A Collection of Results on Nonlinear Dispersive Equations, Banach Lattices and Phase Retrieval

by

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Abstract

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This thesis collects various linear and nonlinear techniques developed by the author and his collaborators to attack problems in Function Space Theory, Phase Retrieval and PDE. The thesis begins with an analysis of the *generalized derivative nonlinear Schrödinger equation*

$$\begin{cases} i\partial_t u + \partial_x^2 u = i|u|^{2\sigma}\partial_x u, \\ u(0) = u_0. \end{cases} \tag{gDNLS}$$

This is a canonical model of a quasilinear dispersive PDE, i.e., a dispersive PDE where one expects continuity but not uniform continuity of the data-to-solution map. As opposed to the semilinear case where Strichartz and contraction mapping arguments are directly applicable, the well-posedness theory for such quasilinear PDE is largely open. In Chapter 2, we study (gDNLS) when $\sigma < 1$, which is the regime where local well-posedness is hardest to establish. Our main result establishes global well-posedness in the energy space H^1, as long as σ is not too small.

In Chapter 3 we transition to the water waves problem. That is, we consider the motion of water when the interface between the water and the air is free to move. In this case, we do not consider the well-posedness problem, but rather the existence of special solutions. Our primary interest is in solitary waves, which are waves that travel across the ocean's

surface at constant speed while never changing shape. When modelling water waves, the fundamental physical parameters are the gravity, surface tension, and fluid depth. It is then an interesting question to identify which combinations of parameters lead to a given physical phenomenon. For solitary waves in two dimensions, we discuss the complete solution to the existence/non-existence problem. More specifically, we prove non-existence of solitary waves when surface tension and depth are arbitrary but gravity is zero, which was the only case that had not yet yielded a solution.

Chapter 4 is dedicated to the phase retrieval problem; that is, the determination of a function f up to unavoidable ambiguity from $|f|$. In a recent article, Calderbank, Daubechies, Freeman and Freeman dispelled of the prevailing belief that phase retrieval in infinite dimensions is inherently unstable. Motivated by this, Chapter 4 contains an extensive study of the stability of phase retrieval, for both real and complex scalars. In particular, we give the first construction of an infinite-dimensional subspace $E \subseteq L^2(\mu; \mathbb{C})$ with the property that for any $f, g \in E$, if $|f|$ is approximately equal to $|g|$ with respect to the L^2 norm, then there exists a unimodular scalar λ such that f is approximately equal to λg.

Recall that a basis of a Banach space E is a sequence (f_n) in E such that every $f \in E$ admits a unique sequence of scalars (a_n) satisfying $f = \sum_{n=1}^{\infty} a_n f_n$. The goal of Chapter 5 is to study bases (f_n) of $L^p(\mathbb{R})$ consisting entirely of non-negative functions. Such non-negative coordinate systems are of relevance in both Functional Analysis and Applied Mathematics. However, constructing them is notoriously difficult, as can be extrapolated from the following fact: For any non-negative basis (f_n) of $L^p(\mathbb{R})$ there exists a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that $(f_{\sigma(n)})$ is *not* a basis of $L^p(\mathbb{R})$. Overcoming this issue, in Chapter 5 we give the first construction of a non-negative basis of $L^2(\mathbb{R})$.

Chapter 6 is devoted to free Banach lattices. Given a Banach space E, one can generate a Banach lattice FBL$[E]$ so that every operator $T : E \to X$ into a Banach lattice X uniquely extends to FBL$[E]$ as a lattice homomorphism of the same norm. The correspondence $E \mapsto$ FBL$[E]$ provides an indispensable link between Banach space theory and Banach lattice theory. In Chapter 6, we give a convenient functional representation of FBL$[E]$ and its p-convex variants, and then deeply study these spaces. In particular, we study how properties of an operator $T : E \to F$ between Banach spaces transfer to the associated lattice homomorphism $\overline{T} : \mathrm{FBL}[E] \to \mathrm{FBL}[F]$. Special consideration is devoted to the case when the operator T is an isomorphic embedding, which leads us to examine extension properties of operators into ℓ_p, and several classical Banach space properties such as being a G.T. space. A detailed investigation of basic sequences and sublattices of free Banach lattices is also provided. Among other things, this allows us to settle an a priori unrelated question, providing the first instance of a subspace of a Banach lattice without bibasic sequences. Along the way, a dictionary between Banach space properties of E and Banach lattice properties of $\mathrm{FBL}[E]$ is assembled. For example, we characterize the existence of lattice copies of ℓ_1 in $\mathrm{FBL}[E]$ and show that $\mathrm{FBL}[E]$ has an upper p-estimate if and only if id_{E^*} is $(q,1)$-summing ($\frac{1}{p} + \frac{1}{q} = 1$).

To Grandma.

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Acknowledgments

Firstly, I would like to thank my advisor, Daniel Tataru, for introducing me to the field of dispersive and fluid equations. The weekly meetings with him were very beneficial, especially during the pandemic. In addition to learning a tremendous amount of PDE from Daniel, I learnt the standard to which one should strive for when doing mathematics. His insistence on obtaining optimal, complete results has been an important influence on all of my papers.

Secondly, I would like to thank Ben Pineau, with whom all of my PDE and phase retrieval papers are joint. Ben has been an incredible help, both as a friend and a collaborator. Many of the key insights in our joint works are due to him, and I would not have been nearly as successful without him.

I am lucky to have had several remarkable influences, both mathematical and otherwise. Two of particular note are Dan Freeman and Pedro Tradacete. I met Dan at the conference *Recent Advances in Functional Analysis* in 2018. There I found out that he had made some progress on one of the problems I was thinking about. Despite knowing I was only a first year graduate student and that he was ahead of me on the problem, he kindly offered to collaborate with me. Over the last five years I have visited Dan and his wife Nikki three times, and they have been wonderfully welcoming and kind. We have written several papers, though I will never forget how the first one began. On a similar note, I would like to thank Pedro Tradacete for all he has done for my career. As with Dan, I met Pedro when I was a beginning student, but this did not deter him from treating me as an equal. He has brought me to Spain five times, and been a wonderful collaborator. He is one of the major reasons I was a plenary speaker at the conference *Workshop on Banach spaces and Banach lattices II, ICMAT, 2022*, and is the reason I am one of the main speakers at the three week summer school *ICMAT-IMAG Doc-Course in Functional Analysis, 2023*. It is safe to say that Pedro's view towards young students and his impact on the Functional Analysis community has been particularly exemplary. Many thanks also go to Timur Oikhberg for being a wonderful collaborator, Vladimir Troitsky for introducing me to research, Sung-Jin Oh for all his help during my PhD, and the Functional Analysis team at Texas A&M for turning a two week visit into two years of mathematical material. Finally, and above all, I would like to thank my grandma. She was a remarkable role model, and I would not be who I am without her.

On a more technical note, this thesis includes results from several articles, joint with various authors. This includes:

(i) The article [277], which is titled *Global well-posedness for the generalized derivative nonlinear Schrödinger equation.* This article is the main source of material for Chapter 2 of this thesis, and is joint with Ben Pineau.

(ii) The article [177], which is titled *No pure capillary solitary waves exist in 2D finite depth.* This article is the main source of material for Chapter 3 of this thesis, and is joint with Mihaela Ifrim, Ben Pineau and Daniel Tataru.

(iii) The article [115], which is titled *Stable phase retrieval in function spaces.* This article is one of the two main sources of material for Chapter 4 of this thesis, and is joint with Dan Freeman, Timur Oikhberg and Ben Pineau.

(iv) The article [82], which is titled *Examples of Hölder-stable phase retrieval.* This article is the other main source of material for Chapter 4 of this thesis, and is joint with Michael Christ and Ben Pineau.

(v) The article [116], which is titled *A Schauder basis for L_2 consisting of non-negative functions.* This article is the main source of material for Chapter 5 of this thesis, and is joint with Dan Freeman and Alex Powell.

(vi) The article [262], which is titled *Free Banach lattices.* This article is the main source of material for Chapter 6 of this thesis, and is joint with Timur Oikhberg, Pedro Tradacete and Vladimir Troitsky.

I thank all of the above authors for their collaboration - I am very grateful for your enormous help. I would also like to thank Antonio Avilés, Gonzalo Martínez-Cervantes, Abraham Rueda Zoca and Mary Angelica Tursi for various insights and interesting discussions.

Omitted from this thesis are my articles [51, 185, 198, 314, 315, 316, 317]. I would like to express my sincerest thanks to my collaborators on these works as well. Even more importantly, I would like to thank those who are *still* collaborating with me. This includes, but is not limited to, Dan Freeman (phase retrieval), Enrique García-Sánchez (upper p-estimates and weak-L_p), Mihaela Ifrim (free boundary Euler), Denny Leung (upper p-estimates and weak-L_p), Sung-Jin Oh (Hall MHD), Timur Oikhberg (phase retrieval and free Banach lattices), Ben Pineau (phase retrieval, free boundary Euler, Hall MHD and nonlinear Schrödinger equations), Daniel Tataru (free boundary Euler) and Pedro Tradacete (Free Banach lattices, upper p-estimates and weak-L_p). Although I have decided to exclude all works-in-progress from this thesis, they represent a significant portion of the research conducted during my PhD. As before, I would like to thank all of the people I have just mentioned for their contributions to these upcoming articles. In many respects, they deserve more credit than I.

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Chapter 1

Introduction

The objective of this thesis is to present a variety of linear and nonlinear tools developed over the past five years to analyze a diverse set of problems in PDE, Functional Analysis, Harmonic Analysis and Applied Mathematics. The text below is divided into five main chapters. Each of these chapters analyzes a different problem, though the reader will notice several overarching themes. Below, we briefly sketch the main topics - detailed introductions will be given at the the beginning of each chapter. For the most part, the thesis is modular, and individual chapters can be read in any order. However, some care has been taken to make the notation consistent. A notable exception is that the symbols L^p and L_p will both be used to represent the Lebesgue spaces. Generally speaking, L^p is used in PDE contexts, and L_p in Function Space Theory.

1.1 Overview of the thesis

One of the most fundamental partial differential equations is the *nonlinear Schrödinger equation*

$$i\partial_t u + \Delta u = \pm|u|^\alpha u. \tag{1.1.1}$$

When considering the local theory of (1.1.1), the most important question is to find all $s \in \mathbb{R}$ such that (1.1.1) is locally well-posed in the L^2-based Sobolev space $H^s(\mathbb{R}^d)$. For smooth nonlinearities (i.e. $\alpha \in 2\mathbb{N}$), regularity persists, so that well-posedness in $H^{s_1}(\mathbb{R}^d)$ implies well-posedness in $H^{s_2}(\mathbb{R}^d)$ whenever $s_1 \leq s_2$. For this reason, the goal is often to find the lowest value of s such that (1.1.1) is well-posed in $H^s(\mathbb{R}^d)$. An additional benefit of a good low regularity well-posedness theory is that it often automatically leads to a global result. Indeed, conservation laws tend to be available in the spaces $L^2(\mathbb{R}^d)$ and $H^1(\mathbb{R}^d)$, meaning that in subcritical cases, a good local theory in one of these spaces often immediately propagates into global well-posedness. By now, the local theory of (1.1.1) is well understood. The reader may consult Tao's book [313] for the classical results.

A natural next step in this direction is to study the *generalized derivative nonlinear Schrödinger equation*

$$i\partial_t u + \partial_x^2 u = i|u|^{2\sigma}\partial_x u, \tag{1.1.2}$$

which is the quasilinear cousin of (1.1.1). We refer the reader to the introduction of Chapter 2 for a discussion of the physical scenarios (1.1.2) models, and the remarkably diverse soliton solutions it possesses.

The derivative in the nonlinearity in (1.1.2) causes major difficulties when considering local well-posedness, and only the case $\sigma = 1$ has a complete theory. This is due to the fact that when $\sigma = 1$ (1.1.2) becomes completely integrable, which allows one to use tools from this field. For local well-posedness, the analysis of (1.1.2) is most difficult when $\sigma < 1$. A basic reason for this is that when $\sigma < 1$ the nonlinearity is extremely rough, and lacks the decay necessary for global smoothing type estimates. These two features pose considerable difficulty, and rule out the possibility of naively applying standard tools for closing low-regularity estimates.

When $\sigma < 1$, the key threshold for $H^s(\mathbb{R})$ well-posedness of (1.1.2) is $s = 1$. Indeed, well-posedness in the energy space $H^1(\mathbb{R})$ would have two immediate consequences. First, it would make the soliton stability results in [232] rigorous, as [232] needs to assume $H^1(\mathbb{R})$ well-posedness to state and prove their results. Secondly, since (1.1.2) is L^2-subcritical when $\sigma < 1$, a H^1 local well-posedness theory would immediately yield global existence. In Chapter 2, we give the first H^1 well-posedness result for (1.1.2) when $\sigma < 1$:

Theorem 1.1.1. Let $\sigma \in (\frac{\sqrt{3}}{2}, 1)$. Then (1.1.2) is globally well-posed in $H^1(\mathbb{R})$.

Since the introduction of Chapter 2 contains a detailed outline of the proof of Theorem 1.1.1, we will not repeat it here. The restriction $\sigma > \frac{\sqrt{3}}{2}$ is technical, and comes from balancing the gains yielded by our paradifferential gauge transformation with the losses incurred from the rough nonlinearity. In principle, Theorem 1.1.1 should hold for $\sigma \in (\frac{1}{2}, 1)$, but proving this would likely require new tools.

Since when $\sigma < 1$ the nonlinearity in (1.1.2) is extremely rough, the question of finding the highest s for which (1.1.2) is well-posed in $H^s(\mathbb{R})$ becomes a relevant issue. In Chapter 2, we also prove well-posedness in $H^s(\mathbb{R})$ for s up to 4σ. This threshold is significant, as the nonlinearity in (1.1.2) is only $C^{1,2\sigma-1}$-Hölder continuous. In other words, the 4σ threshold is twice as large as a naive energy estimate would suggest. Again, since the introduction of Chapter 2 contains a detailed sketch of the proof, we will not repeat the main ideas of the high regularity theorem here. What is worth noting is that - by significantly expanding upon the ideas in Chapter 2 - Ben Pineau and I recently found the *sharp* high regularity well-posedness threshold for (1.1.2). Moreover, our analysis applies equally to (1.1.1), as long as the power is not too low ($\alpha > 1/2$ is plenty). This result will appear in a forthcoming work.

In Chapter 3, we consider the existence/non-existence problem for solitary water waves. Solitary water waves are localized disturbances of a fluid surface which travel at constant speed and with a fixed profile. Such waves were first observed by Russell in the mid–19th century [293], and are fundamental features of many water wave models. When modelling water waves, the fundamental physical parameters are the gravity, surface tension, and fluid depth. It is then an interesting question to identify which combinations of parameters lead to a given physical phenomenon. For solitary waves in two dimensions, all possible combinations had yielded a solution, except one. In Chapter 3, we fill in the missing case. More precisely, we prove the following theorem:

Theorem 1.1.2. No solitary waves exist in finite depth for the pure capillary irrotational water wave problem in 2D, even without the assumption that the free surface is a graph.

We note that the non-existence of *infinite* depth pure capillary irrotational solitary water waves in 2D was proven in [174]. To explain the proof of the finite depth result stated in Theorem 1.1.2, we briefly recall the water waves equations. Below, we denote the water domain at time t by $\Omega(t) \subseteq \mathbb{R}^d$, and assume that $\Omega(t)$ has a flat finite bottom $\{y = -h\}$. For the sake of simplicity, we let $\eta(x,t)$ denote the height of the free surface as a function of the horizontal coordinate, so that

$$\Omega(t) = \{(x,y) \in \mathbb{R}^d : -h < y < \eta(x,t)\}. \tag{1.1.3}$$

However, we emphasize that in Theorem 1.1.2 we do not need to assume that the free surface is a graph. This is important, as it is known that *periodic* pure capillary travelling waves exist, and that their free surfaces need not be graphs over the horizontal coordinate.

We denote by u the fluid velocity and by p the pressure. The vector field u solves Euler's equations inside $\Omega(t)$,

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p - g e_d, \\ \text{div } u = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1.4}$$

and the bottom boundary is impenetrable:

$$u \cdot e_d = 0 \quad \text{when } y = -h. \tag{1.1.5}$$

On the upper boundary the atmospheric pressure is normalized to zero and we have the dynamic boundary condition

$$p = -\sigma \mathbf{H}(\eta) \quad \text{on } \Gamma(t) := \{y = \eta(x)\} \tag{1.1.6}$$

as well as the kinematic boundary condition

$$\partial_t + u \cdot \nabla \quad \text{is tangent to } \bigcup \Gamma(t). \tag{1.1.7}$$

Here $g \geq 0$ represents the gravity, $\sigma \geq 0$ represents the surface tension coefficient, and $\mathbf{H}(\eta)$ is the mean curvature. The kinematic boundary condition requires that the normal velocity of the free surface be given by $v \cdot n_{\Gamma(t)}$, with $n_{\Gamma(t)}$ the outward unit normal to $\Gamma(t)$; the dynamic boundary condition represents the balance of forces at the fluid-vacuum interface.

We adhere to the classical assumption that the flow is irrotational, so that we can write u in terms of a velocity potential ϕ as $u = \nabla\phi$. It is easy to see that ϕ is a harmonic function whose normal derivative is zero on the bottom. Thus, ϕ is determined by its trace $\psi = \phi|_{\Gamma(t)}$ on the free boundary $\Gamma(t)$. A solitary wave is then a solution to the above equations that decays at infinity, and whose profile is uniformly translating in the horizontal direction with velocity c, i.e., $\phi(x, y, t) = \phi_0(x - ct, y)$, $\eta(x, y, t) = \eta_0(x - ct, y)$, and $p(x, y, t) = p_0(x - ct, y)$. This ansatz leads to a complicated nonlinear elliptic system.

To prove Theorem 1.1.2, three key insights are needed. First, one needs to "flatten" the domain, and reduce geometry to algebra. Although there are several ways to do this, the optimal approach is to Riemann map $\Omega(t)$ onto the strip $\{-h < y < 0\}$. The reason one does this is because ϕ is harmonic, and conformal maps preserve this key property. However, note that this forces us to work in two dimensions; no non-existence results are known in higher dimensions.

Now that the equations are defined on the strip, and appropriate Laplace equations are satisfied, one can reduce the problem to the upper boundary. This leads to the so-called "holomorphic" formulation of the water waves problem. The resulting equations are highly nonlinear and nonlocal. However, remarkably, the solitary wave ansatz simplifies these equations dramatically. Indeed, one arrives at the equation

$$-\frac{c^2}{2}\frac{(W_\alpha+\overline{W_\alpha}+W_\alpha\overline{W_\alpha})}{|1+W_\alpha|^2}+g\Im(W)+\frac{i\sigma}{1+W_\alpha}\partial_\alpha\left(\frac{1+W_\alpha}{|1+W_\alpha|}\right)=0, \tag{1.1.8}$$

with W denoting the holomorphic variable, and α the horizontal coordinate.

