_T(t) + 1 \mid t \in T \} = \rho_T(\emptyset) + 1.$$

A binary relation \prec on a set Ω is said to be *well-founded* if there is no infinite sequence of elements $p_n \in \Omega$ so that

$$\dots \prec p_3 \prec p_2 \prec p_1.$$

In this case, we may similarly define a rank $\rho_\prec: \Omega \rightarrow \text{Ord}$ by

$$\rho_\prec(p) = 0 \Leftrightarrow p \text{ is minimal, i.e., } q \not\prec p \text{ for all } q \in \Omega$$

and

$$\rho_\prec(p) = \sup \{ \rho_\prec(q) + 1 \mid q \prec p \}.$$

As for trees, we set $\rho(\prec) = \sup \{ \rho_\prec(p) + 1 \mid p \in \Omega \}$. For example, if T is a well-founded tree, we may let \prec be the relation \supseteq on T , that is, for $s, t \in T$, we have $s \prec t$ if t is a proper initial segment of s . Then $\rho_T = \rho_\prec$.

We let Tr_Σ denote the set of all trees $T \subseteq \Sigma^{<\mathbb{N}}$ and WF_Σ denote the subset of all well-founded trees. If Σ is a countable set, then Tr_Σ is a closed subset of the Polish space $\{0, 1\}^{\Sigma^{<\mathbb{N}}}$, whereas WF_Σ is a coanalytic subset of Tr_Σ . Moreover, by [10, Exercise 34.6], ρ is a coanalytic rank on WF_Σ . In particular, this means that, for all $\lambda < \omega_1$, the set

$$WF_\Sigma^\lambda = \{T \in Tr_\Sigma \mid \rho(T) \leq \lambda\}$$

is Borel.

Set

$$X_\downarrow = \{(z_n)_{n=1}^\infty \in X^\mathbb{N} \mid z_1 \geq z_2 \geq \dots \geq 0\}$$

and define, for every sequence $(z_n) \in X_\downarrow$, a tree $\Psi((z_n)) \subseteq (X_+ \setminus \{0\})^{<\mathbb{N}}$ by

$$(y_1, \dots, y_k) \in \Psi((z_n)) \Leftrightarrow$$

$$\forall i \forall n \quad \|(y_i - z_n)^+\| < \frac{\|y_i\|}{2^i} \quad \& \quad \|y_{i+1} - y_i\| < \frac{\|y_i\|}{2^i}.$$

Lemma 4.4. *Let $P \subseteq X_+ \setminus \{0\}$ be a countable norm-dense subset. Then, for all $(z_n)_{n=1}^\infty \in X_\downarrow$, the following conditions are equivalent.*

- (1) $\inf_n z_n = 0$,
- (2) $\Psi((z_n))$ is well-founded,
- (3) $\Psi((z_n)) \cap P^{<\mathbb{N}}$ is well-founded.

Proof. Note first that the implication (2) \Rightarrow (3) is trivial.

(3) \Rightarrow (1): Assume that 0 is not the infimum of the sequence (z_n) , that is, that there is some $y \in X_+$ satisfying

$$z_1 \geq z_2 \geq \dots \geq y > 0.$$

Pick then elements $y_i \in B(y, \frac{\|y\|}{4^i}) \cap P$, which implies that

$$\|y_{i+1} - y_i\| \leq \|y_{i+1} - y\| + \|y - y_i\| < \frac{\|y\|}{4^{i+1}} + \frac{\|y\|}{4^i} \leq \frac{\|y\|}{2^i}$$

for all i . Furthermore,

$$\|(y_i - z_n)^+\| \leq \|(y - z_n)^+\| + \|y - y_i\| < \frac{\|y_i\|}{2^i},$$

showing that (y_1, y_2, \dots) is an infinite branch of $\Psi((z_n)) \cap P^{<\mathbb{N}}$.