The next key insight is that when $g=0$, (1.1.8) satisfies a Pohozaev-type identity. More precisely, one begins by justifying a change of variables of the form $\log(1+W_\alpha):=U+iV=U-i\mathcal{T}_hU$. Here, $\mathcal{T}_h$ denotes the Tilbert transform, which encodes that W is holomorphic. After this, one multiplies the resulting equation by α, and integrates. Eventually, one arrives at the identity:

$$2c^2\int_{\mathbb{R}}(\cosh(U)-1)d\alpha=-\frac{h}{2}\int_{\mathbb{R}}(|U_\alpha|^2-|\mathcal{T}_hU_\alpha|^2)d\alpha.$$

By Plancherel,

$$-\frac{h}{2}\int_{\mathbb{R}}(|U_\alpha|^2-|\mathcal{T}_hU_\alpha|^2)d\alpha=-\frac{h}{2}\int_{\mathbb{R}}|\xi|^2|\widehat{U}|^2\text{sech}^2(h\xi)\leq 0.$$

However, $\cosh(U)-1\geq 0$. Therefore, we deduce that

$$\cosh(U)=1,$$

so that $U\equiv 0$. This formally implies that no solitary waves exist. However, this computation is not rigorous, as it is not compatible with the function spaces imposed on W and U. Therefore, the last step is to justify the above identity. For this, one replaces multiplication by α by an appropriate truncation of this operator. One then analyzes the Tilbert transform in a paradifferential fashion, viewing it as the Hilbert transform at high frequency and a derivative at low frequency. Finally, one proves some basic estimates that justify passing to the limit. With these ideas in mind, the proof of Theorem 1.1.2 is very easy, taking only a couple of pages. The complete proof is given in Chapter 3.

In Chapter 4, we study the phase retrieval problem. Let (Ω, Σ, μ) be a measure space, $C \geq 1$, and $1 \leq p \leq \infty$. A subspace $E \subseteq L_p(\mu)$ is said to do *C-stable phase retrieval* if for all $f, g \in E$ we have

$$\inf_{|\lambda|=1} \|f - \lambda g\|_{L_p} \leq C \||f| - |g|\|_{L_p}. \tag{1.1.9}$$

In (1.1.9), the infimum is taken over all unimodular scalars λ, and $|f|$ denotes the modulus of f, defined for $t \in \Omega$ by $|f|(t) := |f(t)|$. In the case of real-valued functions, there are only two unimodular scalars, ± 1, so that

$$\inf_{|\lambda|=1} \|f - \lambda g\|_{L_p} = \min\{\|f - g\|_{L_p}, \|f + g\|_{L_p}\}.$$

However, if the scalar field is complex, one must consider all elements of the unit circle when computing the infimum. This makes complex phase retrieval problems much more difficult than those over the real field.

Note that if $f, g \in E$ and $|f| = |g|$, then (1.1.9) forces f to be a multiple of g. Since E is a subspace, this tells us that the equality $|f| = |g|$ holds for $f, g \in E$ if and only if $f = \lambda g$ for some unimodular scalar λ. The recovery of f up to global phase from $|f|$ is called *phase retrieval*. In other words, defining $f \sim g$ if f is a unimodular multiple of g, phase retrieval asks that the map $|f| \mapsto f/\sim$ be well-defined on E. The inequality (1.1.9) asks not only that this map be well-defined, but also that it be C-Lipschitz. Phase retrieval problems appear in several applied circumstances and have applications in frame theory, applied harmonic analysis, imaging (crystallography, ptychography) and audio processing. Historically, the study of phase retrieval in mathematical physics dates back to at least 1933 when in his seminal work *Die allgemeinen Prinzipien der Wellenmechanik* [272] W. Pauli asked whether a wave function is uniquely determined by the probability densities of position and momentum. In other words, Pauli asked whether $|f|$ and $|\widehat{f}|$ determine $f \in L_2(\mathbb{R})$ up to multiplication by a unimodular scalar. Such a question arises when trying to reconstruct necessary mathematical information from physical experiments. Indeed, in the mathematical formulation of quantum mechanics one works with a normalized wave function f that solves the Schrödinger equation. One then interprets $|f|^2$ and $|\widehat{f}|^2$ as the probability density of position and momentum, respectively, for the associated physical system. Conversely, physical experiment allows one to measure the position and momentum of particles, hence, in principle, identify $|f|$ and $|\widehat{f}|$. However, is such knowledge sufficient to recover the wave function

f up to global phase?[1] For general $L_2(\mathbb{R})$ functions, a negative answer to this question was given in 1944. Nevertheless, for rather large subspaces $E \subseteq L_2(\mathbb{R})$ such a reconstruction is possible, as we describe in Chapter 4.

The inequality (1.1.9) leads to many interesting mathematical questions, requiring various linear and nonlinear techniques from function space theory, harmonic analysis, and probability to solve. However, the study of subspaces satisfying (1.1.9) is also in its infancy. Indeed, until last year there were *no* examples of infinite dimensional subspaces where (1.1.9) was known to hold. Moreover, there were several scenarios where phase retrieval was proven to be possible, but necessarily highly unstable [7, 70]. This is in contrast to the finite dimensional case, where it is known that phase retrieval is automatically stable, and that "most" subspaces of proportional dimension do phase retrieval.

In the recent article [71], Calderbank, Daubechies, Freeman and Freeman were able to construct the first examples of infinite dimensional subspaces of real-valued $L_2(\mathbb{R})$ which do stable phase retrieval. The complex case was left open, but will be solved in Chapter 4. Indeed, in Chapter 4 we will give very simple and natural constructions of infinite dimensional subspaces satisfying (1.1.9). Notably, this includes the following:

Theorem 1.1.3. The closed subspace of $L_2([0,1];\mathbb{R})$ generated by $\{\sin(2\pi 4^n x) : n \in \mathbb{N}\}$ does stable phase retrieval.

Theorem 1.1.4. For a probability measure μ, the closed span of a sequence $(r_n) \subseteq L_4(\mu;\mathbb{R})$ of mean zero iid random variables does stable phase retrieval in $L_4(\mu;\mathbb{R})$ if and only if $|r_n|$ is not identically constant.

The proofs of the above results will be presented in Chapter 4. For now, note that the subspaces in Theorems 1.1.3 and 1.1.4 cannot do complex phase retrieval. Indeed, a subspace E containing two linearly independent real vectors can never do complex phase retrieval. To see this, simply note that if $f, g \in E$ are real and linearly independent, then $f + ig$ and $f - ig$ have the same modulus, but are not linearly dependent. Nevertheless, in the complex case, we have a satisfactory analogue of Theorem 1.1.3:

[1] Note that all linear and most nonlinear Schrödinger equations are phase invariant, so if f is a normalized wave function then so is λf for all unimodular scalars λ.

Theorem 1.1.5. Let $P \in L_2([0,1];\mathbb{C})$ be a trigonometric polynomial of the form

$$P(x) = \sum_{k=1}^{N} a_k e^{2\pi i k x},\ a_k \in \mathbb{C}.$$

If $|P|$ is not constant and $A > 2N$, then the closed span of $\{P(A^n x) : n \in \mathbb{N}\}$ does stable phase retrieval.

The techniques used to prove Theorems 1.1.3 to 1.1.5 derive from classical harmonic analysis. They are similar, in a loose sense, to certain combinatorial constructions of $\Lambda(p)$-sets proposed by Rudin [292]. As we shall explain in Chapter 4, there are various other natural examples of subspaces doing stable phase retrieval. Notably, this includes a complex analogue of Theorem 1.1.4, and certain variants of Theorem 1.1.3 with "4^n" replaced by a subset of $\mathbb{N}$ of maximal density. Chapter 4 will also discuss various structural properties of the class of SPR subspaces. To give a flavour of this topic, a handful of these structural results are stated below. However, this is very much only the tip of the iceberg - much more information can be found in Chapter 4.

The inequality (1.1.9) requires that the phase recovery map be well-defined and Lipschitz. Although Theorems 1.1.3 to 1.1.5 are stated as establishing (1.1.9), the arguments in the original paper [82] we wrote on this were not able to establish Lipschitz continuity of the phase recovery map. More precisely, in [82] it was shown that the subspaces in Theorems 1.1.3 to 1.1.5 do phase retrieval, and that the phase recovery map is Hölder continuous on the unit ball of E. However, this deficiency in regularity was removed in our more recent paper [115], via the following theorem:

Theorem 1.1.6. For a subspace $E \subseteq L_p(\mu)$, Hölder stability of the phase recovery map is equivalent to Lipschitz stability. More precisely, for $0 < \gamma \leq 1$, the inequality

$$\inf_{|\lambda|=1} \|f - \lambda g\|_{L_p} \leq C \||f| - |g|\|_{L_p}^{\gamma} \left(\|f\|_{L_p} + \|g\|_{L_p}\right)^{1-\gamma},\ \forall f, g \in E$$

implies the inequality (1.1.9) with constant $(4C)^{\frac{1}{\gamma}}$.

The proof of Theorem 1.1.6 - presented in Chapter 4 - is based on an equally interesting observation; namely, that instabilities in phase retrieval can be witnessed on orthogonal vectors:

Theorem 1.1.7. Let $f, g \in L_2(\mu)$. Then there exists $f', g' \in \operatorname{span}\{f, g\}$ such that $f' \perp g'$ and

$$\||f'| - |g'|\|_{L_2} \leq \||f| - |g|\|_{L_2},$$

$$\inf_{|\lambda|=1} \|f - \lambda g\|_{L_2} \leq \inf_{|\lambda|=1} \|f' - \lambda g'\|_{L_2}.$$

In other words, replacing (f, g) by the orthogonal pair (f', g') tightens both sides of the inequality (1.1.9).

Theorem 1.1.7 is new and useful in finite dimensions. In particular, it immediately implies that for finite dimensional E, phase retrieval is automatically stable. Theorem 1.1.7 can also be generalized to L_p-spaces, by noting that for orthogonal vectors (f', g') we have $\inf_{|\lambda|=1} \|f' - \lambda g'\|_{L_2} = (\|f'\|_{L_2}^2 + \|g'\|_{L_2}^2)^{\frac{1}{2}}$. Our general result in L_p says that failure of stable phase retrieval can be witnessed by "well-separated vectors". Such an observation immediately implies Theorem 1.1.6.

The last result we mention here concerns the relationship between doing stable phase retrieval in $L_p(\mu)$ versus doing stable phase retrieval in $L_q(\mu)$. Let $1 \leq q < p < \infty$ and μ be a probability measure. In this case, we have $L_p(\mu) \subseteq L_q(\mu)$, and from the Kadec-Pelcynski theory, we also know that if $E \subseteq L_p(\mu)$ does stable phase retrieval, then $\|\cdot\|_{L_p} \sim \|\cdot\|_{L_q}$ on E. In particular, we may view E as a closed subspace of $L_q(\mu)$. This leads to the natural question of whether E doing stable phase retrieval as a subspace of $L_p(\mu)$ implies that E does stable phase retrieval as a subspace of $L_q(\mu)$. For $p \geq 2$, the answer is negative:

Theorem 1.1.8. For $p \geq 2$, there exists a closed subspace $E \subseteq L_p([0,1];\mathbb{R})$ that does stable phase retrieval, but fails to do stable phase retrieval in $L_q([0,1];\mathbb{R})$ for any $1 \leq q < p$.

Surprisingly, the condition $p \geq 2$ in Theorem 1.1.8 is sharp: If $E \subseteq L_p(\mu)$ does stable phase retrieval and $p < 2$, then E does stable phase retrieval in $L_q(\mu)$ for all $1 \leq q \leq p$. Moreover, we can precisely characterize stable phase retrieval in $L_p(\mu)$, $p < 2$, in a measure-theoretic fashion, as follows:

Theorem 1.1.9. Let (Ω, μ) be a probability space and let E be a closed infinite dimensional subspace of $L_p(\mu;\mathbb{R})$. The following are equivalent for $p < 2$:

(i) E does stable phase retrieval in $L_p(\mu)$.

(ii) E does stable phase retrieval in $L_1(\mu)$ and $\|\cdot\|_{L_p} \sim \|\cdot\|_{L_1}$ on E.

(iii) There exists $\alpha > 0$ such that for all $f, g \in E$,

$$\mu(\{t \in \Omega : |f(t)| \geq \alpha \|f\|_{L_p} \text{ and } |g(t)| \geq \alpha \|g\|_{L_p}\}) > \alpha. \tag{1.1.10}$$

As mentioned above, proofs of all these results can be found in Chapter 4. Chapter 4 also contains various other results of a similar nature.

Chapter 5 investigates non-negative bases in function spaces. Recall that a *basis* of a Banach space E is a sequence $(f_n)_{n=1}^{\infty}$ of elements of E such that for every $f \in E$ there exists a unique sequence of scalars (a_n) such that $f = \sum_{n=1}^{\infty} a_n f_n$. Given a Banach space E, one often wants to build a good basis of E, so that the identification $f \leftrightarrow (a_n)$ preserves as much of the structure of E as possible. For example, if E is a Hilbert space, one tends to work with orthonormal bases.

A longstanding question asked whether $L_2(\mathbb{R})$ admits a basis (f_n) with each of the f_n's being a non-negative function. As is easy to see, such a basis (f_n) cannot be orthonormal. However, the redeeming property of such a basis is that any sequence (a_n) of non-negative numbers necessarily represents a non-negative function $f = \sum_{n=1}^{\infty} a_n f_n$. A newfound interest in non-negative bases arose after an engineer inquired about vanishing moment conditions for wavelet bases, and the extent to which non-negativity obstructs properties of more general signal representations [285, p. 5784]. Shortly after [285] appeared, Johnson and Schechtman [193] showed that $L_1(\mathbb{R})$ admits a basis of non-negative functions. However, they were unable to solve the original problem in $L_2(\mathbb{R})$. In Chapter 5, we tackle this problem:

Theorem 1.1.10. $L_2(\mathbb{R})$ has a basis (f_n) consisting of non-negative functions.

The proof of Theorem 1.1.10 is rather involved, as can be extrapolated from the following fact: For any non-negative basis (f_n) of $L_2(\mathbb{R})$ there exists a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that $(f_{\sigma(n)})$ is *not* a basis of $L_2(\mathbb{R})$. Since Hilbert spaces have a remarkably uniform structure, it is very difficult to build conditional systems in them. Nevertheless, Theorem 1.1.10 shows that this is indeed possible. Chapter 5 also contains other interesting results on coordinate systems, including how to build a Schauder frame from any set with dense span.

In Chapter 6 we construct and analyze free Banach lattices. Given a Banach space E and $p \in [1, \infty]$ the *free p-convex Banach lattice over E* is a p-convex Banach lattice $\mathrm{FBL}^{(p)}[E]$ together with a linear isometry $\phi_E \colon E \to \mathrm{FBL}^{(p)}[E]$ such that, for every p-convex Banach lattice X and every bounded linear operator $T\colon E \to X$, there is a unique linear lattice homomorphism $\widehat{T}\colon \mathrm{FBL}^{(p)}[E] \to X$ making the following diagram commute:

$$\begin{tikzcd} \mathrm{FBL}^{(p)}[E] \arrow[dr, dashed, "\exists!\, \widehat{T}"] & \\ E \arrow[u, "\phi_E"] \arrow[r, "T"'] & X. \end{tikzcd}$$

Free Banach lattices provide a fundamental tool for understanding the interplay between Banach space and Banach lattice properties. In particular, spaces of the form $\mathrm{FBL}[\ell_2(A)]$ for an uncountable set A are used in [26, Section 5] to resolve an open problem of Diestel.

We begin Chapter 6 by giving an explicit description of $\mathrm{FBL}^{(p)}[E]$ as a sublattice of the vector lattice of real-valued functions defined on the dual Banach space E^* of E. Indeed, for any function $f\colon E^* \to \mathbb{R}$, define

$$\|f\|_{\mathrm{FBL}^{(p)}[E]} = \sup\Big\{ \bigg(\sum_{k=1}^n \big|f(x_k^*)\big|^p\bigg)^{\frac{1}{p}} \ :\ n \in \mathbb{N},\, x_1^*, \dots, x_n^* \in E^*,\, \sup_{x\in B_E} \sum_{k=1}^n \big|x_k^*(x)\big|^p \leqslant 1 \Big\}.$$

The set $H_p[E]$ of positively homogeneous functions $f\colon E^* \to \mathbb{R}$ with $\|f\|_{\mathrm{FBL}^{(p)}[E]} < \infty$ can be checked to be a Banach lattice with respect to the pointwise operations and this norm. Moreover, as we will show, $\mathrm{FBL}^{(p)}[E]$ is simply the closure in $H_p[E]$ of the sublattice generated by the set $\{\delta_x \ :\ x \in E\}$, where $\delta_x\colon E^* \to \mathbb{R}$ is the evaluation map given by $\delta_x(x^*) = x^*(x)$, together with the linear isometry $\phi_E\colon E \to \mathrm{FBL}[E]$ defined by $\phi_E(x) = \delta_x$. Using this representation, in Chapter 6 we provide a comprehensive analysis of the fine structure of $\mathrm{FBL}^{(p)}[E]$, as well as the correspondence $E \mapsto \mathrm{FBL}^{(p)}[E]$. Moreover, we apply our knowledge to solve problems that are a priori unrelated to free Banach lattices.

Chapter 2

Derivative nonlinear Schrödinger equations

2.1 Introduction

In this chapter - based on the joint work [277] with Ben Pineau - we consider the *generalized derivative nonlinear Schrödinger equation*:

$$\begin{cases} i\partial_t u + \partial_x^2 u = i|u|^{2\sigma}\partial_x u, \\ u(0) = u_0, \end{cases} \tag{gDNLS}$$

where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ and $\sigma > 0$. We will be particularly interested in the case $\sigma < 1$, as this is where H^s local well-posedness is most difficult. We begin with a brief history of this family of equations, and some of its closely related analogues.

The (gDNLS) equations originate from the study of the so-called *derivative nonlinear Schrödinger equation*:

$$\begin{cases} i\partial_t u + \partial_x^2 u = i|u|^2\partial_x u, \\ u(0) = u_0, \end{cases} \tag{DNLS}$$

which corresponds to (gDNLS) with $\sigma = 1$. Physically, (DNLS) derives from the one-dimensional compressible magneto-hydrodynamic equation in the presence of the Hall effect, and the propagation of circular polarized nonlinear Alfvén waves in magnetized plasmas [247, 249, 271]. It also appears as a model for ultrashort optical pulses [4, 250], as well as in various other physical scenarios [78, 187, 296]. Mathematically, (DNLS) also has many interesting features. For example, like the 1D cubic NLS, it is completely integrable [199]. However, it scales like the 1D quintic NLS, which makes it L^2 critical. Moreover, although at first glance (DNLS) looks to be semilinear, it is known that uniform continuity of the solution map fails in H^s as long as $s < \frac{1}{2}$ (see [52, 305]). Therefore, this PDE has a clear quasilinear flavour.

In recent years, the (gDNLS) family of equations has seen increasing interest, stemming from the 2013 article of Liu, Simpson and Sulem [232]. One of the original motivations of [232] was to shed light on the global well-posedness of (DNLS) in the energy space H^1, which was an important open problem. However, in an interesting turn of events, Bahouri and Perelman [39] managed to prove global well-posedness for the (DNLS) equation *before* the global well-posedness of (gDNLS) could be established for any $\sigma \neq 1$. In this chapter we make progress towards resolving one half of the program of Liu, Simpson and Sulem by proving that (gDNLS) is globally well-posed in H^1 for $\sigma \in (\frac{\sqrt{3}}{2}, 1)$. Note that, although completed shortly after each other, our result for $\sigma < 1$ and the $\sigma = 1$ result of [39] are completely independent, and the methods used differ quite dramatically. Indeed, for $\sigma = 1$, local well-posedness in H^1 has been known for a long time [161], and can be established by employing a suitable gauge transformation, and standard Strichartz estimates. In fact, the smoothing properties of the equation are suitable to lower the well-posedness threshold to $H^{\frac{1}{2}}$ as in [306]. Global well-posedness, however, is considerably harder, as the problem is L^2 critical. For this reason, Bahouri and Perelman (as well as Harrop-Griffiths, Killip, Ntekoume and Vişan [154, 155, 206] in their subsequent work) crucially rely on the complete integrability of (DNLS). In the case $\sigma < 1$, the main difficulties are reversed. Establishing local well-posedness is difficult because of the lack of decay and roughness of the nonlinearity. On the other hand, one expects to be able to easily propagate any reasonable H^1 local well-posedness theory in time to obtain a global result. This is because when $\sigma < 1$ the problem becomes L^2 subcritical, and one expects to be able to use the conserved energy and mass of the problem to control the H^1 norm of a solution.

Another motivation for (gDNLS) is the rich family of soliton solutions, which is actually where the majority of [232]'s efforts were focused. Assuming a suitable H^1 well-posedness theory, the authors of [232] were able to use the abstract theory of Grillakis, Shatah and Strauss [135, 136] to investigate the orbital stability of the solitons. However, an H^1 well-posedness theory for $\sigma < 1$ had not been known until now.

When $\sigma < 1$, one can view (gDNLS) as a prototypical model for a quasilinear dispersive equation with a rough, low power nonlinearity (see [226] for a KdV analogue). Such nonlinearities in the context of semilinear NLS type equations are becoming increasingly well-understood [77, 324], and at modest regularity local well-posedness can usually be proven by a combination of regularization and perturbative arguments. However, the combination of derivative and low power coefficient in the nonlinearity of (gDNLS) causes many interesting technical issues, several of which are yet to be fully understood. One issue for low regularity well-posedness is that the coefficient $|u|^{2\sigma}$ in the nonlinearity is less than quadratic in order. Because of this, the smoothing properties of the linear part of the Schrödinger equation are seemingly not strong enough to directly compensate for the apparent derivative loss which occurs because of the u_x term in the nonlinearity. Another tool to avoid derivative loss - which has been successfully employed in the case $\sigma > 1$ in [151, 160] - is a gauge transformation. This technique allows one to re-normalize the equation to effectively remove the worst interactions in the derivative nonlinearity. However, again, it seems one can only directly apply this method when $\sigma \geq 1$ (i.e. $|u|^{2\sigma}$ is of quadratic order or higher), as in the case $\sigma < 1$ negative powers of $|u|$ eventually appear in the analysis. This is related to the roughness of the nonlinearity, and will be elaborated on further when we outline the proof of our results.

To contrast, the Benjamin-Ono equation,

$$\begin{cases} u_t + Hu_{xx} = uu_x, \\ u(0) = u_0, \end{cases} \tag{2.1.1}$$

has a similar low power derivative nonlinearity uu_x, and as with (gDNLS), the linear part of the equation does not have strong enough smoothing properties to directly compensate for the derivative loss in the nonlinearity. Nevertheless, H^1 well-posedness for this equation was established several years ago in [312]. One should note, however, that the Benjamin-Ono nonlinearity has a much nicer algebraic structure than that of (gDNLS) (it is smooth and multilinear), which makes the equation more amenable to normal form type techniques (such as cubic corrections or a gauge transformation). Moreover, Christ [81] showed that Schrödinger's equation with Benjamin-Ono's nonlinearity is ill-posed in any reasonable sense, so the analogies between these equations are at best heuristic. For (gDNLS), our solution to the above difficulties will be to introduce a family of *partial* gauge transformation adapted to each dyadic frequency scale and the corresponding paradifferential flow - which removes the portion of the nonlinearity which is large in a pointwise sense, on a scale which is balanced against the corresponding frequency localization scale of the nonlinearity. This will then be combined with smoothing and maximal function type arguments to attain the H_x^1 well-posedness threshold.

Another novel issue in the study of (gDNLS) is that the nonlinearity has only a finite degree of Hölder regularity, and so one does not expect to be able to construct smooth solutions from regular data. In our case, the nonlinearity is only $C^{1,2\sigma-1}$ Hölder continuous. We expect therefore to only be able to differentiate the equation with respect to some parameter "2σ times" to obtain estimates. To maximize the potential regularity of solutions, we note that the scaling of the Schrödinger equation suggests that we can convert L_x^2 based estimates for one time derivative of a solution to estimates for two spatial derivatives. Therefore, it is advantageous to differentiate (gDNLS) in time rather than in space, and then convert time derivative estimates into estimates for spatial derivatives of a solution. After a single time differentiation, we are left with $2\sigma - 1$ degrees of regularity on the nonlinearity. By working with fractional space derivatives, one expects to be able to prove an energy estimate for the $H_x^{1+2\sigma}$ norm of a solution. However, working with fractional time derivatives (after suitably localizing in time), one expects to improve this further, and prove well-posedness in H_x^s up to $s = 2 \cdot 1 + 2 \cdot (2\sigma - 1) = 4\sigma$. A similar heuristic argument applies to any dispersion generalized equation with rough nonlinearity, where one can convert time derivative estimates into estimates on a certain number of spatial derivatives, perhaps modulo some perturbative terms coming from the nonlinearity. In general, we expect this heuristic to give rather sharp results, but this is not even known for semilinear NLS equations with rough nonlinearities [77, 324], and is essentially unexplored in the quasilinear setting.

Finally, let us recall some basic symmetry properties of (gDNLS) as well as some conservation laws, which we will use to propagate our local well-posedness result to a global one. First, we have the scaling transformation

$$u(t,x) \mapsto u_\lambda(t,x) := \lambda^{\frac{1}{2\sigma}} u(\lambda^2 t, \lambda x), \quad \lambda > 0,$$

which makes the critical Sobolev index $s_c = \frac{1}{2} - \frac{1}{2\sigma}$. In particular, the problem is L^2 subcritical when $\sigma < 1$. Moreover, (gDNLS) admits the following conserved quantities:

$$M(u) = \frac{1}{2}\int_{\mathbb{R}} |u|^2 dx, \tag{2.1.2}$$

$$P(u) = \frac{1}{2} Re \int_{\mathbb{R}} i\overline{u} u_x dx, \tag{2.1.3}$$

$$E(u) = \frac{1}{2}\int_{\mathbb{R}} |u_x|^2 dx + \frac{1}{2(\sigma+1)} Re \int_{\mathbb{R}} i|u|^{2\sigma}\overline{u}u_x dx, \tag{2.1.4}$$

which are the mass, momentum and energy, respectively. Unlike the standard NLS, (DNLS) doesn't enjoy the Galilean invariance nor the pseudo-conformal invariance symmetries, the latter being relevant for avoiding blowup. We also note that a simple change of variables allows us to change the sign of the nonlinearity in (gDNLS) and arrive at

$$i\partial_t u + \partial_x^2 u + i|u|^{2\sigma}\partial_x u = 0. \tag{2.1.5}$$

This latter equation is more common in the study of the solitary waves of (gDNLS).

Results

The main result of this chapter is global well-posedness of (gDNLS) in $H^s(\mathbb{R})$ when $\frac{\sqrt{3}}{2} < \sigma < 1$ and $s \in [1, 4\sigma)$. However, we divide this theorem into a "low-regularity" part and a "high-regularity" part, to maximize the range of σ. The high-regularity result is as follows:

Theorem 2.1.1. (High-Regularity) Let $\frac{1}{2} < \sigma < 1$ and let $2 - \sigma < s < 4\sigma$. Then (gDNLS) is locally well-posed in $H^s(\mathbb{R})$.

As mentioned, for a restricted range of σ, we can lower the well-posedness threshold down to H^1, where the conserved energy also gives global well-posedness:

Theorem 2.1.2. Let $\frac{\sqrt{3}}{2} < \sigma < 1$ and let $1 \leq s < 4\sigma$. Then (gDNLS) is globally well-posed in $H^s(\mathbb{R})$.

Remark 2.1.3. As a special case, Theorem 2.1.1 shows in particular that we have local well-posedness in H^s for $\frac{3}{2} \leq s \leq 2$. Therefore, we recover the only previously known local well-posedness results for (gDNLS) when $\sigma < 1$; namely, we recover the H^2 result of [160] and improve the result of [297], which used weighted Sobolev spaces.

Remark 2.1.4. In both Theorem 2.1.1 and Theorem 2.1.2, well-posedness is to be interpreted in the usual quasilinear fashion, including existence, uniqueness and continuous dependence on the data. More specifically, given an appropriate Sobolev index s and time $T > 0$, we first build a function space X_T^s that continuously embeds into $C([-T,T]; H_x^s)$. We then show that for each $u_0 \in H_x^s$ there exists a unique solution u to (gDNLS) that lies in X_T^s and satisfies $u(t = 0) = u_0$. Finally, we show that the data to solution map is continuous, even as a map from H_x^s to the stronger topology X_T^s.

Remark 2.1.5. Since (DNLS) is known to be globally well-posed in $H^{\frac{1}{2}}$, one may wonder why we only consider H^s well-posedness when $s \geq 1$. This is, in fact, not necessary. For each $\sigma \in (\frac{\sqrt{3}}{2}, 1)$, we expect that technical modifications of our proof should establish H^s well-posedness of (gDNLS) in a range $s \in [l(\sigma), 4\sigma)$ with $l(\sigma) < 1$ and $l(\sigma) \to \frac{1}{2}$ as $\sigma \to 1$. We avoid doing this for the sake of simplicity. It remains an open problem to prove well/ill-posedness in $H^{\frac{1}{2}}$ for any $\frac{1}{2} < \sigma < 1$, and to find the smallest $\sigma \in (0, 1)$ such that (gDNLS) is well-posed in H^1.

History on well-posedness and solitons

There is a vast literature devoted to the well-posedness of (DNLS), as it took several decades for the regularity to approach current thresholds, and for global results to emerge. We begin our review with the work of Tsutsumi and Fukuda [322, 323] who studied the well-posedness in $H^s(\mathbb{R})$ for $s > \frac{3}{2}$ by classical energy methods and parabolic regularization. The well-posedness in $H^1(\mathbb{R})$ was reached by Hayashi [161] by applying a gauge transformation to overcome the derivative loss, and Strichartz estimates to close a-priori estimates. The $H^1(\mathbb{R})$-solution was shown to be global by Hayashi and Ozawa [162], as long as the initial data satisfies $\|u_0\|_{L^2}^2 < 2\pi$. Later, Wu [331] improved this global result by relaxing the smallness condition to $\|u_0\|_{L^2}^2 < 4\pi$, which is natural in view of the soliton structure.

Below the energy space, there are also many results for (DNLS). Takaoka [306] proved local well-posedness in $H^s(\mathbb{R})$ when $s \geq \frac{1}{2}$ by the Fourier restriction method. This was complemented by a result of Biagioni and Linares [52] which notes that the solution map from $H^s(\mathbb{R})$ to $C([-T, T]; H^s(\mathbb{R}))$ cannot be locally uniformly continuous when $s < \frac{1}{2}$. By using the I-method, Colliander, Keel, Staffilani, Takaoka and Tao [84, 85] proved that the $H^s(\mathbb{R})$-solution is global if $s > \frac{1}{2}$ and $\|u_0\|_{L^2}^2 < 2\pi$. Guo and Wu [148] were later able to strengthen this result by proving that $H^{\frac{1}{2}}(\mathbb{R})$-solutions are global if $\|u_0\|_{L^2}^2 < 4\pi$. For an incomplete list of well-posedness results for (DNLS) on the torus, see [159, 253] and references therein.

There are also many works that use the complete integrability of the (DNLS) equation. The breakthrough result is [39], which establishes global well-posedness in $H^{\frac{1}{2}}(\mathbb{R})$. However, [39] was preceded by many results - see, e.g., [186, 274, 275] - highlights of which include a global well-posedness result in the weighted Sobolev space $H^{2,2}(\mathbb{R})$, and progress towards the soliton resolution conjecture. Moreover, although $H^{\frac{1}{2}}$ regularity is necessary for uniform continuity of the solution map, [154, 155, 206] are able to lower the global well-posedness threshold all the way to the critical Sobolev space L^2, definitively resolving the well-posedness theory for (DNLS) on the line. On the other hand, blowup for (DNLS) on non-standard domains (for example, the half-line with the Dirichlet boundary condition) is known to be possible [310, 330].

For (gDNLS), the literature on well-posedness is also quite large, though the results are far less definitive. As mentioned, (gDNLS) was popularized by [232], though well-posedness was not considered in that article. Possibly the first well-posedness result was by Hao, who in [151] was able to prove local well-posedness in $H^{\frac{1}{2}}(\mathbb{R})$ intersected with an appropriate Strichartz space for $\sigma \geq \frac{5}{2}$. Ambrose and Simpson [13] proved the existence and uniqueness of solutions $u \in C([0,T); H^2(\mathbb{T}))$ and the existence of solutions $u \in L^\infty([0,T), H^1(\mathbb{T}))$ for $\sigma \geq 1$. The uniqueness of $H^1(\mathbb{T})$-solutions was left unresolved, as the proof uses a compactness argument. Existence and uniqueness in $H^{\frac{1}{2}}(\mathbb{R})$ was proved by Santos in [297] for $\sigma > 1$, by utilizing global smoothing and maximal function estimates. A result in weighted Sobolev spaces was also proved in [297] for the case $\frac{1}{2} < \sigma < 1$, as adding weights helps compensate for the low power in the nonlinearity. In terms of $H^s(\mathbb{R})$ spaces, [160] proves local well-posedness in H^2 when $\sigma \geq \frac{1}{2}$, local well-posedness in H^1 when $\sigma \geq 1$, existence of weak solutions when $\sigma < 1$, and certain unconditional uniqueness results at high regularity. See [251] for more on unconditional uniqueness. The (gDNLS) equation with extremely rough nonlinearities $0 < \sigma < \frac{1}{2}$ is studied in [225, 227], but not in standard Sobolev spaces H^s.

We now turn to the history on stability of solitons. This is also a vast subject, and (gDNLS) is not the only generalization of (DNLS) whose solitons have been considered. For the sake of unification, therefore, let us consider the equation

$$i\partial_t u + \partial_x^2 u + i|u|^{2\sigma}\partial_x u + b|u|^{4\sigma}u = 0, \ x \in \mathbb{R}, \tag{2.1.6}$$

which is just a Schrödinger equation with a scale-invariant combination of derivative and power nonlinearities. Direct calculation verifies that the soliton solutions of (2.1.6) are given by

$$u_{\omega,c}(t,x) = e^{i\omega t}\phi_{\omega,c}(x - ct)$$

where

$$\phi_{\omega,c}(x) = \Phi_{\omega,c}(x)e^{i\theta_{\omega,c}(x)}, \qquad \theta_{\omega,c}(x) = \frac{c}{2}x - \frac{1}{2\sigma+2}\int_{-\infty}^{x} \Phi_{\omega,c}(y)^{2\sigma}dy,$$

and, using the notation $\gamma = 1 + \frac{(2\sigma+2)^2}{2\sigma+1} b$, the function $\Phi_{\omega,c}$ is real valued with $\Phi_{\omega,c}(x)^{2\sigma}$ given by

$$\begin{cases} \dfrac{(\sigma+1)(4\omega - c^2)}{\sqrt{c^2 + \gamma(4\omega - c^2)} \cosh\left(\sigma\sqrt{4\omega - c^2}x\right) - c} & \gamma > 0, \quad -2\sqrt{\omega} < c < 2\sqrt{\omega}, \\ \dfrac{2(\sigma+1)c}{(\sigma c x)^2 + \gamma} & \gamma > 0, \qquad c = 2\sqrt{\omega}, \\ \dfrac{(\sigma+1)(4\omega - c^2)}{\sqrt{c^2 + \gamma(4\omega - c^2)} \cosh\left(\sigma\sqrt{4\omega - c^2}x\right) - c} & \gamma \le 0, \quad -2\sqrt{\omega} < c < -2\sqrt{-\gamma/(1-\gamma)}\sqrt{\omega}. \end{cases}$$

These solitons are, of course, related to the Hamiltonian structure of (2.1.6), as well as to the conservation of mass, energy and momentum, which we leave to the reader to compute.

As expected, the story on soliton stability for (2.1.6) begins with the (DNLS) equation. Indeed, in [145], Guo and Wu proved that the soliton solutions of (DNLS) are orbitally stable when $\omega > \frac{c^2}{4}$ and $c < 0$ by applying the abstract theory of Grillakis, Shatah, and Strauss [135, 136]. Colin and Ohta [83] removed the condition $c < 0$ and proved that $u_{\omega,c}$ is orbitally stable when $\omega > \frac{c^2}{4}$ by applying the variational characterization of solitons as in Shatah [301]. The endpoint case $c = 2\sqrt{\omega}$ is only partially resolved; progress was made by Kwon and Wu in [217], but with certain caveats, such as a non-standard definition of orbital stability. For the study of periodic travelling waves, we refer to [79, 150, 156, 159] and references therein.