(1) \Rightarrow (2): Suppose that $\Psi((z_n))$ is ill-founded, i.e., that $\Psi((z_n))$ has an infinite branch (y_1, y_2, \dots) . Then the conditions $\|y_{i+1} - y_i\| < \frac{\|y_i\|}{2^i}$ ensure that (y_i) is a Cauchy sequence converging to some $y > 0$. Furthermore, as $\|(y_i - z_n)^+\| < \frac{\|y_i\|}{2^i}$ for all n and i , we find that also

$$\|y - (z_n \wedge y)\| = \|(y - z_n)^+\| = 0,$$

i.e., that $y \leq z_n$ for all n . So, y is a strictly positive lower bound for (z_n) . \square

Lemma 4.5. *The following conditions are equivalent.*

- (1) The set

$$X_{\downarrow 0} = \{(z_n)_{n=1}^\infty \in X_\downarrow \mid \inf_n z_n = 0\}$$

is Borel,

$$(2) \quad \sup \left\{ \rho \left(\Psi((z_n)) \right) \mid (z_n)_{n=1}^\infty \in X_{\downarrow 0} \right\} < \omega_1.$$

Proof. (1) \Rightarrow (2): Note that, if $X_{\downarrow 0}$ is Borel, so is the set

$$\Omega = \left\{ ((z_n)_{n=1}^\infty, (y_1, \dots, y_m)) \in X_{\downarrow 0} \times (X_+ \setminus \{0\})^{<\mathbb{N}} \mid (y_1, \dots, y_m) \in \Psi((z_n)) \right\}.$$

We may then define a Borel quasiordering \prec on Ω by setting

$$\begin{aligned} ((z_n)_{n=1}^\infty, (y_1, \dots, y_m)) &\prec ((u_n)_{n=1}^\infty, (v_1, \dots, v_k)) \\ \Leftrightarrow (z_n)_{n=1}^\infty &= (u_n)_{n=1}^\infty \quad \& \quad k < m \quad \& \quad (v_1, \dots, v_k) = (y_1, \dots, y_k) \end{aligned}$$

and observe that \prec is well-founded by Lemma 4.4. It then follows from the boundedness theorem for analytic well-founded relations [10, Theorem 31.1] that

$$\begin{aligned} \sup \left\{ \rho \left(\Psi((z_n)) \right) \mid (z_n)_{n=1}^\infty \in X_{\downarrow 0} \right\} &= \sup \left\{ \rho_{\Psi((z_n))}(\emptyset) + 1 \mid (z_n)_{n=1}^\infty \in X_{\downarrow 0} \right\} \\ &= \sup \left\{ \rho_{\prec}((z_n), \emptyset) + 1 \mid (z_n)_{n=1}^\infty \in X_{\downarrow 0} \right\} \\ &= \rho(\prec) \\ &< \omega_1. \end{aligned}$$

(2) \Rightarrow (1): Observe that, if $P \subseteq X_+ \setminus \{0\}$ is a fixed countable norm-dense subset and

$$\sup \left\{ \rho \left(\Psi((z_n)) \right) \mid (z_n)_{n=1}^\infty \in X_{\downarrow 0} \right\} < \omega_1,$$

then also

$$\lambda = \sup \left\{ \rho \left(\Psi((z_n)) \cap P^{<\mathbb{N}} \right) \mid (z_n)_{n=1}^\infty \in X_{\downarrow 0} \right\} < \omega_1.$$

Note now that the map $X_{\downarrow} \xrightarrow{\Theta} Tr_P$ defined by

$$\Theta((z_n)) = \Psi((z_n)) \cap P^{<\mathbb{N}}$$

is Borel measurable and satisfies

$$(z_n) \in X_{\downarrow 0} \quad \Leftrightarrow \quad \Theta((z_n)) \in WF_P^\lambda.$$

Because WF_P^λ is Borel, this shows that also $X_{\downarrow 0}$ is Borel. \square