For (gDNLS), the story on soliton stability is much richer. In [232] it was shown that the solitary waves $u_{\omega,c}$ are orbitally stable if $-2\sqrt{\omega} < c < 2z_0\sqrt{\omega}$, and orbitally unstable if $2z_0\sqrt{\omega} < c < 2\sqrt{\omega}$ when $1 < \sigma < 2$. Here the constant $z_0 = z_0(\sigma) \in (-1, 1)$ is the solution to

$$F_\sigma(z) := (\sigma-1)^2 \left(\int_0^\infty (\cosh y - z)^{-\frac{1}{\sigma}} dy \right)^2 - \left(\int_0^\infty (\cosh y - z)^{-\frac{1}{\sigma}-1} (z \cosh y - 1) dy \right)^2 = 0.$$

Moreover, [232] proves that all solitary waves with $\omega > \frac{c^2}{4}$ are orbitally unstable when $\sigma \ge 2$ and orbitally stable when $0 < \sigma < 1$. As mentioned previously, these results are conditional on an appropriate well-posedness theory; there is also a minor numerical portion to the proof. In the borderline case when $c = 2z_0\sqrt{\omega}$ and $1 < \sigma < 2$, Fukaya ([119], see also [147]) proved orbital instability of the solitons. This completes the study of orbital stability of the solitons of (gDNLS), except in the case of the algebraic soliton, which requires special attention [146, 224].

In the case $\sigma = 1$, $b \neq 0$, there are also many works on soliton stability for (2.1.6), e.g. [83, 120, 157, 158, 159, 258, 259, 261]. On the other hand, there are no results in the case $\sigma \neq 1$, $b \neq 0$, as it seems the explicit formulas for the solitons were not previously known. We also mention that from the point of view of low regularity well-posedness, the additional term $b|u|^{4\sigma}u$ in (2.1.6) is both perturbative and maintains scaling, so in our usual range $\frac{\sqrt{3}}{2} < \sigma < 1$ our proof can easily be modified to establish global well-posedness in H^1, regardless of the size or sign of b. To contrast, recall that the known proof of global well-posedness in the case $\sigma = 1$, $b = 0$ is rather delicate; global well-posedness could, in principle, fail to persist once the effect of the focusing NLS is added. For state of the art global results when $\sigma = 1$, $b \neq 0$ we mention [159], which establishes global well-posedness below the soliton thresholds. In particular, (2.1.6) in the case $\sigma = 1$, $b \leq -\frac{3}{16}$ has been known to be globally well-posed for some time now, as at this point the energy becomes coercive, after a suitable gauge transformation.

Outline of the proofs

Here, we outline the key ideas in the proof of Theorem 2.1.1 and Theorem 2.1.2. We begin with a discussion of the low-regularity argument. Before describing the proof, however, it is instructive to discuss why the gauge transformation used in [160] combined with standard Strichartz estimates will not work. The following discussion is mostly heuristic and for the purpose of motivation only.

Firstly, by a standard energy estimate, one obtains for (regular enough) solutions to (gDNLS),

$$\|u\|_{L_T^\infty H_x^1} \lesssim \|u_0\|_{H_x^1} \exp\left(\int_0^T \|u\|_{L_x^\infty}^{2\sigma-1} \|u_x\|_{L_x^\infty}\right). \tag{2.1.7}$$

Therefore, one expects to be able to prove suitable H^1 bounds for solutions to (gDNLS) as long as one can estimate the Strichartz norm, $\|u_x\|_{L_T^1 L_x^\infty}$. However, applying Strichartz estimates directly to (gDNLS) leads to a loss of a derivative. Therefore, one might naïvely try to do some sort of gauge transformation to remove the $|u|^{2\sigma}u_x$ term in the equation, which is responsible for this loss. Indeed, if one (formally) defines

$$\Phi(t,x) = -\frac{1}{2}\int_{-\infty}^{x} |u|^{2\sigma} dy \tag{2.1.8}$$

and then

$$w = e^{i\Phi}u, \tag{2.1.9}$$

this leads to an equation for w of the form

$$iw_t + \partial_x^2 w = (-\partial_t\Phi + i\partial_x^2\Phi - (\partial_x\Phi)^2)w. \tag{2.1.10}$$

At first glance, it looks like one can prove Strichartz estimates for w_x without losing derivatives, to obtain the corresponding bound for $\|u_x\|_{L^1_T L^\infty_x}$. Unfortunately, if we expand $\partial_t\Phi$, we get

$$\begin{aligned}\partial_t\Phi &= -\sigma\int_{-\infty}^x \mathrm{Re}(|u|^{2\sigma-2}\overline{u}u_t)dy \\ &= -\sigma\int_{-\infty}^x \mathrm{Re}(|u|^{2\sigma-2}\overline{u}i\partial_x^2 u)dy - \sigma\int_{-\infty}^x \mathrm{Re}(|u|^{4\sigma-2}\overline{u}u_x)dy.\end{aligned} \tag{2.1.11}$$

The first term above is problematic. To avoid losing derivatives, we are forced to integrate by parts off one derivative. However, since $|u|^{2\sigma-2}\overline{u}$ is not C^1 when $\sigma < 1$, this will inevitably introduce negative powers of u, so this approach will not work.

While the above calculations are not particularly useful for closing low-regularity estimates, they do clearly identify the main enemies in trying to close Strichartz estimates for the gauge transformed equation. That is, the portion of u which is small or vanishes will prevent us from closing Strichartz estimates for w. Therefore, it is natural to try to somehow perform a gauge transformation which only removes some portion of the derivative nonlinearity $|u|^{2\sigma}u_x$, which corresponds to a part of u for which u is bounded away from zero. Doing this is somewhat subtle. We can't simply fix a universal constant $\varepsilon > 0$, and remove the portion of the nonlinearity for which $|u| > \varepsilon$. This is because when the u_x factor in $|u|^{2\sigma}u_x$ is at very high frequency (compared to ε), we will still lose derivatives in the Strichartz estimate. To work around this issue, we perform a paradifferential expansion of the equation. That is, for each $j > 0$, we project onto frequencies of size $\sim 2^j$ and obtain

$$(i\partial_t + \partial_x^2)P_j u = iP_{<j-4}|u|^{2\sigma}P_j u_x + g_j \tag{2.1.12}$$

where g_j is a perturbative term. The idea now is to split the coefficient $P_{<j-4}|u|^{2\sigma} = P_{<j-4}|u_s|^{2\sigma} + P_{<j-4}|u_l|^{2\sigma}$, where u_l corresponds to the portion of u which is bounded away from zero (where the lower bound depends on the frequency parameter j), and u_s is the remaining portion of u which is bounded above by some small j dependent parameter. We

then try to do a gauge transformation by defining

$$\Phi_j = -\frac{1}{2}\int_{-\infty}^{x} P_{<j-4}|u_l|^{2\sigma} dy \tag{2.1.13}$$

and

$$w_j = e^{i\Phi_j} P_j u. \tag{2.1.14}$$

This leads to an equation for w_j of the form,

$$(i\partial_t + \partial_x^2)w_j = (-\partial_t\Phi_j + i\partial_x^2\Phi_j - (\partial_x\Phi_j)^2)w_j + e^{i\Phi_j} g_j + ie^{i\Phi_j} P_{<j-4}|u_s|^{2\sigma} P_j u_x. \tag{2.1.15}$$

The point now is that the negative powers of u that arise in the $\partial_t\Phi_j$ term are bounded above by some parameter depending on the frequency scale 2^j. To avoid derivative loss, we would like this parameter to be as small as possible (i.e. u_l should be bounded below by a (j dependent) constant which is as large as possible). However, we still have to contend with the remainder of the original derivative nonlinearity, $ie^{i\Phi_j} P_{<j-4}|u_s|^{2\sigma} P_j u_x$, which is expected to cause derivative loss unless u_s is sufficiently small (depending on j). Therefore, we have to compromise between potential losses incurred by the $\partial_t\Phi_j$ term, and the remaining derivative nonlinearity. Unfortunately, by optimizing the appropriate splitting of u, it turns out that we will still lose $1-\sigma$ derivatives in estimating the Strichartz norm $\|u_x\|_{L^1_T L^\infty_x}$, and therefore, one only expects to be able to control $\|u_x\|_{L^1_T L^\infty_x}$ by $\|u\|_{L^\infty_T H^{2-\sigma}_x}$. As mentioned, while this is certainly an improvement over previous results [160, 297], this method is not quite robust enough to get well-posedness down to the energy space.

To get H^1 well-posedness, we combine this modified gauge transformation (and Strichartz estimates) with smoothing and maximal function type estimates, as in Propositions 2.2.3 and 2.2.4. However, we modify these Strichartz and maximal function norms (see the definition of Y^s_T below) to reflect the loss of $1-\sigma$ derivatives compared to the $L^\infty_T H^1_x$ norm, as mentioned above. That is, we build this deficiency into the function spaces where we construct solutions. In particular, the Strichartz ($L^1_T L^\infty_x$) component of the norm involves no more than σ derivatives. Therefore, the energy estimate (2.1.7) described above is no longer appropriate to close a priori estimates in H^1. Hence, the energy estimate has to be modified accordingly so that the control parameter (i.e. the Strichartz component) does not lead to a loss of derivatives (in excess of the H^1 norm) in the Strichartz/maximal function component of the estimate. It is actually this part of the argument that leads to the restriction on σ, which we will elaborate on later.

Next, we outline the proof of the high regularity well-posedness. As mentioned previously, the $C^{1,2\sigma-1}$ Hölder regularity of the function $z \mapsto |z|^{2\sigma}$ effectively limits the number of times one can differentiate the equation to obtain H^s estimates. A direct energy estimate, which involves differentiating the equation s times in the spatial variable (i.e. applying D_x^s to the equation) limits the range for which one can obtain estimates to $s \leq 2\sigma$. In [160], the authors managed to bypass this issue in the case $s = 2$ by instead obtaining an L_x^2 energy estimate for the time derivative $\partial_t u$. The point is that doing this only requires one to differentiate the nonlinearity a single time. Once an appropriate L_x^2 estimate is obtained, H_x^2 energy estimates for the solution can then be obtained by observing that up to an error of size $O(\|u\|_{L_T^\infty H_x^1}^{2\sigma+1})$, the equation gives,

$$\|(\partial_x^2 u)(t)\|_{L_x^2} \sim \|(\partial_t u)(t)\|_{L_x^2}. \tag{2.1.16}$$

In this chapter, we generalize this approach to derivatives of fractional order. It turns out that (after suitably localizing a solution in time), one can morally obtain an estimate (up to a suitable error term) essentially of the form

$$\|D_t^{\frac{s}{2}} u\|_{L_T^\infty L_x^2} \sim \|D_x^s u\|_{L_T^\infty L_x^2} \tag{2.1.17}$$

where $1 < s < 4\sigma$. The main idea for proving this estimate is a modulation type analysis. Namely, when the space-time Fourier transform of a solution u (after suitably localizing in time) is supported close to the characteristic hypersurface (or in the low modulation region), $\tau = -\xi^2$, one expects to be able to directly compare $D_t^{\frac{s}{2}} u$ and $D_x^s u$. On the other hand, when the space-time Fourier transform is supported far away from the hypersurface (or in the high modulation region), one expects to be able to control $D_t^{\frac{s}{2}} u$ and $D_x^s u$ in L_x^2 by a lower order error term stemming from the nonlinearity of the equation. This latter high modulation control can be loosely thought of as a space-time elliptic estimate.

With a method for suitably comparing space and time derivatives of a solution in hand, it then essentially suffices to obtain an energy estimate for $D_t^{\frac{s}{2}} u$ when u is localized near the characteristic hypersurface (which is precisely where one expects to be able to compare $D_t^{\frac{s}{2}} u$ to $D_x^s u$). Therefore, in light of the $C^{1,2\sigma-1}$ regularity of the nonlinearity, we should be able to obtain H_x^s estimates for a solution as long as $\frac{s}{2} < 2\sigma$. This explains the upper threshold of 4σ for our result. As hinted at earlier, the lower threshold of $2 - \sigma$ is explained by the fact that such an energy estimate closes as long as one can control $\|u_x\|_{L_T^1 L_x^\infty}$. Our low regularity estimates allow us to control this term by the $L_T^\infty H_x^s$ norm of u, as long as $s > 2 - \sigma$, where σ lies in the full range $(\frac{1}{2}, 1)$. This should be contrasted with the H^1

case where we employ a more complicated functional setting and only deal with a restricted range of σ. For clarity, we have chosen to present our high regularity results in the simplest possible functional setting, which is why the lower bound of $2-\sigma$ appears in Theorem 2.1.1, as it comes naturally from our previous estimates. Since $2-\sigma < \frac{3}{2}$ when $\sigma > \frac{1}{2}$, this is a reasonable lower threshold for the high regularity result (as it encompasses the range for which $\|u_x\|_{L^1_T L^\infty_x}$ can be controlled by Sobolev embedding). Nonetheless, we emphasize that the main novelty in Theorem 2.1.1 is the upper threshold $s < 4\sigma$.

2.2 Preliminaries

In this section we settle notation and recall some standard tools.

Littlewood-Paley decomposition

First, we recall the standard Littlewood-Paley decomposition. For this, let ϕ_0 be a radial function in $C_0^\infty(\mathbb{R})$ that satisfies

$$0 \leq \phi_0 \leq 1, \quad \phi_0(\xi) = 1 \text{ for } |\xi| \leq 1, \quad \phi_0(\xi) = 0 \text{ for } |\xi| \geq \frac{7}{6}.$$

Let $\phi(\xi) := \phi_0(\xi) - \phi_0(2\xi)$. For $j \in \mathbb{Z}$, define

$$\widehat{P_{\leq j} f}(\xi) = \phi_0(2^{-j}\xi)\widehat{f}(\xi),$$
$$\widehat{P_j f}(\xi) = \phi(2^{-j}\xi)\widehat{f}(\xi).$$

We will denote $P_{>j} = I - P_{\leq j}$, where I is the identity. Similarly, we define $P_{[a,b]} = \sum_{a \leq j \leq b} P_j$. We will also use the notation $\tilde{P}_j$, $\tilde{P}_{<j}$, $\tilde{P}_{>j}$ to denote a slightly enlarged or shrunken frequency localization. For example, we may denote $P_{[j-3,j+3]}$ by $\tilde{P}_j$.

Next, we recall a useful bookkeeping device. Following [176, 311], we denote by $L(\phi_1, \dots, \phi_n)$ a translation invariant expression of the form

$$L(\phi_1, \dots, \phi_n)(x) = \int K(y)\phi_1(x+y_1)\cdots\phi_n(x+y_n)dy,$$

where $K \in L^1$. Of interest is the following Leibniz type rule from [176, 311] which will make certain commutator expressions simpler to estimate:

Lemma 2.2.1. (Leibniz rule for P_j). We have the commutator identity

$$[P_j, f]g = L(\partial_x f, 2^{-j}g). \tag{2.2.1}$$

Frequency envelopes

One way we will employ the Littlewood-Paley projections is to define frequency envelopes, which are another nice bookkeeping device introduced by Tao [311]. To define these, suppose we are given a Sobolev type space X such that

$$\|P_{\leq 0}u\|_X^2 + \sum_{j=1}^{\infty} \|P_j u\|_X^2 \sim \|u\|_X^2. \tag{2.2.2}$$

A frequency envelope for u in X is a positive sequence $(a_j)_{j\in\mathbb{N}_0}$ such that

$$\|P_{\leq 0}u\|_X \lesssim a_0\|u\|_X, \quad \|P_j u\|_X \lesssim a_j\|u\|_X, \quad \sum_{j=0}^{\infty} a_j^2 \lesssim 1. \tag{2.2.3}$$

We say that a frequency envelope is admissible if $a_0 \approx 1$ and it is slowly varying, meaning that

$$a_j \leq 2^{\delta|j-k|}a_k, \quad j,k \geq 0, \quad 0 < \delta \ll 1.$$

An admissible frequency envelope always exists, say by

$$a_j = 2^{-\delta j} + \|u\|_X^{-1} \max_{k\geq 0} 2^{-\delta|j-k|}\|P_k u\|_X. \tag{2.2.4}$$

In (2.2.4) - and in the definitions of the X_T^s and H_x^s frequency envelope formulas defined later - there is a slight notational conflict, and $P_0 u$ should really be interpreted as $P_{\leq 0}u$.

Remark 2.2.2. Frequency envelopes will be particularly convenient for expediting the proof of continuous dependence later on.

Strichartz and maximal function estimates

Next we recall some standard linear estimates for the Schrödinger equation on the line, which will play a key role in our analysis. We begin with the relevant maximal function and Strichartz estimates for the linear Schrödinger flow:

Proposition 2.2.3. (Homogeneous Strichartz and maximal function estimates) For $v \in \mathcal{S}(\mathbb{R})$, $\theta \in [0,1]$ and $T \in (0,1)$, we have for $j > 0$

$$\begin{aligned} &\|e^{it\partial_x^2}v\|_{L_T^{\frac{4}{\theta}}L_x^{\frac{2}{1-\theta}}} \lesssim \|v\|_{L^2}, \\ &\|e^{it\partial_x^2}P_j v\|_{L_x^{\frac{2}{1-\theta}}L_T^{\frac{2}{\theta}}} \lesssim 2^{j(\frac{1}{2}-\theta)}\|v\|_{L^2}. \end{aligned} \tag{2.2.5}$$

Proof. See [204, Lemma 3.1]. ☐

We will also need the inhomogeneous versions of these estimates. Here $D_x^s := |\partial_x|^s$, $\langle D_x\rangle^s := (1+|\partial_x|^2)^{\frac{s}{2}}$, and $|\partial_x| := H\partial_x$ where H is the Hilbert transform, $\widehat{Hu} = -i\operatorname{sgn}(\xi)\widehat{u}$. We further note that both Propositions 2.2.3 and 2.2.4 hold for $j=0$, with the interpretation $P_0 = P_{\leq 0}$.

Proposition 2.2.4. (Inhomogeneous Strichartz and maximal function estimates) For $f \in \mathcal{S}(\mathbb{R}^2)$, $\theta \in [0,1]$ and $T \in (0,1)$, we have for $j > 0$

$$\begin{aligned}
&\left\| \int_0^t e^{i(t-s)\partial_x^2} f(s)ds \right\|_{L_T^{\frac{4}{\theta}} L_x^{\frac{2}{1-\theta}}} \lesssim \|f\|_{L_T^{(\frac{4}{\theta})'} L_x^{(\frac{2}{1-\theta})'}},\\
&\left\| \langle D_x\rangle^{\frac{\theta}{2}} \int_0^t e^{i(t-s)\partial_x^2} f(s)ds \right\|_{L_T^\infty L_x^2} \lesssim \|f\|_{L_x^{p(\theta)} L_T^{q(\theta)}},\\
&\left\| D_x^{\frac{1+\theta}{2}} \int_0^t e^{i(t-s)\partial_x^2} f(s)ds \right\|_{L_x^\infty L_T^2} \lesssim \|f\|_{L_x^{p(\theta)} L_T^{q(\theta)}},\\
&\left\| \langle D_x\rangle^{\frac{\theta}{2}} \int_0^t e^{i(t-s)\partial_x^2} P_j f(s)ds \right\|_{L_x^2 L_T^\infty} \lesssim 2^{\frac{j}{2}} \|f\|_{L_x^{p(\theta)} L_T^{q(\theta)}},\\
&\left\| \int_0^t e^{i(t-s)\partial_x^2} P_j f(s)ds \right\|_{L_x^{\frac{2}{1-\theta}} L_T^{\frac{2}{\theta}}} \lesssim 2^{j(\frac{1}{2}-\theta)} \|f\|_{L_T^1 L_x^2},
\end{aligned} \tag{2.2.6}$$

where

$$\frac{1}{p(\theta)} = \frac{3+\theta}{4}, \qquad \frac{1}{q(\theta)} = \frac{3-\theta}{4}. \tag{2.2.7}$$

Proof. See [204, Lemma 3.4 and Remark 3.7]. ☐

The following fractional Leibniz rules will also be useful for some of the following estimates:

Proposition 2.2.5. Let $\alpha \in (0,1)$, $\alpha_1, \alpha_2 \in [0,\alpha]$, $p, p_1, p_2, q, q_1, q_2 \in (1,\infty)$ satisfy $\alpha_1+\alpha_2 = \alpha$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then

$$\|D_x^\alpha(fg) - D_x^\alpha f g - f D_x^\alpha g\|_{L_x^p L_T^q} \lesssim \|D_x^{\alpha_1} f\|_{L_x^{p_1} L_T^{q_1}} \|D_x^{\alpha_2} g\|_{L_x^{p_2} L_T^{q_2}}. \tag{2.2.8}$$

The endpoint cases $q_1 = \infty, \alpha_1 = 0$ as well as $(p,q) = (1,2)$ are also allowed.

Proof. See [202, Lemma 2.6] or [204, Lemma 3.8]. ☐

Another variant of the fractional Leibniz rule for L_x^p spaces is as follows:

Proposition 2.2.6. Let $\alpha \in (0,1)$, $\alpha_1, \alpha_2 \in (0, \alpha)$ and $p \in [1, \infty)$, $1 < p_1, p_2 < \infty$ satisfy $\alpha_1 + \alpha_2 = \alpha$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then

$$\|D_x^\alpha (fg) - D_x^\alpha f g - f D_x^\alpha g\|_{L_x^p} \lesssim \|D_x^{\alpha_1} f\|_{L_x^{p_1}} \|D_x^{\alpha_2} g\|_{L_x^{p_2}}. \tag{2.2.9}$$

The endpoint case $\alpha_2 = 0$, $1 < p_2 \leq \infty$ is also allowed if $p > 1$.

Proof. See [202, Lemma 2.6]. □

Next, we need a vector-valued Moser type estimate which will be convenient when derivatives fall on $|u|^{2\sigma}$.

Proposition 2.2.7. Let $F \in C^1(\mathbb{C})$. Let $\alpha \in (0,1)$, $p, q, p_1, p_2, q_2 \in (1, \infty)$ and $q_1 \in (1, \infty]$ with

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}. \tag{2.2.10}$$

Then

$$\|D_x^\alpha F(u)\|_{L_x^p L_T^q} \lesssim \|F'(u)\|_{L_x^{p_1} L_T^{q_1}} \|D_x^\alpha u\|_{L_x^{p_2} L_T^{q_2}}. \tag{2.2.11}$$

Proof. See Theorem A.6 of [203]. □

We also recall the scalar version of the above estimate,

Proposition 2.2.8. Let $F \in C^1(\mathbb{C})$, $u \in L^\infty(\mathbb{R})$, $\alpha \in (0,1)$, $1 < p, q, r < \infty$, and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then

$$\|D_x^\alpha F(u)\|_{L^r} \lesssim \|F'(u)\|_{L^p} \|D_x^\alpha u\|_{L^q}. \tag{2.2.12}$$

Proof. See [80], Proposition 3.1. □

We will also make use of not only the standard Bernstein estimates (see, for example, [313, (A.2)-(A.6), page 333]) but the following vector-valued version:

Proposition 2.2.9. Let $1 \leq p, q \leq \infty$, $j > 0$ and $s \in \mathbb{R}$. Then we have

$$\|D_x^s P_j u\|_{L_x^p L_T^q} \sim 2^{js} \|P_j u\|_{L_x^p L_T^q}. \tag{2.2.13}$$

Proof. Let $\tilde{P}_j$ have corresponding multiplier $\tilde{\phi}_j$, where, as in the preliminaries on Littlewood-Paley theory, we have $\tilde{\phi}_j(\xi) = \tilde{\phi}(2^{-j}\xi)$. Notice that

$$D_x^s(\tilde{P}_j P_j u) = (D_x^s \mathcal{F}^{-1} \tilde{\phi}_j) * P_j u.$$

For each x, we have the inequality

$$\|D_x^s P_j u\|_{L_T^q} \leq |D_x^s \mathcal{F}^{-1}\tilde{\phi}_j| * \|P_j u\|_{L_T^q}.$$

Hence, applying L_x^p and Young's inequality, we have

$$\|D_x^s P_j u\|_{L_x^p L_T^q} \leq \|D_x^s \mathcal{F}^{-1}\tilde{\phi}_j\|_{L_x^1} \|P_j u\|_{L_x^p L_T^q} \lesssim 2^{js} \|P_j u\|_{L_x^p L_T^q}.$$

On the other hand,

$$2^{js}\|P_j u\|_{L_x^p L_T^q} = 2^{js}\|D_x^{-s} D_x^s P_j u\|_{L_x^p L_T^q} \lesssim \|D_x^s P_j u\|_{L_x^p L_T^q}.$$

□

A useful lemma

Finally, we need a Hölder estimate, which we will use to extract all of the $C^{1,2\sigma-1}$-regularity that our nonlinearity offers. We will use this lemma, e.g., when derivatives fall on $|u|^{2\sigma-2}\overline{u}$, or more generally on terms with regularity $C^{0,\alpha}$ for $0 < \alpha < 1$.

To set notation, for $\alpha \in (0,1]$ and $1 \leq p \leq \infty$ define the Hölder space $\dot{\Lambda}_\alpha^p(\mathbb{R})$ by

$$\|u\|_{\dot{\Lambda}_\alpha^p} := \sup_{|h|>0} \frac{\|u(\cdot + h) - u(\cdot)\|_{L^p}}{|h|^\alpha}. \tag{2.2.14}$$

This is just the usual homogeneous Hölder space $\dot{C}^{0,\alpha}$ when $p = \infty$.

Lemma 2.2.10. Suppose that $F \in \dot{C}^{0,\alpha}(\mathbb{C})$. Then for every $0 < \beta < \alpha < 1$, $p \in [1,\infty]$ with $\alpha p \geq 1$, we have

$$\|F(u)\|_{\dot{\Lambda}_\beta^p} \lesssim \|F\|_{\dot{C}^{0,\alpha}} \|u\|^\alpha_{W^{\frac{\beta}{\alpha},p\alpha}}. \tag{2.2.15}$$

Proof. We have

$$\begin{aligned}\frac{|F(u(x+h)) - F(u(x))|}{|h|^\beta} &= \frac{|F(u(x+h)) - F(u(x))|}{|u(x+h)-u(x)|^\alpha}\left(\frac{|u(x+h)-u(x)|}{|h|^{\frac{\beta}{\alpha}}}\right)^\alpha \\ &\leq \|F\|_{\dot{C}^{0,\alpha}}\left(\frac{|u(x+h)-u(x)|}{|h|^{\frac{\beta}{\alpha}}}\right)^\alpha.\end{aligned} \tag{2.2.16}$$

Hence,

$$
\begin{aligned}
\|F(u)\|_{\dot{\Lambda}^p_\beta} &\le \|F\|_{\dot{C}^{0,\alpha}} \sup_{|h|>0} \| \left(\frac{|u(x+h)-u(x)|}{|h|^{\frac{\beta}{\alpha}}} \right)^{\alpha} \|_{L^p} \\
&\le \|F\|_{\dot{C}^{0,\alpha}} \|u\|^{\alpha}_{\dot{\Lambda}^{p\alpha}_{\frac{\beta}{\alpha}}} \\
&\lesssim \|F\|_{\dot{C}^{0,\alpha}} \|u\|^{\alpha}_{W^{\frac{\beta}{\alpha},p\alpha}}
\end{aligned}
\tag{2.2.17}
$$

where the last line follows from a standard embedding (c.f. [313, Exercise A.21]). □

We also have the following very useful corollary of the above lemma which we will use extensively.

Corollary 2.2.11. Suppose that $F \in \dot{C}^{0,\alpha}(\mathbb{C})$ with $F(0) = 0$. Then for every $0 < \beta < \alpha < 1$, $p \in [1, \infty]$ with $\alpha p \ge 1$ and $\varepsilon \in (0, \alpha - \beta)$, we have

$$
\|F(u)\|_{W^{\beta,p}} \lesssim_{\varepsilon} \|F\|_{\dot{C}^{0,\alpha}} \|u\|^{\alpha}_{W^{\frac{\beta}{\alpha}+\varepsilon,p\alpha}}. \tag{2.2.18}
$$

Proof. This follows from the embedding (c.f. [313, Exercise A.21]),

$$
\|F(u)\|_{W^{\beta,p}} \lesssim_{\varepsilon} \|F(u)\|_{L^p} + \|F(u)\|_{\dot{\Lambda}^p_{\beta+\alpha\varepsilon}} \tag{2.2.19}
$$

and Lemma 2.2.10 as well as the fact that

$$
\|F(u)\|_{L^p} \lesssim \|F\|_{\dot{C}^{0,\alpha}} \|u\|^{\alpha}_{L^{p\alpha}}. \tag{2.2.20}
$$

□

Remark 2.2.12. It is easy to see that $F(z) = \overline{z}|z|^{2\sigma-2}$ meets the hypothesis of the above corollary (c.f. [127, Lemma 2.4]). The price to pay when using Corollary 2.2.11 is that there is a sort of "loss of regularity" when derivatives fall on $F(u)$ in the sense that a derivative of order $0 < s < 2\sigma - 1$ will be amplified by a factor of $\frac{1}{2\sigma-1}$.

2.3 Low regularity estimates

Now, we proceed with the proof of Theorem 2.1.2. By the scaling symmetry $u_\lambda(t,x) := \lambda^{\frac{1}{2\sigma}} u(\lambda^2 t, \lambda x)$, we see that the L^2_x norm is subcritical with respect to scaling. Hence, we will assume without loss of generality throughout that for some small $0 < \varepsilon \ll 1$ the initial data satisfies $\|u_0\|_{H^s_x} \le \varepsilon$. We then will obtain local well-posedness on the time interval $[-T, T]$ where $T \lesssim 1$ is fixed.

Function spaces

We now define the spaces where we seek solutions. To begin, we define our baseline Strichartz type space Y_T^0 via

$$\begin{aligned}\|u\|_{Y_T^0} := &\left(\sum_{j>0}\|P_j D_x^{\sigma-1}u\|^2_{L_T^4L_x^\infty}\right)^{\frac{1}{2}} + \left(\sum_{j>0}\|P_j D_x^{\sigma-\frac{1}{2}}u\|^2_{L_x^\infty L_T^2}\right)^{\frac{1}{2}} \\ &+ \left(\sum_{j>0}\|P_j D_x^{\sigma-\frac{3}{2}}u\|^2_{L_x^2L_T^\infty}\right)^{\frac{1}{2}} + \|P_{\leq 0}u\|_{L_x^2L_T^\infty}.\end{aligned} \tag{2.3.1}$$

Then we define the space X_T^0 by:

$$\|u\|_{X_T^0} := \left(\sum_{j>0}\|P_j u\|^2_{L_T^\infty L_x^2}\right)^{\frac{1}{2}} + \|P_{\leq 0}u\|_{L_T^\infty L_x^2} + \|u\|_{Y_T^0}. \tag{2.3.2}$$

For higher Sobolev indices, $s \geq 0$, we define the spaces X_T^s and Y_T^s by

$$\|u\|_{Y_T^s} := \|\langle D_x\rangle^s u\|_{Y_T^0}, \quad \|u\|_{X_T^s} := \|\langle D_x\rangle^s u\|_{X_T^0}. \tag{2.3.3}$$

One should observe that we trivially have $\|u\|_{C([-T,T];H_x^s)} \leq \|u\|_{X_T^s}$.

Remark 2.3.1. One might wonder why the above Y_T^s space is not defined in a more standard way, where one replaces σ with 1. Indeed, one can see from the proof of the following estimates that by using this stronger norm, one will incur a loss of $1-\sigma$ derivatives in excess of the $L_T^\infty H_x^s$ norm. The function spaces defined above account for this loss.

Finally, it will be convenient to define the weaker norm S_T^s which just involves the purely Strichartz components of the X_T^s norm. Namely,

$$\|u\|_{S_T^s} = \|P_{\leq 0}u\|_{L_T^\infty L_x^2} + \left(\sum_{j>0}\|P_j\langle D_x\rangle^s u\|^2_{L_T^\infty L_x^2}\right)^{\frac{1}{2}} + \left(\sum_{j>0}\|P_j\langle D_x\rangle^{s-1+\sigma}u\|^2_{L_T^4L_x^\infty}\right)^{\frac{1}{2}}. \tag{2.3.4}$$

The behavior of the S_T^1 norm will be relevant for continuing a local solution to a global one when $\sigma \in (\frac{\sqrt{3}}{2}, 1)$ in both the low and high regularity regimes.

X_T^s frequency envelopes

It is easy to see that for $s \geq 0$, we have

$$\|P_{\leq 0}u\|^2_{X_T^s} + \sum_{j=1}^{\infty}\|P_j u\|^2_{X_T^s} \sim \|u\|^2_{X_T^s}. \tag{2.3.5}$$

Hence, for $u \in X^s_T$, we use b_j to denote the X^s_T frequency envelope for u defined by

$$b_j = 2^{-\delta j} + \|u\|_{X^s_T}^{-1} \max_{k\geq 0} 2^{-\delta|j-k|}\|P_k u\|_{X^s_T} \tag{2.3.6}$$

where δ is some small, but fixed, positive parameter. Similarly, for $v \in H^s_x$, we use a_j to denote the H^s_x frequency envelope for v defined by

$$a_j = 2^{-\delta j} + \|v\|_{H^s_x}^{-1} \max_{k\geq 0} 2^{-\delta|j-k|}\|P_k v\|_{H^s_x}. \tag{2.3.7}$$

Unless otherwise stated, X^s_T and H^s_x frequency envelopes will always be defined by the above formulae.

Remark 2.3.2. In an identical fashion, one can also define S^s_T frequency envelopes.

Next, we state a technical lemma which will be useful for tracking the contributions of the rough part of the nonlinearity in (gDNLS) when derivatives fall on it.

Lemma 2.3.3. (Moser type estimate) Let $s \in [1, \frac{3}{2}]$, $\sigma \in (\frac{1}{2}, 1)$, $0 < T \lesssim 1$ and let b_j be a X^s_T frequency envelope for u. Write $\alpha = s - 1 + \sigma < 2\sigma$. For $j > 0$, we have the following Moser type estimate,

$$\|D^\alpha_x P_j |u|^{2\sigma}\|_{L^2_T L^\infty_x} \lesssim b_j \|u\|_{S^1_T}^{2\sigma-1}\|u\|_{X^s_T}. \tag{2.3.8}$$

Proof. There are two cases to consider. First assume $\alpha > 1$. We have

$$\begin{aligned}\|D^\alpha_x P_j(|u|^{2\sigma})\|_{L^2_T L^\infty_x} &\lesssim \|P_j D^{\alpha-1}_x(|u|^{2\sigma-2}\overline{u}u_x)\|_{L^2_T L^\infty_x} \\ &\lesssim \|P_j D^{\alpha-1}_x(P_{<j-4}(|u|^{2\sigma-2}\overline{u})u_x)\|_{L^2_T L^\infty_x} + \|P_j D^{\alpha-1}_x(P_{\geq j-4}(|u|^{2\sigma-2}\overline{u})u_x)\|_{L^2_T L^\infty_x}.\end{aligned} \tag{2.3.9}$$

For the first term, we have by Bernstein,

$$\begin{aligned}\|P_j D^{\alpha-1}_x(P_{<j-4}(|u|^{2\sigma-2}\overline{u})u_x)\|_{L^2_T L^\infty_x} &= \|P_j D^{\alpha-1}_x(P_{<j-4}(|u|^{2\sigma-2}\overline{u})\tilde{P}_j u_x)\|_{L^2_T L^\infty_x} \\ &\lesssim 2^{j(\alpha-1)}\|u\|^{2\sigma-1}_{L^\infty_T L^\infty_x}\|\tilde{P}_j u_x\|_{L^2_T L^\infty_x} \\ &\lesssim \|u\|^{2\sigma-1}_{S^1_T}\|D^\alpha_x \tilde{P}_j u\|_{L^2_T L^\infty_x} \\ &\lesssim b_j\|u\|^{2\sigma-1}_{S^1_T}\|u\|_{X^s_T}.\end{aligned} \tag{2.3.10}$$

For the second term, we have for $\delta > 0$ small (under the additional assumption that $2^{-\delta j} \lesssim b_j$)

$$\begin{aligned}\|P_j D^{\alpha-1}_x(P_{\geq j-4}(|u|^{2\sigma-2}\overline{u})u_x)\|_{L^2_T L^\infty_x} &\lesssim 2^{j(\alpha-1)}\|P_{\geq j-4}(|u|^{2\sigma-2}\overline{u})u_x\|_{L^2_T L^\infty_x} \\ &\lesssim 2^{j(\alpha-1)}\|P_{\geq j-4}(|u|^{2\sigma-2}\overline{u})\|_{L^4_T L^\infty_x}\|u_x\|_{L^4_T L^\infty_x} \\ &\lesssim b_j\|D^{\alpha-1+\delta}_x(|u|^{2\sigma-2}\overline{u})\|_{L^4_T L^\infty_x}\|u_x\|_{L^4_T L^\infty_x} \\ &\lesssim b_j\|D^{\alpha-1+\delta}_x(|u|^{2\sigma-2}\overline{u})\|_{L^4_T L^\infty_x}\|u\|_{X^s_T}.\end{aligned} \tag{2.3.11}$$

It now suffices to show that

$$\|D_x^{\alpha-1+\delta}(|u|^{2\sigma-2}\overline{u})\|_{L_T^4 L_x^\infty} \lesssim \|u\|_{S_T^1}^{2\sigma-1}.$$

For this we fix $\varepsilon > 0$ small and invoke Corollary 2.2.11 and the fact that $2\sigma - 1 < 1$,

$$\begin{aligned}\|D_x^{\alpha-1+\delta}(|u|^{2\sigma-2}\overline{u})\|_{L_T^4 L_x^\infty} &\lesssim_T \|\langle D_x\rangle^{\frac{\alpha-1+\delta+\varepsilon}{2\sigma-1}} u\|_{L_T^4 L_x^\infty}^{2\sigma-1} \\ &\lesssim \|u\|_{S_T^1}^{2\sigma-1}\end{aligned} \tag{2.3.12}$$

where in the last line we take ε, δ small enough and used that $\frac{\alpha-1}{2\sigma-1} < \sigma$ when $s \in [1, \frac{3}{2}]$ and $\sigma \in (\frac{1}{2}, 1)$.