For every ordinal α and $(z_n)_{n=1}^\infty \in X_{\downarrow}$, we define a game $G_\alpha[(z_n)]$ between two players I and II as follows. Players I and II alternate in playing ordinals β_i and vectors $y_i \in X_+ \setminus \{0\}$,

$$\begin{array}{cccccc} \mathbf{I} & \beta_1 & \beta_2 & \dots & \beta_{k-1} & \beta_k \\ \mathbf{II} & y_1 & y_2 & \dots & y_{k-1} & y_k \end{array}$$

and where the positions played are subject to the conditions

$$\alpha > \beta_1 > \beta_2 > \dots > \beta_{k-1} > \beta_k \geq 0$$

and

$$\|y_{i+1} - y_i\| < \frac{\|y_i\|}{2^i}.$$

The game ends when I plays $\beta_k = 0$ and II plays its response y_k . This will eventually happen as the ordinals are well-ordered. Player II is then said to *win* a run of the game provided that

$$\|(y_i - z_n)^+\| < \frac{\|y_i\|}{2^i}$$

for all $n \geq 1$ and $1 \leq i \leq k$. Otherwise player I wins.

Example 4.6 (Banach lattices with the Fatou property). Suppose X is a Banach lattice with the Fatou property, that is, whenever we have elements $0 \leq x_1 \leq x_2 \leq \dots \leq x$ with $x = \sup_n x_n$, then $\|x\| = \sup_n \|x_n\|$. Assume also that $(z_n) \in X_{\downarrow 0}$. Then it is easy to see that I has a winning strategy in the game $G_1[(z_n)]$. Indeed, I simply plays $\beta_1 = 0$, to which II responds with some vector y_1 . If II wins this run of the game, we must have

$$\|y_1 - (z_n \wedge y_1)\| = \|(y_1 - z_n)^+\| < \frac{\|y_1\|}{2}$$

for all n and

$$0 \leq y_1 - (z_1 \wedge y_1) \leq y_1 - (z_2 \wedge y_1) \leq y_1 - (z_3 \wedge y_1) \leq \dots \leq y_1 = \sup_n (y_1 - (z_n \wedge y_1)),$$

which contradicts the Fatou property. This means that a winning strategy for I in $G_1[(z_n)]$ is simply to play $\beta_1 = 0$.

Lemma 4.7. *For every $(z_n)_{n=1}^\infty \in X_{\downarrow}$ and every ordinal α ,*

$$\rho(\Psi((z_n))) \leq \alpha \quad \Leftrightarrow \quad \text{I has a winning strategy in the game } G_\alpha[(z_n)].$$

Proof. A straightforward inspection shows that

$$\rho(\Psi((z_n))) > \alpha \quad \Leftrightarrow \quad \text{II has a winning strategy in the game } G_\alpha[(z_n)].$$

Note also that, because the rules and winning conditions in both games are Borel, the games are determined, that is, either player I or II has a winning strategy [10, Theorem 20.6]. Therefore,

$$\begin{aligned} \rho(\Psi((z_n))) \leq \alpha &\quad \Leftrightarrow \quad \text{II has no winning strategy in the game } G_\alpha[(z_n)] \\ &\quad \Leftrightarrow \quad \text{I has a winning strategy in the game } G_\alpha[(z_n)] \end{aligned}$$

as claimed. \square

Definition 4.8. *The separable Banach lattice X is said to be α -Fatou provided that, for all $(z_n) \in X_{\downarrow}$,*

$$\begin{aligned} \inf_n z_n = 0 &\quad \Leftrightarrow \quad \text{I has a winning strategy in the game } G_\alpha[(z_n)] \\ &\quad \Leftrightarrow \quad \rho(\Psi((z_n))) \leq \alpha. \end{aligned}$$

By Example 4.6, Banach lattices with the Fatou property are 1-Fatou.

Proof of Theorem 1.18. (1) \Rightarrow (2): Suppose that (1) holds, i.e., that $X_{\downarrow 0}$ is Borel. Then, for all sequences (x_n) and vectors x , we have

$$x_n \xrightarrow[n]{\sigma\sigma} x \quad \Leftrightarrow \quad \exists(z_n) \in X_{\downarrow 0} \quad \forall m \quad \forall^\infty n \quad x_n \leq z_m,$$

which is clearly Σ_1^1 .