This handles the case $\alpha > 1$. Next, we assume $0 < \alpha \leq 1$. For this, we write

$$P_j|u|^{2\sigma} = P_j|P_{<j}u|^{2\sigma} + P_j(|u|^{2\sigma} - |P_{<j}u|^{2\sigma}). \tag{2.3.13}$$

We have for the first term,

$$\begin{aligned}\|D_x^\alpha P_j|P_{<j}u|^{2\sigma}\|_{L_x^\infty} &\lesssim 2^{j(\alpha-1)}\|P_j(|P_{<j}u|^{2\sigma-2}\overline{P_{<j}u}P_{<j}u_x)\|_{L_x^\infty} \\ &\lesssim 2^{j(\alpha-1)}\|P_j(P_{<j-4}(|P_{<j}u|^{2\sigma-2}\overline{P_{<j}u})\tilde{P}_j u_x)\|_{L_x^\infty} \\ &+ 2^{j(\alpha-1)}\|P_j(P_{\geq j-4}(|P_{<j}u|^{2\sigma-2}\overline{P_{<j}u})P_{<j}u_x)\|_{L_x^\infty} \\ &\lesssim \|u\|_{L_x^\infty}^{2\sigma-1}\|\tilde{P}_j D_x^\alpha u\|_{L_x^\infty} + 2^{-j\delta}\|D_x^{2\delta}(|P_{<j}u|^{2\sigma-2}\overline{P_{<j}u})\|_{L_x^\infty}\|D_x^{\alpha-1-\delta}u_x\|_{L_x^\infty}.\end{aligned} \tag{2.3.14}$$

Hence, by taking δ small enough, using Corollary 2.2.11, and the fact that $2^{-j\delta} \lesssim b_j$, we obtain

$$\|D_x^\alpha P_j|P_{<j}u|^{2\sigma}\|_{L_T^2 L_x^\infty} \lesssim_T b_j\|u\|_{S_T^1}^{2\sigma-1}\|u\|_{X_T^s}. \tag{2.3.15}$$

Next, we estimate

$$\begin{aligned}\|P_j D_x^\alpha(|u|^{2\sigma} - |P_{<j}u|^{2\sigma})\|_{L_T^2 L_x^\infty} &\lesssim 2^{j\alpha}\|u\|_{L_T^\infty L_x^\infty}^{2\sigma-1}\sum_{k\geq j}\|P_k u\|_{L_T^2 L_x^\infty} \\ &\lesssim \|u\|_{S_T^1}^{2\sigma-1}\|u\|_{X_T^s}\sum_{k\geq j}2^{-\alpha|k-j|}b_k \\ &\lesssim b_j\|u\|_{S_T^1}^{2\sigma-1}\|u\|_{X_T^s}\end{aligned} \tag{2.3.16}$$

where in the last line, we used the slowly varying property of b_j. This completes the proof. $\square$

Remark 2.3.4. By repeating the proof almost verbatim, and taking b_j instead to be a S^s_T frequency envelope for u, we can modify the conclusion of the lemma to

$$\|D^\alpha_x P_j|u|^{2\sigma}\|_{L^2_T L^\infty_x} \lesssim b_j \|u\|^{2\sigma-1}_{S^1_T}\|u\|_{S^s_T}. \tag{2.3.17}$$

Remark 2.3.5. The $\|u\|^{2\sigma-1}_{S^1_T}$ coefficient in the estimate (2.3.8) could be optimized in terms of the parameters s and σ. We do not pursue this, for the sake of simplicity and also because it does not improve any of the later estimates in an important way.

Uniform bounds

In this subsection, we prove a priori estimates for solutions to (gDNLS). First, we prove uniform X^s_T bounds:

Proposition 2.3.6. Let $0 < \varepsilon \ll 1$, $s \in [1, \frac{3}{2}]$, $\sigma \in (\frac{\sqrt{3}}{2}, 1)$ and let $u_0 \in H^s_x$ with $\|u_0\|_{H^s_x} \leq \varepsilon$. Let $T \lesssim 1$. Suppose $u \in X^s_T$ solves the equation,

$$\begin{cases} (i\partial_t + \partial_x^2)u = i|u|^{2\sigma}\partial_x u, \\ u(0) = u_0. \end{cases} \tag{2.3.18}$$

Furthermore, let a_j and b_j be a H^s_x and X^s_T frequency envelope for u_0 and u (on the time interval $[0, T]$), respectively, as defined in Section 2.3. Then we have the following X^s_T estimates for $j > 0$,

a) (Frequency localized X^s_T bound)

$$\|P_j u\|_{X^s_T} \lesssim_{\|u\|_{S^1_T}} a_j\|u_0\|_{H^s_x} + T^{\frac{1}{2}} b_j(1 + \|u\|^{4\sigma}_{S^1_T})\|u\|_{X^s_T} + T^{\frac{1-\sigma}{2}} b_j \|u\|^\sigma_{X^1_T}\|u\|_{X^s_T}. \tag{2.3.19}$$

b) (Uniform X^s_T bound)

$$\|u\|_{X^s_T} \lesssim_{\|u\|_{X^1_T}} \|u_0\|_{H^s_x} \leq \varepsilon. \tag{2.3.20}$$

We will also need the following result:

Proposition 2.3.7. Let $0 < \varepsilon \ll 1$ and σ, T and s be as in Proposition 2.3.6. Suppose $v \in X^0_T$ is a solution to the equation,

$$\begin{cases} (i\partial_t + \partial_x^2)v = i|w|^{2\sigma}\partial_x v + g\partial_x a v + \overline{g}\partial_x a\overline{v}, \\ v(0) = v_0, \end{cases} \tag{2.3.21}$$

for some $w \in X^1_T$ solving (gDNLS) (with possibly different initial data), $g \in Z := Z_T := L_x^{\frac{2}{2\sigma-1}} L_T^\infty \cap L_T^\infty W_x^{\frac{3}{4\sigma}-\frac{1}{2}+\varepsilon,\infty} \cap L_T^4 W_x^{\frac{3}{2}-\sigma+\varepsilon,\infty}$ and $a \in X^1_T$, all with sufficiently small norm $\ll 1$. Then v satisfies the bound

$$\|v\|_{X^0_T} \lesssim \|v_0\|_{L^2}. \tag{2.3.22}$$

Remark 2.3.8. In practice g will correspond to terms which are of similar regularity to the term $|u|^{2\sigma-1}$. For such terms to lie in Z (specifically the latter two components of this norm), we will need $\sigma > \frac{\sqrt{3}}{2}$. This will be elaborated on later in the proof.

Remark 2.3.9. Proposition 2.3.7 will be useful for establishing difference estimates for solutions in the weaker topology, X^0_T. This will allow us to show uniqueness for X^1_T solutions, and to prove a weak Lipschitz type bound for the solution map.

We begin with the proof of Proposition 2.3.6. We divide the relevant estimates into two parts. First, we control the Y^s_T component of the norm. Then we do an energy type estimate to control the $L^\infty_T H^s_x$ component. For this purpose, we have the following lemmas:

Lemma 2.3.10. (Y^s_T estimate) Let $s \in [1, \frac{3}{2}]$, $\sigma \in (\frac{1}{2}, 1)$ and let u, T, a_j and b_j be as in Proposition 2.3.6. Then for $j > 0$ we have

$$\|P_j u\|_{Y^s_T} \lesssim_{\|u\|_{S^1_T}} a_j \|u_0\|_{H^s_x} + T^{\frac{1}{2}} b_j (1 + \|u\|^{4\sigma}_{S^1_T}) \|u\|_{X^s_T}. \tag{2.3.23}$$

Lemma 2.3.11. ($L^\infty_T H^s_x$ estimate) Let s, σ, T, a_j, b_j and u be as in *Proposition* 2.3.6. Then for $j > 0$ we have

$$\|P_j u\|_{L^\infty_T H^s_x} \lesssim a_j \|u_0\|_{H^s_x} + T^{\frac{1-\sigma}{2}} b_j \|u\|^{\sigma}_{X^1_T} \|u\|_{X^s_T}. \tag{2.3.24}$$

Proof. We begin with the proof of Lemma 2.3.10. For this purpose, let us apply P_j to (2.3.18) and write

$$(i\partial_t + \partial_x^2) u_j = i P_{<j-4} |u|^{2\sigma} \partial_x u_j + g_j \tag{2.3.25}$$

where

$$g_j = i P_j (P_{\geq j-4} |u|^{2\sigma} \partial_x u) + i [P_j, P_{<j-4} |u|^{2\sigma}] \partial_x u. \tag{2.3.26}$$

The term

$$i P_{<j-4} |u|^{2\sigma} \partial_x u_j \tag{2.3.27}$$

which corresponds to the worst interactions between $\partial_x u$ and $|u|^{2\sigma}$ is non-perturbative, and can lead to loss of derivatives in the Y^s_T estimates for u_j. It is desirable to remove as much of this bad interaction as possible. As mentioned earlier, one might try to remove it entirely with a gauge transformation, but this will not work, because the function $z \mapsto |z|^{2\sigma}$ is not smooth enough. Fortunately, in some sense, formally, the worst terms introduced by a gauge transformation are only poorly behaved when u is small (i.e. sufficiently close to 0). On the other hand, if u is sufficiently small (on a scale depending on j), then we expect to be able to treat the associated part of the term (2.3.27) perturbatively. One then expects to be able to remove the other part (in which u is bounded away from zero) with a gauge transformation, and gain some mileage.

With this strategy in mind, let φ be a smooth compactly supported function on $\mathbb{R}$ with $\varphi = 1$ on the unit interval and zero outside $(-2, 2)$. Likewise, define $\chi = 1 - \varphi$. We want to tailor these functions to a particular frequency, which we do by defining the rescaled functions $\varphi_j(x) = \varphi(2^j x)$ and $\chi_j(x) = \chi(2^j x)$. Next, we further rewrite (2.3.25) as the following equation,

$$(i\partial_t + \partial_x^2)u_j = iP_{<j-4}[\chi_j(|u|^2)|u|^{2\sigma}]\partial_x u_j + iP_{<j-4}[\varphi_j(|u|^2)|u|^{2\sigma}]\partial_x u_j + g_j. \tag{2.3.28}$$

Remark 2.3.12. One might wonder whether one can modify the 2^j scale in the definition of φ_j to $2^{j\alpha}$ for some $\alpha > 0$. It turns out that $\alpha = 1$ is the optimal choice, as one can ascertain from repeating the estimates below with this new parameter α. This optimization is obtained by balancing the contributions from the terms I_j^1 and I_j^3 in the below estimates.

Now, we do a partial gauge transformation to remove $iP_{<j-4}[\chi_j(|u|^2)|u|^{2\sigma}]\partial_x u_j$, which corresponds to the part of (2.3.27) for which the coefficient $|u|^{2\sigma}$ is bounded below by $2^{-j\sigma}$. Indeed, define

$$\Phi_j(t, x) := -\frac{1}{2}P_{<j-4}\partial_x^{-1}[\chi_j(|u|^2)|u|^{2\sigma}] \tag{2.3.29}$$

where

$$(\partial_x^{-1} f)(x) := \int_{-\infty}^{x} f(y)dy \tag{2.3.30}$$

and then define

$$w_j := u_j e^{i\Phi_j}. \tag{2.3.31}$$

Before proceeding, we need the following technical estimate which relates u_j to w_j.

Lemma 2.3.13. Let S refer to any of the four spaces, $L_T^\infty L_x^2$, $L_x^\infty L_T^2$, $L_x^2 L_T^\infty$, or $L_T^4 L_x^\infty$. Let $\beta \in (-1, 1)$ and $0 < \varepsilon \ll 1$. Then for $j > 0$, we have

$$\|\langle D_x\rangle^\beta u_j\|_S \lesssim_\varepsilon (1 + \|u\|_{S_T^1})^{2\sigma}(\|\langle D_x\rangle^\beta \tilde{P}_j w_j\|_S + \|\langle D_x\rangle^{\beta-\varepsilon} w_j\|_S). \tag{2.3.32}$$

Remark 2.3.14. As a brief remark, the range on β accounts for (more than) the greatest range of derivatives allowed in any component of the $X_T^{1-\sigma}$ norm, which will correspond to the situation in which we apply the estimate. Strictly speaking, this is overkill, but it lets us avoid dealing with several individual cases. Also, the $\beta - \varepsilon$ factor in the second term in the above estimate is to compensate for terms in which w_j is not frequency localized. In particular, later when applying Proposition 2.2.4, the ε will allow us to sum up the individual frequency dyadic contributions of w_j.

Proof. We have using the fact that u_j is frequency localized to frequency $\sim 2^j$,

$$\begin{aligned}\|\langle D_x\rangle^\beta u_j\|_S &= \|\langle D_x\rangle^\beta \tilde{P}_j(e^{-i\Phi_j}w_j)\|_S \\ &\lesssim \|D_x^\beta \tilde{P}_j(P_{<j-2}e^{-i\Phi_j}\tilde{P}_j w_j)\|_S + \|D_x^\beta \tilde{P}_j(P_{\geq j-2}e^{-i\Phi_j}w_j)\|_S.\end{aligned} \tag{2.3.33}$$

For the first term, we have by the (vector-valued) Bernstein's inequality

$$\|D_x^\beta \tilde{P}_j(P_{<j-2}e^{-i\Phi_j}\tilde{P}_j w_j)\|_S \lesssim \|D_x^\beta \tilde{P}_j w_j\|_S. \tag{2.3.34}$$

For the second term, we have from Bernstein's inequality (and since $j > 0$),

$$\begin{aligned}\|D_x^\beta \tilde{P}_j(P_{\geq j-2}e^{-i\Phi_j}w_j)\|_S &\lesssim 2^{j\beta}\|\tilde{P}_j(P_{\geq j-2}e^{-i\Phi_j}w_j)\|_S \\ &\lesssim 2^{j\beta}\|P_{\geq j-2}e^{-i\Phi_j}\|_{L_T^\infty L_x^\infty}\|P_{<j+2}w_j\|_S + 2^{j\beta}\sum_{k\geq j}\|\tilde{P}_k e^{-i\Phi_j}\|_{L_T^\infty L_x^\infty}\|\tilde{P}_k w_j\|_S \\ &\lesssim_\varepsilon \|P_{\geq j-2}D_x^{|\beta|+2\varepsilon}e^{-i\Phi_j}\|_{L_T^\infty L_x^\infty}\|\langle D_x\rangle^{\beta-\varepsilon}w_j\|_S\end{aligned} \tag{2.3.35}$$

where $\varepsilon > 0$ is small enough so that for instance, $|\beta| + 2\varepsilon < 1$. Then we have by Bernstein,

$$\|P_{\geq j-2}D_x^{|\beta|+2\varepsilon}e^{-i\Phi_j}\|_{L_T^\infty L_x^\infty} \lesssim \|\partial_x P_{\geq j-2}e^{-i\Phi_j}\|_{L_T^\infty L_x^\infty} \lesssim \|u\|_{S_T^1}^{2\sigma}. \tag{2.3.36}$$

Combining the above estimates completes the proof. □

Given Lemma 2.3.13, we are in a position to convert estimates for w_j into estimates for u_j. A direct computation shows that w_j satisfies the following equation:

$$\begin{cases}(i\partial_t + \partial_x^2)w_j &= ie^{i\Phi_j}P_{<j-4}[\varphi_j(|u|^2)|u|^{2\sigma}]\partial_x u_j + (-\partial_t\Phi_j + i\partial_x^2\Phi_j - (\partial_x\Phi_j)^2)w_j + e^{i\Phi_j}g_j, \\ \qquad w_j(0) &= e^{i\Phi_j}u_j(0).\end{cases} \tag{2.3.37}$$

The goal is to prove a priori estimates for w_j - and hence u_j - in Y_T^s. We observe a couple of useful facts. First, by Bernstein, we have $\|u_j\|_{Y_T^s} \lesssim 2^{j(\sigma+s-1)}\|u_j\|_{Y_T^{1-\sigma}}$. Secondly, we obviously have $\|gw_j\|_{L^1_T L^2_x} = \|gu_j\|_{L^1_T L^2_x}$ for measurable functions, g. Using these observations, Lemma 2.3.13, the maximal function estimates and the usual Strichartz estimates from Propositions 2.2.3 and 2.2.4 we have that

$$\begin{aligned}\frac{\|u_j\|_{Y^s_T}}{(1+\|u\|_{S^1_T})^{2\sigma}} &\lesssim \|u_j(0)\|_{H^s_x} + 2^{j(\sigma+s-1)}\|P_{<j-4}[\varphi_j(|u|^2)|u|^{2\sigma}]\partial_x u_j\|_{L^1_T L^2_x} + 2^{j(\sigma+s-1)}\|g_j\|_{L^1_T L^2_x}\\ &+ 2^{j(\sigma+s-1)}\|\partial_t \Phi_j u_j\|_{L^1_T L^2_x} + 2^{j(\sigma+s-1)}\|\partial_x^2 \Phi_j u_j\|_{L^1_T L^2_x} + 2^{j(\sigma+s-1)}\|(\partial_x \Phi_j)^2 u_j\|_{L^1_T L^2_x}\\ &:= \|u_j(0)\|_{H^s_x} + I_1^j + I_2^j + I_3^j + I_4^j + I_5^j.\end{aligned} \tag{2.3.38}$$

We now estimate each of the above terms.