(2) \Rightarrow (1): Observe that, for a sequence $x_1 \geq x_2 \geq \dots \geq 0$, we have

$$\forall y > 0 \exists n \quad y \not\leq x_n \quad \Leftrightarrow \quad \inf_n x_n = 0 \quad \Leftrightarrow \quad x_n \xrightarrow[n]{\sigma\sigma} 0.$$

The first expression is clearly Π_1^1 , so, if the last is Σ_1^1 , then these equivalent expressions are all Borel and hence $X_{\downarrow 0}$ is a Borel set.

(1) \Leftrightarrow (3): Just observe that, by Lemma 4.5, the set $X_{\downarrow 0}$ is Borel if and only if there is some $\alpha < \omega_1$ so that

$$\rho\left(\Psi((z_n))\right) \leq \alpha$$

for all $(z_n) \in X_{\downarrow 0}$. \square

Let us also note [8, Theorem 2.6]. The authors there operate with a slightly stronger Fatou property, namely, the conjunction of σ -monotonic completeness and what we have termed the Fatou property here. Thus, translated in to our terminology, [8, Theorem 2.6] states that a sequence (e_n) in a σ -monotonically complete, σ -order continuous Banach lattice $X = [e_n]$ with the Fatou property is simultaneously a Schauder and a σ -order basis for X if and only if it satisfies condition (3) of Corollary 1.3.

We note that, in fact, a stronger result holds. Indeed, if (e_n) is a sequence in a σ -order continuous Banach lattice $X = [e_n]$, then σ -order bases are just uniform bases and the latter are automatically Schauder bases. So, (e_n) is simultaneously a Schauder and a σ -order basis for X if and only if it satisfies condition (3) of Corollary 1.3.

5. EXAMPLES OF SPACES WITH HIGHER ORDER FATOU PROPERTIES

Our next task is to show that the hierarchy of α -Fatou properties does not collapse. That is, we will construct spaces that are α -Fatou, but only for larger and larger $\alpha < \omega_1$.

Theorem 5.1. *For every countable ordinal α , there is a separable Banach lattice with a countable π -basis that fails to be α -Fatou.*

Proof. Our proof goes by induction on $1 \leq \alpha < \omega_1$. For each α , we will construct a separable Banach lattice $(X_\alpha, \|\cdot\|_\alpha)$ with a countable π -basis $B_\alpha \subseteq X_\alpha$ so that X_α fails the α -Fatou property, together with a Banach lattice homomorphism

$$X_\alpha \xrightarrow{\phi_\alpha} \mathbb{R}$$

of norm 1. Furthermore, setting

$$S_\alpha = \{x \in X_\alpha^+ \mid \|x\|_\alpha = \phi_\alpha(x) = 1\},$$

we will construct a sequence in S_α ,

$$z_1^\alpha \geq z_2^\alpha \geq \dots > 0 = \inf_n z_n^\alpha$$

and a tree $T_\alpha \subseteq \Psi((z_n^\alpha)) \cap S_\alpha^{<\mathbb{N}}$ so that

$$\rho(T_\alpha) > \alpha,$$

which implies that

$$\rho\left(\Psi((z_n^\alpha))\right) > \alpha$$

and thus that X_α fails the α -Fatou property.

Base case, $\alpha = 1$. We let X_1 be the space c of convergent sequences endowed with the equivalent renorming

$$\|(t_1, t_2, \dots)\|_1 = \max\left\{\frac{1}{3}\|(t_1, t_2, \dots)\|_{\ell_\infty}, \left|\lim_n t_n\right|\right\}$$

and define ϕ_1 by

$$\phi_1((t_1, t_2, \dots)) = \lim_n t_n.$$

Also, for each n , set

$$z_n^1 = \underbrace{(0, 0, \dots, 0)}_{n \text{ times}}, 1, 1, 1, \dots \in S_1.$$

Finally, note that because $y = (1, 1, 1, \dots) \in \Psi((z_n^\alpha))$ and $\|y\|_1 = \phi_1(y) = 1$, we may simply let

$$T_1 = \{\emptyset, y\},$$

whereby $\rho(T_1) = 2 > 1$.

Successor case. Suppose that $(X_\alpha, \|\cdot\|_\alpha)$, B_α , ϕ_α , (z_n^α) and T_α have been defined as above. We then let $X_{\alpha+1} = X_\alpha$ with the new norm

$$\|x\|_{\alpha+1} = \max \left\{ \frac{1}{5} \|x\|_\alpha, |\phi_\alpha(x)| \right\}$$

and set $\phi_{\alpha+1} = \phi_\alpha$, $B_{\alpha+1} = B_\alpha$ and $z_n^{\alpha+1} = z_n^\alpha$ for all n . Note that $\|\cdot\|_{\alpha+1} \leq \|\cdot\|_\alpha$, $\|\phi_{\alpha+1}\| = 1$ and that

$$\|x\|_{\alpha+1} = \|x\|_\alpha \Leftrightarrow \|x\|_\alpha = |\phi_\alpha(x)|,$$

from which it follows that $S_\alpha \subseteq S_{\alpha+1}$. Finally, let

$$T_{\alpha+1} = \{(z_1^\alpha, y_1, \dots, y_n) \mid (y_1, \dots, y_n) \in T_\alpha\} \cup \{\emptyset\} \subseteq S_\alpha^{<N} \subseteq S_{\alpha+1}^{<N}$$

and note that $\rho(T_{\alpha+1}) > \rho(T_\alpha) > \alpha$. To see that $T_{\alpha+1} \subseteq \Psi((z_n^{\alpha+1}))$, note first that, for all $x, y \in S_{\alpha+1}$, we have

$$\phi_\alpha((x - y)^+) = (\phi_\alpha(x - y))^+ = 0^+ = 0 = \phi_\alpha(x - y),$$

whereby $\|(x - y)^+\|_{\alpha+1} = \frac{1}{5} \|(x - y)^+\|_\alpha$ and $\|x - y\|_{\alpha+1} = \frac{1}{5} \|x - y\|_\alpha$. Thus, for all n ,

$$\begin{aligned} \|(z_1^\alpha - z_n^{\alpha+1})^+\|_{\alpha+1} &= \frac{1}{5} \|(z_1^\alpha - z_n^{\alpha+1})^+\|_\alpha \\ &= \frac{1}{5} \|(z_1^\alpha - z_n^\alpha)^+\|_\alpha \\ &\leq \frac{1}{5} \|z_1^\alpha\|_\alpha \\ &= \frac{1}{5} \|z_1^\alpha\|_{\alpha+1} \\ &< \frac{\|z_1^\alpha\|_{\alpha+1}}{2^1} \end{aligned}$$

and

$$\begin{aligned} \|(y_i - z_n^{\alpha+1})^+\|_{\alpha+1} &= \frac{1}{5} \|(y_i - z_n^{\alpha+1})^+\|_\alpha \\ &= \frac{1}{5} \|(y_i - z_n^\alpha)^+\|_\alpha \\ &< \frac{1}{5} \frac{\|y_i\|_\alpha}{2^i} \\ &< \frac{\|y_i\|_{\alpha+1}}{2^{i+1}}. \end{aligned}$$

Similarly,

$$\|y_1 - z_1^\alpha\|_{\alpha+1} = \frac{1}{5} \|y_1 - z_1^\alpha\|_\alpha \leq \frac{2}{5} < \frac{\|z_1^\alpha\|_{\alpha+1}}{2^1}$$

and

$$\|y_{i+1} - y_i\|_{\alpha+1} = \frac{1}{5} \|y_{i+1} - y_i\|_{\alpha} < \frac{1}{5} \frac{\|y_i\|_{\alpha}}{2^i} < \frac{\|y_i\|_{\alpha+1}}{2^{i+1}}.$$