Estimate for I_1^j

By Bernstein and the fact that $|u| \lesssim 2^{-\frac{j}{2}}$ on the support of φ_j,

$$\begin{aligned}2^{j(\sigma+s-1)}\|P_{<j-4}[\varphi_j(|u|^2)|u|^{2\sigma}]\partial_x u_j\|_{L^1_T L^2_x} &\lesssim 2^{j(\sigma+s-1)}\|\varphi_j(|u|^2)|u|^{2\sigma}\|_{L^1_T L^\infty_x}\|\partial_x u_j\|_{L^\infty_T L^2_x}\\ &\lesssim T\|u_j\|_{L^\infty_T H^s_x}\\ &\lesssim T b_j \|u\|_{X^s_T}.\end{aligned} \tag{2.3.39}$$

Estimate for I_2^j

We have

$$g_j = iP_j(P_{\geq j-4}|u|^{2\sigma}\partial_x u) + i[P_j, P_{<j-4}|u|^{2\sigma}]\partial_x \tilde{P}_j u \tag{2.3.40}$$

where $\tilde{P}_j$ is a "fattened" projection to frequency $\sim 2^j$. By the standard Littlewood-Paley trichotomy, we write

$$\begin{aligned}P_j(P_{\geq j-4}|u|^{2\sigma}\partial_x u) &= P_j(\tilde{P}_j|u|^{2\sigma}\partial_x \tilde{P}_{<j}u) + P_j(\tilde{P}_j|u|^{2\sigma}\tilde{P}_j\partial_x u)\\ &+ \sum_{k>j} P_j(\tilde{P}_k|u|^{2\sigma}\tilde{P}_k\partial_x u).\end{aligned} \tag{2.3.41}$$

For the first term, we have by the Moser estimate (2.3.8) and Bernstein's inequality,

$$\begin{aligned}2^{j(\sigma+s-1)}\|P_j(\tilde{P}_j|u|^{2\sigma}\partial_x\tilde{P}_{<j}u)\|_{L^1_T L^2_x} &\lesssim 2^{j(\sigma+s-1)}\|\tilde{P}_j|u|^{2\sigma}\|_{L^1_T L^\infty_x}\|\partial_x u\|_{L^\infty_T L^2_x}\\ &\lesssim \|\tilde{P}_j D_x^{\sigma+s-1}|u|^{2\sigma}\|_{L^1_T L^\infty_x}\|\partial_x u\|_{L^\infty_T L^2_x}\\ &\lesssim T^{\frac{1}{2}} b_j \|u\|^{2\sigma}_{S^1_T}\|u\|_{X^s_T}.\end{aligned} \tag{2.3.42}$$

The second term is dealt with similarly. For the third term, we have by Bernstein's inequality

$$\begin{aligned}
&2^{j(\sigma+s-1)}\|\sum_{k>j} P_j(\tilde{P}_k|u|^{2\sigma}\tilde{P}_k\partial_x u)\|_{L^1_T L^2_x}\\
&\lesssim T^{\frac{3}{4}}\sum_{k>j}\|\tilde{P}_k u\|_{L^4_T L^\infty_x} 2^{j(\sigma+s-1)}2^k\|\tilde{P}_k|u|^{2\sigma}\|_{L^\infty_T L^2_x}\\
&\lesssim T^{\frac{3}{4}}\sum_{k>j}2^{(j-k)(\sigma+s-1)}\|D_x^{\sigma+s-1}\tilde{P}_k u\|_{L^4_T L^\infty_x}\|\tilde{P}_k\partial_x|u|^{2\sigma}\|_{L^\infty_T L^2_x}\\
&\lesssim T^{\frac{3}{4}}\|u\|^{2\sigma}_{S^1_T}\|u\|_{X^s_T}\sum_{k>j}2^{-(\sigma+s-1)|k-j|}b_k\\
&\lesssim T^{\frac{3}{4}}b_j\|u\|_{X^s_T}\|u\|^{2\sigma}_{S^1_T}\sum_{k>j}2^{-(\sigma+s-1-\delta)|k-j|}\\
&\lesssim T^{\frac{3}{4}}b_j\|u\|_{X^s_T}\|u\|^{2\sigma}_{S^1_T}.
\end{aligned}\tag{2.3.43}$$

For the commutator term, we have by Lemma 2.2.1

$$2^{j(\sigma+s-1)}[P_j, P_{<j-4}|u|^{2\sigma}]\partial_x\tilde{P}_j u = 2^{j(\sigma+s-2)}L(\partial_x P_{<j-4}|u|^{2\sigma}, \tilde{P}_j\partial_x u)\tag{2.3.44}$$

for some appropriate translation invariant expression L.

This term is easily estimated by

$$\begin{aligned}
2^{j(\sigma+s-2)}\|L(\partial_x P_{<j-4}|u|^{2\sigma}, \tilde{P}_j\partial_x u)\|_{L^1_T L^2_x} &\lesssim 2^{j(\sigma+s-2)}\|\partial_x P_{<j-4}|u|^{2\sigma}\|_{L^\infty_T L^2_x}\|\tilde{P}_j\partial_x u\|_{L^1_T L^\infty_x}\\
&\lesssim \|u\|^{2\sigma-1}_{L^\infty_T L^\infty_x}\|\partial_x u\|_{L^\infty_T L^2_x}\|\tilde{P}_j D_x^{\sigma+s-1}u\|_{L^1_T L^\infty_x}\\
&\lesssim b_j T^{\frac{3}{4}}\|u\|^{2\sigma}_{S^1_T}\|u\|_{X^s_T}.
\end{aligned}\tag{2.3.45}$$

Hence, we have

$$I_2^j \lesssim T^{\frac{1}{2}}b_j\|u\|_{X^s_T}\|u\|^{2\sigma}_{S^1_T}.\tag{2.3.46}$$

Estimate for I_3^j

We expand

$$\partial_t\Phi_j = -\frac{1}{2}P_{<j-4}\partial_x^{-1}[2^j\chi'(2^j|u|^2)\partial_t|u|^2|u|^{2\sigma}] - \frac{1}{2}P_{<j-4}\partial_x^{-1}[\chi_j(|u|^2)\partial_t|u|^{2\sigma}] =: J_1 + J_2.\tag{2.3.47}$$

We have

$$
\begin{aligned}
J_1 &= -\frac{1}{2} P_{<j-4}\partial_x^{-1}[2^j\chi'(2^j|u|^2)\partial_t|u|^2|u|^{2\sigma}] \\
&= -P_{<j-4}\partial_x^{-1}[2^j\chi'(2^j|u|^2)\mathrm{Re}(\overline{u}u_t)|u|^{2\sigma}] \\
&= -P_{<j-4}\partial_x^{-1}[2^j\chi'(2^j|u|^2)\mathrm{Re}(i\overline{u}u_{xx})|u|^{2\sigma}] - P_{<j-4}\partial_x^{-1}[2^j\chi'(2^j|u|^2)\mathrm{Re}(\overline{u}|u|^{2\sigma}u_x)|u|^{2\sigma}] \\
&= -P_{<j-4}\partial_x^{-1}[2^j\chi'(2^j|u|^2)\partial_x\mathrm{Re}(i\overline{u}u_x)|u|^{2\sigma}] - P_{<j-4}\partial_x^{-1}[2^j\chi'(2^j|u|^2)\mathrm{Re}(\overline{u}|u|^{2\sigma}u_x)|u|^{2\sigma}] \\
&:= K_1 + K_2.
\end{aligned}
\tag{2.3.48}
$$

For the first term, K_1, in (2.3.48) we write

$$
\begin{aligned}
-P_{<j-4}\partial_x^{-1}[2^j\chi'(2^j|u|^2)\partial_x\mathrm{Re}(i\overline{u}u_x)|u|^{2\sigma}] &= -P_{<j-4}[2^j\chi'(2^j|u|^2)\mathrm{Re}(i\overline{u}u_x)|u|^{2\sigma}] \\
&\quad + P_{<j-4}\partial_x^{-1}[2^{2j}\chi''(2^j|u|^2)\partial_x|u|^2\mathrm{Re}(i\overline{u}u_x)|u|^{2\sigma}] \\
&\quad + P_{<j-4}\partial_x^{-1}[2^j\chi'(2^j|u|^2)\mathrm{Re}(i\overline{u}u_x)\partial_x|u|^{2\sigma}].
\end{aligned}
\tag{2.3.49}
$$

We have for the first term in (2.3.49)

$$
\begin{aligned}
\|P_{<j-4}[2^j\chi'(2^j|u|^2)\mathrm{Re}(i\overline{u}u_x)|u|^{2\sigma}]\|_{L_T^\infty L_x^2} &\lesssim 2^j\|\chi'(2^j|u|^2)\mathrm{Re}(i\overline{u}u_x)|u|^{2\sigma}\|_{L_T^\infty L_x^2} \\
&\lesssim \|u\|_{L_T^\infty L_x^\infty}^{2\sigma-1}\|u_x\|_{L_T^\infty L_x^2}
\end{aligned}
\tag{2.3.50}
$$

where we used the fact that

$$
\|\chi'(2^j|u|^2)|u|^{2\sigma+1}\|_{L_T^\infty L_x^\infty} = \|\varphi'(2^j|u|^2)|u|^{2\sigma+1}\|_{L_T^\infty L_x^\infty} \lesssim 2^{-j}\|u\|_{L_T^\infty L_x^\infty}^{2\sigma-1}. \tag{2.3.51}
$$

Now, for the second term in (2.3.49), we have

$$
\begin{aligned}
2^{2j}\|P_{<j-4}\partial_x^{-1}[\chi''(2^j|u|^2)\partial_x|u|^2\mathrm{Re}(i\overline{u}u_x)|u|^{2\sigma}]\|_{L_T^\infty L_x^\infty} &\lesssim 2^{2j}\|\chi''(2^j|u|^2)\partial_x|u|^2\mathrm{Re}(i\overline{u}u_x)|u|^{2\sigma}\|_{L_T^\infty L_x^1} \\
&\lesssim 2^{2j}\|\varphi''(2^j|u|^2)\mathrm{Re}(\overline{u}u_x)\mathrm{Re}(i\overline{u}u_x)|u|^{2\sigma}\|_{L_T^\infty L_x^1} \\
&\lesssim 2^{j(1-\sigma)}\|u_x\|_{L_T^\infty L_x^2}^2.
\end{aligned}
\tag{2.3.52}
$$

The third term in (2.3.49) is estimated similarly to the second term.

Hence, we obtain that $2^{j(\sigma+s-1)}\|K_1u_j\|_{L_T^1L_x^2}$ is estimated by

$$
\begin{aligned}
&2^{j(\sigma+s-1)}2^{j(1-\sigma)}T\|u_x\|_{L_T^\infty L_x^2}^2\|u_j\|_{L_T^\infty L_x^2} + 2^{j(\sigma+s-1)}\|u\|_{L_T^\infty L_x^\infty}^{2\sigma-1}T^{\frac{3}{4}}\|u_x\|_{L_T^\infty L_x^2}\|u_j\|_{L_T^4L_x^\infty} \\
&\lesssim T\|u_x\|_{L_T^\infty L_x^2}^2\|D_x^s u_j\|_{L_T^\infty L_x^2} + T^{\frac{3}{4}}\|u\|_{L_T^\infty L_x^\infty}^{2\sigma-1}\|u_x\|_{L_T^\infty L_x^2}\|D_x^{\sigma+s-1}u_j\|_{L_T^4L_x^\infty}.
\end{aligned}
\tag{2.3.53}
$$

Next, we estimate K_2. We have by Cauchy Schwarz, and Sobolev embedding,

$$\begin{aligned}\|P_{<j-4}\partial_x^{-1}[2^j\chi'(2^j|u|^2)\mathrm{Re}(\overline{u}|u|^{2\sigma}u_x)|u|^{2\sigma}]\|_{L_T^\infty L_x^\infty} &\lesssim 2^j\|\varphi'(2^j|u|^2)\mathrm{Re}(\overline{u}|u|^{2\sigma}u_x)|u|^{2\sigma}\|_{L_T^\infty L_x^1}\\ &\lesssim 2^{j(\frac{1}{2}-\sigma)}\|u\|_{L_T^\infty L_x^{4\sigma}}^{2\sigma}\|u_x\|_{L_T^\infty L_x^2}\\ &\lesssim 2^{j(\frac{1}{2}-\sigma)}\|u\|_{S_T^1}^{2\sigma}\|u_x\|_{L_T^\infty L_x^2}\\ &\lesssim \|u\|_{S_T^1}^{2\sigma}\|u_x\|_{L_T^\infty L_x^2}\end{aligned} \tag{2.3.54}$$

where we used the fact that $\sigma \geq \frac{1}{2}$.

Hence, we finally obtain the estimate,

$$\begin{aligned}2^{j(\sigma+s-1)}\|u_jJ_1\|_{L_T^1L_x^2} &\lesssim T\|u_x\|_{L_T^\infty L_x^2}^2\|D_x^su_j\|_{L_T^\infty L_x^2} + T^{\frac{3}{4}}\|u\|_{L_T^\infty L_x^\infty}^{2\sigma-1}\|u_x\|_{L_T^\infty L_x^2}\|D_x^{\sigma+s-1}u_j\|_{L_T^4L_x^\infty}\\ &\quad + T\|u\|_{S_T^1}^{2\sigma}\|u_x\|_{L_T^\infty L_x^2}\|D_x^{\sigma+s-1}u_j\|_{L_T^\infty L_x^2}\\ &\lesssim T^{\frac{3}{4}}(1+\|u\|_{S_T^1}^{4\sigma})\|u_j\|_{X_T^s}.\end{aligned} \tag{2.3.55}$$

Next, we turn to the estimate for J_2. We have

$$\begin{aligned}J_2 &= -\frac{1}{2}P_{<j-4}\partial_x^{-1}[\chi_j(|u|^2)\partial_t|u|^{2\sigma}]\\ &= -\sigma P_{<j-4}\partial_x^{-1}[\chi_j(|u|^2)|u|^{2\sigma-2}\mathrm{Re}(\overline{u}u_t)]\\ &= -\sigma P_{<j-4}\partial_x^{-1}[\chi_j(|u|^2)|u|^{2\sigma-2}\mathrm{Re}(i\overline{u}u_{xx})] - \sigma P_{<j-4}\partial_x^{-1}[\chi_j(|u|^2)|u|^{2\sigma-2}\mathrm{Re}(\overline{u}|u|^{2\sigma}u_x)]\\ &:= K_3 + K_4.\end{aligned} \tag{2.3.56}$$

For the first term, we have

$$\begin{aligned}K_3 &= -\sigma P_{<j-4}[\chi_j(|u|^2)|u|^{2\sigma-2}\mathrm{Re}(i\overline{u}u_x)] + \sigma P_{<j-4}\partial_x^{-1}[\chi_j(|u|^2)\partial_x|u|^{2\sigma-2}\mathrm{Re}(i\overline{u}u_x)]\\ &\quad - 2^j\sigma P_{<j-4}\partial_x^{-1}[\varphi'(2^j|u|^2)\partial_x|u|^2|u|^{2\sigma-2}\mathrm{Re}(i\overline{u}u_x)]\\ &= K_{3,1} + K_{3,2} + K_{3,3}.\end{aligned} \tag{2.3.57}$$

We now must estimate each of the above terms. For the first two terms, we have

$$\|K_{3,1}\|_{L_T^\infty L_x^2} \lesssim \|u\|_{L_T^\infty L_x^\infty}^{2\sigma-1}\|u_x\|_{L_T^\infty L_x^2} \tag{2.3.58}$$

and

$$\begin{aligned}\|K_{3,2}\|_{L_T^\infty L_x^\infty} &\lesssim \|\chi_j(|u|^2)\partial_x|u|^{2\sigma-2}\mathrm{Re}(i\overline{u}u_x)\|_{L_T^\infty L_x^1}\\ &\lesssim \|\chi_j(|u|^2)|u|^{2\sigma-4}\mathrm{Re}(\overline{u}u_x)\mathrm{Re}(i\overline{u}u_x)\|_{L_T^\infty L_x^1}\\ &\lesssim 2^{j(1-\sigma)}\|u_x\|_{L_T^\infty L_x^2}^2\end{aligned} \tag{2.3.59}$$

where we used the fact that

$$\chi_j(|u|^2)|u|^{2\sigma-2} \lesssim 2^{j(1-\sigma)}. \tag{2.3.60}$$

Remark 2.3.15. It should be emphasized that the main point of the partial gauge transformation is to be able to estimate the term $K_{3,2}$ above, which involves negative powers of $|u|$.

Now, we turn to the estimate for $K_{3,3}$. We have

$$\begin{aligned}\|K_{3,3}\|_{L_T^\infty L_x^\infty} &\lesssim 2^j\|\varphi'(2^j|u|^2)|u|^{2\sigma-2}\mathrm{Re}(\overline{u}u_x)\mathrm{Re}(i\overline{u}u_x)\|_{L_T^\infty L_x^1} \\ &\lesssim 2^{j(1-\sigma)}\|u_x\|_{L_T^\infty L_x^2}^2.\end{aligned} \tag{2.3.61}$$

Hence, we have

$$2^{j(\sigma+s-1)}\|K_3u_j\|_{L_T^1L_x^2} \lesssim T^{\frac{3}{4}}\|u\|_{L_T^\infty L_x^\infty}^{2\sigma-1}\|u_x\|_{L_T^\infty L_x^2}\|D_x^{\sigma+s-1}u_j\|_{L_T^4L_x^\infty} + T\|u_x\|_{L_T^\infty L_x^2}^2\|D_x^su_j\|_{L_T^\infty L_x^2}. \tag{2.3.62}$$

Finally, we estimate K_4. We have

$$\begin{aligned}\|K_4\|_{L_T^\infty L_x^\infty} &\lesssim \|P_{<j-4}\partial_x^{-1}[\chi_j(|u|^2)|u|^{2\sigma-2}\mathrm{Re}(\overline{u}|u|^{2\sigma}u_x)]\|_{L_T^\infty L_x^\infty} \\ &\lesssim \|\chi_j(|u|^2)|u|^{2\sigma-2}\mathrm{Re}(\overline{u}|u|^{2\sigma}u_x)\|_{L_T^\infty L_x^1} \\ &\lesssim \|u\|_{L_T^\infty L_x^\infty}^{4\sigma-2}\|u_x\|_{L_T^\infty L_x^2}\|u\|_{L_T^\infty L_x^2}.\end{aligned} \tag{2.3.63}$$

Hence, combining with the estimate for K_3, we obtain

$$\begin{aligned}2^{j(\sigma+s-1)}\|u_jJ_2\|_{L_T^1L_x^2} &\lesssim T^{\frac{3}{4}}\|u\|_{L_T^\infty L_x^\infty}^{2\sigma-1}\|u_x\|_{L_T^\infty L_x^2}\|D_x^{\sigma+s-1}u_j\|_{L_T^4L_x^\infty} + T\|u_x\|_{L_T^\infty L_x^2}^2\|D_x^su_j\|_{L_T^\infty L_x^2} \\ &\quad + T\|u\|_{L_T^\infty L_x^\infty}^{4\sigma-2}\|u_x\|_{L_T^\infty L_x^2}\|u\|_{L_T^\infty L_x^2}\|D_x^{\sigma+s-1}u_j\|_{L_T^\infty L_x^2} \\ &\lesssim_T T^{\frac{3}{4}}(1+\|u\|_{S_T^1}^{4\sigma})\|u_j\|_{X_T^s}.\end{aligned} \tag{2.3.64}$$