Limit case. Suppose that $\alpha_1 < \alpha_2 < \dots < \alpha = \lim_n \alpha_n$. Assume also that the construction has been done for all ordinals smaller than α . We then let

$$X_{\alpha} = \left\{ x = (x_1, x_2, \dots) \in \prod_n X_{\alpha_n} \mid \lim_n \phi_{\alpha_n}(2^n x_n) \text{ exists} \ \& \ \lim_n \|x_n\|_{\alpha_n} = 0 \right\}$$

equipped with the norm

$$\|x\|_{\alpha} = \max \left\{ \sup_n \|x_n\|_{\alpha_n}, \left| \lim_n \phi_{\alpha_n}(2^n x_n) \right| \right\}$$

and define the homomorphism ϕ_{α} by

$$\phi_{\alpha}(x) = \lim_n \phi_{\alpha_n}(2^n x_n).$$

Observe that the set B_{α} of elements of the form $(0, \dots, 0, b, 0, 0, \dots)$ with $b \in B_{\alpha_n}$ in the n th position forms a countable π -basis for X_{α} . Furthermore, for every k , we define

$$z_k^{\alpha} = \left(\frac{z_k^{\alpha_1}}{2^1}, \frac{z_k^{\alpha_2}}{2^2}, \frac{z_k^{\alpha_3}}{2^3}, \dots \right) \in S_{\alpha}$$

and remark that $z_1^{\alpha} \geq z_2^{\alpha} \geq \dots > 0 = \inf_k z_k^{\alpha}$.

Now, suppose that $k \in \mathbb{N}$ is given and that, for every $m \geq k$, we have some vector $y_m \in S_{\alpha_m}$. Then also

$$\left(\underbrace{0, 0, \dots, 0}_{k-1}, \frac{y_k}{2^k}, \frac{y_{k+1}}{2^{k+1}}, \frac{y_{k+2}}{2^{k+2}}, \dots \right) \in S_{\alpha}.$$

We therefore let T_{α} be the tree consisting of all finite strings of the form

$$\left(\left(\underbrace{0, 0, \dots, 0}_{k-1}, \frac{y_k^1}{2^k}, \frac{y_{k+1}^1}{2^{k+1}}, \frac{y_{k+2}^1}{2^{k+2}}, \dots \right), \dots, \left(\underbrace{0, 0, \dots, 0}_{k-1}, \frac{y_k^n}{2^k}, \frac{y_{k+1}^n}{2^{k+1}}, \frac{y_{k+2}^n}{2^{k+2}}, \dots \right) \right),$$

where $k, n \in \mathbb{N}$ and

$$(y_m^1, y_m^2, \dots, y_m^n) \in T_{\alpha_m}$$

for all $m \geq k$. To see that $T_{\alpha} \subseteq \Psi((z_m^{\alpha}))$, observe first that, for all $1 \leq i < n$,

$$\begin{aligned} & \left\| \left(\underbrace{0, 0, \dots, 0}_{k-1}, \frac{y_k^{i+1}}{2^k}, \frac{y_{k+1}^{i+1}}{2^{k+1}}, \frac{y_{k+2}^{i+1}}{2^{k+2}}, \dots \right) - \left(\underbrace{0, 0, \dots, 0}_{k-1}, \frac{y_k^i}{2^k}, \frac{y_{k+1}^i}{2^{k+1}}, \frac{y_{k+2}^i}{2^{k+2}}, \dots \right) \right\|_{\alpha} \\ &= \left\| \left(\underbrace{0, 0, \dots, 0}_{k-1}, \frac{y_k^{i+1} - y_k^i}{2^k}, \frac{y_{k+1}^{i+1} - y_{k+1}^i}{2^{k+1}}, \frac{y_{k+2}^{i+1} - y_{k+2}^i}{2^{k+2}}, \dots \right) \right\|_{\alpha} \\ &= \sup_{m \geq k} \left\| \frac{y_m^{i+1} - y_m^i}{2^m} \right\|_{\alpha_m} \\ &< \sup_{m \geq k} \frac{1}{2^i} \left\| \frac{y_m^i}{2^m} \right\|_{\alpha_m} \\ &\leq \frac{1}{2^i} \left\| \left(\underbrace{0, 0, \dots, 0}_{k-1}, \frac{y_k^i}{2^k}, \frac{y_{k+1}^i}{2^{k+1}}, \frac{y_{k+2}^i}{2^{k+2}}, \dots \right) \right\|_{\alpha}. \end{aligned}$$

Similarly, for all k and m ,

$$\begin{aligned}
 & \left\| \left(\underbrace{(0, 0, \dots, 0)}_{k-1}, \frac{y_k^i}{2^k}, \frac{y_{k+1}^i}{2^{k+1}}, \frac{y_{k+2}^i}{2^{k+2}}, \dots \right) - z_m^\alpha \right\|_\alpha^+ \\
 &= \left\| \left(\underbrace{(0, 0, \dots, 0)}_{k-1}, \frac{y_k^i}{2^k}, \frac{y_{k+1}^i}{2^{k+1}}, \frac{y_{k+2}^i}{2^{k+2}}, \dots \right) - \left(\frac{z_m^{\alpha_1}}{2^1}, \frac{z_m^{\alpha_2}}{2^2}, \frac{z_m^{\alpha_3}}{2^3}, \dots \right) \right\|_\alpha^+ \\
 &= \left\| \left(\underbrace{(0, 0, \dots, 0)}_{k-1}, \frac{(y_k^i - z_m^{\alpha_k})^+}{2^k}, \frac{(y_{k+1}^i - z_m^{\alpha_{k+1}})^+}{2^{k+1}}, \frac{(y_{k+2}^i - z_m^{\alpha_{k+2}})^+}{2^{k+2}}, \dots \right) \right\|_\alpha \\
 &= \sup_{r \geq k} \left\| \frac{(y_r^i - z_m^{\alpha_r})^+}{2^r} \right\|_{\alpha_r} \\
 &< \sup_{r \geq k} \frac{1}{2^r} \left\| \frac{y_r^i}{2^r} \right\|_{\alpha_r} \\
 &\leq \frac{1}{2^i} \left\| \left(\underbrace{(0, 0, \dots, 0)}_{k-1}, \frac{y_k^i}{2^k}, \frac{y_{k+1}^i}{2^{k+1}}, \frac{y_{k+2}^i}{2^{k+2}}, \dots \right) \right\|_\alpha.
 \end{aligned}$$

Finally, we verify that $\rho(T_\alpha) > \alpha$, i.e., that $\rho_{T_\alpha}(\emptyset) \geq \alpha$. But, if $\beta < \alpha$, find some k so that $\beta < \alpha_k$ and hence $\rho_{T_{\alpha_m}}(\emptyset) > \beta$ for all $m \geq k$. As T_α contains all strings

$$\left(\underbrace{(0, 0, \dots, 0)}_{k-1}, \frac{y_k^1}{2^k}, \frac{y_{k+1}^1}{2^{k+1}}, \frac{y_{k+2}^1}{2^{k+2}}, \dots \right), \dots, \left(\underbrace{(0, 0, \dots, 0)}_{k-1}, \frac{y_k^n}{2^k}, \frac{y_{k+1}^n}{2^{k+1}}, \frac{y_{k+2}^n}{2^{k+2}}, \dots \right),$$

where $(y_m^1, y_m^2, \dots, y_m^n) \in T_{\alpha_m}$, this shows that also $\rho_{T_\alpha}(\emptyset) > \beta$. As $\beta < \alpha$ was arbitrary, we have that $\rho_{T_\alpha}(\emptyset) \geq \alpha$. \square

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