Now combining this with the estimate for J_1 finally yields the desired estimate for I_3^j. Namely, we have

$$\begin{aligned}I_3^j &\lesssim T^{\frac{3}{4}}(1+\|u\|_{S_T^1}^{4\sigma})\|u_j\|_{X_T^s} \\ &\lesssim T^{\frac{3}{4}}b_j(1+\|u\|_{S_T^1}^{4\sigma})\|u\|_{X_T^s}.\end{aligned} \tag{2.3.65}$$

Estimate for I_4^j

This term is straightforward to deal with. Indeed, after expanding $\partial_x^2\Phi_j$ we have

$$\begin{aligned}\|\partial_x^2\Phi_j\|_{L_T^\infty L_x^2} &\lesssim 2^j\|\varphi'(2^j|u|^2)\mathrm{Re}(\overline{u}u_x)|u|^{2\sigma}\|_{L_T^\infty L_x^2} + \|\chi_j(|u|^2)\mathrm{Re}(|u|^{2\sigma-2}\overline{u}u_x)\|_{L_T^\infty L_x^2} \\ &\lesssim \|u\|_{L_T^\infty L_x^\infty}^{2\sigma-1}\|u_x\|_{L_T^\infty L_x^2}.\end{aligned} \tag{2.3.66}$$

Hence,

$$\begin{aligned} I_4^j &\lesssim T^{\frac{3}{4}} \|u\|_{L_T^\infty L_x^\infty}^{2\sigma-1} \|u_x\|_{L_T^\infty L_x^2} \|D_x^{\sigma+s-1} u_j\|_{L_T^4 L_x^\infty} \\ &\lesssim T^{\frac{3}{4}} \|u\|_{S_T^1}^{2\sigma} \|u_j\|_{X_T^s} \\ &\lesssim T^{\frac{3}{4}} b_j \|u\|_{S_T^1}^{2\sigma} \|u\|_{X_T^s}. \end{aligned} \tag{2.3.67}$$

Estimate for I_5^j

The estimate for I_5^j is also straightforward as it doesn't involve any differentiated terms. Indeed, we have

$$\|\partial_x \Phi_j\|_{L_T^\infty L_x^\infty}^2 \lesssim \|u\|_{L_T^\infty L_x^\infty}^{4\sigma}. \tag{2.3.68}$$

Hence, by Sobolev embedding,

$$\begin{aligned} I_5^j &\lesssim T \|u\|_{L_T^\infty L_x^\infty}^{4\sigma} \|D_x^{\sigma+s-1} u_j\|_{L_T^\infty L_x^2} \\ &\lesssim T \|u\|_{S_T^1}^{4\sigma} \|u_j\|_{X_T^s} \\ &\lesssim T b_j \|u\|_{S_T^1}^{4\sigma} \|u\|_{X_T^s}. \end{aligned} \tag{2.3.69}$$

Now, combining all the estimates above completes the proof of Lemma 2.3.10.

Remark 2.3.16. By taking b_j to instead be a S_T^s frequency envelope for u, and repeating the proof almost verbatim with Remark 2.3.4 in place of (2.3.8), we instead obtain

$$\|P_j u\|_{Y_T^s} \lesssim_{\|u\|_{S_T^1}} a_j \|u_0\|_{H_x^s} + T^{\frac{1}{2}} b_j (1 + \|u\|_{S_T^1}^{4\sigma}) \|u\|_{S_T^s}. \tag{2.3.70}$$

This will be relevant for when we later establish local well-posedness in the high regularity regime $2 - \sigma < s < 4\sigma$ for the full range of $\frac{1}{2} < \sigma < 1$. Specifically, this will be important for establishing a priori bounds in the range $2 - \sigma < s \leq \frac{3}{2}$ when Sobolev embedding is not suitable for controlling the term $\|u_x\|_{L_T^4 L_x^\infty}$. The reason the proof of (2.3.70) is almost identical to the current proof is that we have not yet used the maximal function part of the norm of X_T^s; we will begin using this part of the norm in the proof of Lemma 2.3.11.

Remark 2.3.17. As a second important remark, the estimate (2.3.70) also holds for $T \lesssim 1$ if the nonlinearity $i|u|^{2\sigma} u_x$ is replaced by the spatially regularized and time-truncated nonlinearity $i\eta P_{<k} |u|^{2\sigma} u_x$, where $k \in \mathbb{N}$ and $\eta = \eta(t)$ is a time-dependent cutoff function supported in $(-2, 2)$ and equal to 1 on $[-1, 1]$. This fact won't be relevant for the low regularity construction, but will be important for the high regularity construction in Sections 5 and 6 where the cutoff η is needed for estimating (fractional order) time derivatives of a

solution u to (gDNLS). Since the proof of this estimate is nearly identical to Lemma 2.3.10, we omit the details. Nevertheless, for the sake of completeness, we state this observation in the following lemma.

Lemma 2.3.18. Let $k \in \mathbb{N}$, $\sigma \in (\frac{1}{2}, 1)$, $s \in [1, \frac{3}{2}]$, and $T \lesssim 1$. Let η be a time-dependent cutoff function supported in $(-2, 2)$ with $\eta = 1$ on $[-1, 1]$. Let $v, w \in S^s_T$ with $\|v\|_{S^s_T}$, $\|w\|_{S^s_T} \lesssim 1$. Assume that $u, v \in S^s_T$ solve the equations

$$\begin{cases} (i\partial_t + \partial_x^2)u = i\eta P_{<k}|v|^{2\sigma}\partial_x u, \\ u(0) = u_0, \end{cases} \tag{2.3.71}$$

and

$$\begin{cases} (i\partial_t + \partial_x^2)v = i\eta P_{<k}|w|^{2\sigma}\partial_x v, \\ v(0) = u_0, \end{cases} \tag{2.3.72}$$

respectively. Then u satisfies the estimate

$$\|u\|_{Y^s_T} \lesssim \|u_0\|_{H^s_x} + T^{\frac{1}{2}}\|u\|_{S^s_T}. \tag{2.3.73}$$

As mentioned, the proof of Lemma 2.3.18 proceeds in a nearly identical fashion to Lemma 2.3.10, so we omit the details. The main difference is that Φ_j is replaced by

$$\Phi_j = -\frac{1}{2}\eta(t)P_{<j-4}P_{<k}\partial_x^{-1}[\chi_j(|v|^2)|v|^{2\sigma}]. \tag{2.3.74}$$

The requirement (2.3.72) that v solves an additional (gDNLS) type equation is merely relevant for the I_3^j estimate when time derivatives fall on Φ_j, and hence on v. In practice, Lemma 2.3.18 will be used in the construction of solutions at high regularity in Sections 5, 6 and 7.

Next, we turn to proving Lemma 2.3.11.

Proof. Again, we begin by writing the equation in a paradifferential fashion,

$$i\partial_t u_j + \partial_x^2 u_j = iP_{<j-4}|u|^{2\sigma}\partial_x u_j + iP_j(P_{\geq j-4}|u|^{2\sigma}\partial_x u) + i[P_j, P_{<j-4}|u|^{2\sigma}]\partial_x u. \tag{2.3.75}$$

A simple energy estimate (i.e. multiplying by $-i2^{2js}\overline{u_j}$, taking real part and integrating), and Bernstein's inequality gives

$$\begin{aligned}\|u_j\|^2_{L^\infty_T H^s_x} &\lesssim \|u_j(0)\|^2_{H^s_x} + 2^{2js}\int_0^T \left|\int_{\mathbb{R}} P_{<j-4}|u|^{2\sigma}\partial_x |u_j|^2\right| + 2^{2js}\int_0^T \left|\int_{\mathbb{R}} \overline{u_j} P_j(P_{\geq j-4}|u|^{2\sigma}\partial_x u)\right| \\ &+ 2^{2js}\int_0^T \left|\int_{\mathbb{R}} \overline{u_j}[P_j, P_{<j-4}|u|^{2\sigma}]\partial_x u\right| \\ &:= \|u_j(0)\|^2_{H^s_x} + I_1^j + I_2^j + I_3^j.\end{aligned} \tag{2.3.76}$$

Estimate for I_1^j

For the first term, we integrate by parts and estimate using standard interpolation inequalities, Bernstein's inequality, Hölder's inequality and Proposition 2.2.7

$$\begin{aligned}2^{2js}\int_0^T \left|\int_{\mathbb{R}} |u_j|^2 P_{<j-4}\partial_x |u|^{2\sigma}\right| &\lesssim 2^{2js}\|P_{<j-4}\partial_x|u|^{2\sigma}|u_j|^{2(1-\sigma)}\|_{L^1_x L_T^{\frac{1}{1-\sigma}}}\|u_j\|^{2\sigma}_{L^\infty_x L^2_T} \\ &\lesssim 2^{2js}\|P_{<j-4}\partial_x |u|^{2\sigma}\|_{L_x^{\frac{1}{\sigma}} L_T^{\frac{1}{\varepsilon(1-\sigma)}}}\|u_j\|^{2(1-\sigma)}_{L^2_x L_T^{\frac{2}{1-\varepsilon}}}\|u_j\|^{2\sigma}_{L^\infty_x L^2_T} \\ &\lesssim \|P_{<j-4}(D_x^{\sigma-\frac{1}{2}}|u|^{2\sigma})\|_{L_x^{\frac{1}{\sigma}} L_T^{\frac{1}{\varepsilon(1-\sigma)}}}\|D_x^{s-c_1\varepsilon}u_j\|^{2(1-\sigma)}_{L^2_x L_T^{\frac{2}{1-\varepsilon}}}\|D_x^{s+\frac{3}{4\sigma}-\frac{1}{2}+c_2\varepsilon}u_j\|^{2\sigma}_{L^\infty_x L^2_T} \\ &\lesssim T^{(1-\sigma)(1-\varepsilon)}\|P_{<j-4}(D_x^{\sigma-\frac{1}{2}-\varepsilon}|u|^{2\sigma})\|_{L_x^{\frac{1}{\sigma}} L_T^{\frac{1}{\varepsilon(1-\sigma)}}}\|u_j\|^2_{X^s_T} \\ &\lesssim T^{1-\sigma}\|u\|^{2\sigma-1}_{L^2_x L^\infty_T}\|D_x^{\sigma-\frac{1}{2}-\varepsilon}u\|_{L^2_x L^\infty_T}\|u_j\|^2_{X^s_T} \\ &\lesssim T^{1-\sigma}\|u\|^{2\sigma}_{Y^1_T}\|u_j\|^2_{X^s_T} \\ &\lesssim T^{1-\sigma}b_j^2\|u\|^{2\sigma}_{Y^1_T}\|u\|^2_{X^s_T},\end{aligned} \tag{2.3.77}$$

where c_1, c_2 are fixed positive constants, and $\varepsilon > 0$ is sufficiently small. Observe that going from line 3 to line 4 uses the fact that $\sigma > \frac{\sqrt{3}}{2}$ since $s+\frac{3}{4\sigma}-\frac{1}{2} < s+\sigma-\frac{1}{2}$ precisely when $\sigma > \frac{\sqrt{3}}{2}$.

Estimate for I_2^j

We have by the Littlewood-Paley trichotomy

$$2^{2js}\int_{\mathbb{R}} P_j(P_{\geq j-4}|u|^{2\sigma}\partial_x u)\overline{u_j} = 2^{2js}\int_{\mathbb{R}} \tilde{P}_j(|u|^{2\sigma})\tilde{P}_{<j}\partial_x u\bar{u}_j + 2^{2js}\sum_{k>j}\int_{\mathbb{R}} \overline{u_j}P_j(\tilde{P}_k(|u|^{2\sigma})\tilde{P}_k\partial_x u) \tag{2.3.78}$$

for appropriate "fattened" Littlewood-Paley projections $\tilde{P}_j$. For the first term, using Bernstein's inequality and Hölder's inequality, and that $2^{-\delta j} \lesssim b_j$ we have,

$$\begin{aligned} &2^{2js} \int_0^T \left| \int_{\mathbb{R}} \tilde{P}_j(|u|^{2\sigma}) \tilde{P}_{<j} \partial_x u \overline{u}_j \right| \\ &\quad \lesssim 2^{j(\frac{5}{2}-\sigma+s)} \|\tilde{P}_j |u|^{2\sigma}\|_{L_x^{\frac{2}{2\sigma-1}} L_T^2} \|\tilde{P}_j u\|_{L_x^\infty L_T^2}^{2\sigma-1} \|\tilde{P}_j u\|_{L_x^2 L_T^2}^{2(1-\sigma)} \|\tilde{P}_{<j} D_x^{\sigma+s-\frac{3}{2}} u\|_{L_x^2 L_T^\infty} \\ &\quad \lesssim T^{1-\sigma} \|D_x^{2+\sigma-2\sigma^2+\delta} \tilde{P}_j(|u|^{2\sigma})\|_{L_x^{\frac{2}{2\sigma-1}} L_T^2} \|\tilde{P}_j u\|_{X_T^s} \|u\|_{X_T^s} \\ &\quad \lesssim T^{1-\sigma} b_j \|D_x^{2+\sigma-2\sigma^2+2\delta} \tilde{P}_j(|u|^{2\sigma})\|_{L_x^{\frac{2}{2\sigma-1}} L_T^2} \|\tilde{P}_j u\|_{X_T^s} \|u\|_{X_T^s}. \end{aligned} \tag{2.3.79}$$

Note that the first line follows since $s \in [1, \frac{3}{2}]$. The next step is to estimate the term $\|D_x^{2+\sigma-2\sigma^2+2\delta} \tilde{P}_j(|u|^{2\sigma})\|_{L_x^{\frac{2}{2\sigma-1}} L_T^2}$. For notational convenience, write $2+\sigma-2\sigma^2+2\delta = \alpha$. We employ the Littlewood-Paley trichotomy and then Hölder's and Bernstein's inequality to obtain

$$\begin{aligned} \|D_x^\alpha \tilde{P}_j(|u|^{2\sigma})\|_{L_x^{\frac{2}{2\sigma-1}} L_T^2} &\lesssim \|D_x^{\alpha-1} \tilde{P}_j(|u|^{2\sigma-2}\overline{u} u_x)\|_{L_x^{\frac{2}{2\sigma-1}} L_T^2} \\ &\lesssim \|D_x^{\alpha-1} \tilde{P}_j(\tilde{P}_{<j}(|u|^{2\sigma-2}\overline{u}) \tilde{P}_j u_x)\|_{L_x^{\frac{2}{2\sigma-1}} L_T^2} + \|D_x^{\alpha-1} \tilde{P}_j(\tilde{P}_{>j}(|u|^{2\sigma-2}\overline{u}) u_x)\|_{L_x^{\frac{2}{2\sigma-1}} L_T^2} \\ &\lesssim \|u\|_{L_x^2 L_T^\infty}^{2\sigma-1} \|D_x^\alpha \tilde{P}_j u\|_{L_x^\infty L_T^2} + \|D_x^{\alpha-1}(|u|^{2\sigma-2}\overline{u})\|_{L_T^\infty L_x^\infty} \|u_x\|_{L_x^{\frac{2}{2\sigma-1}} L_T^2}. \end{aligned} \tag{2.3.80}$$

Observe that $\|D_x^\alpha u\|_{L_x^\infty L_T^2} \lesssim \|u\|_{Y_T^1}$ since $\alpha < \sigma + \frac{1}{2}$ when $\sigma > \frac{\sqrt{3}}{2}$. Furthermore, by Corollary 2.2.11 and Sobolev embedding, we have

$$\|D_x^{\alpha-1}(|u|^{2\sigma-2}\overline{u})\|_{L_T^\infty L_x^\infty} \lesssim \|\langle D_x \rangle^{\frac{\alpha-1+\varepsilon}{2\sigma-1}} u\|_{L_T^\infty L_x^\infty}^{2\sigma-1} \lesssim \|u\|_{S_T^1}^{2\sigma-1} \tag{2.3.81}$$

where the last inequality again follows because $\sigma > \frac{\sqrt{3}}{2}$. Furthermore, by interpolating $\|u_x\|_{L_x^{\frac{2}{2\sigma-1}} L_T^2}$ between $L_x^2 L_T^2$ and $L_x^\infty L_T^2$, we see that $\|u_x\|_{L_x^{\frac{2}{2\sigma-1}} L_T^2} \lesssim \|u\|_{X_T^1}$. Hence, we can control (2.3.79) by

$$T^{1-\sigma} b_j^2 \|u\|_{X_T^1}^{2\sigma} \|u\|_{X_T^s}^2. \tag{2.3.82}$$

For the other term in (2.3.78), we have

$$
\begin{aligned}
&2^{2js}\int_0^T \left|\sum_{k>j}\int_{\mathbb{R}} \overline{u_j} P_j(\tilde{P}_k(|u|^{2\sigma})\tilde{P}_k\partial_x u)\right| \\
&\lesssim 2^{js}T^{(1-\sigma)}\|D_x^s u_j\|^{2(1-\sigma)}_{L_T^\infty L_x^2}\|D_x^s u_j\|^{2\sigma-1}_{L_x^\infty L_T^2}\sum_{k>j}2^k\|\tilde{P}_k(|u|^{2\sigma})\|_{L_x^{\frac{2}{2\sigma-1}}L_T^2}\|\tilde{P}_k u\|_{L_x^2 L_T^\infty} \\
&\lesssim 2^{j(s-\frac{1}{2}(1-2\sigma)^2)}T^{(1-\sigma)}\|u_j\|_{X_T^s}\sum_{k>j}2^k\|\tilde{P}_k(|u|^{2\sigma})\|_{L_x^{\frac{2}{2\sigma-1}}L_T^2}\|\tilde{P}_k u\|_{L_x^2 L_T^\infty} \\
&\lesssim 2^{j(s-\frac{1}{2}(1-2\sigma)^2)}T^{(1-\sigma)}\|u_j\|_{X_T^s}\sum_{k>j}2^{k(\frac{3}{2}-\sigma-s+1)}\|\tilde{P}_k(|u|^{2\sigma})\|_{L_x^{\frac{2}{2\sigma-1}}L_T^2}\|\tilde{P}_k D_x^{s+\sigma-\frac{3}{2}}u\|_{L_x^2 L_T^\infty} \\
&\lesssim T^{(1-\sigma)}\|u_j\|_{X_T^s}\sum_{k>j}2^{(j-k)(s-\frac{1}{2}(1-2\sigma)^2)}\|\tilde{P}_k(D_x^{2+\sigma-2\sigma^2}|u|^{2\sigma})\|_{L_x^{\frac{2}{2\sigma-1}}L_T^2}\|\tilde{P}_k D_x^{s+\sigma-\frac{3}{2}}u\|_{L_x^2 L_T^\infty} \\
&\lesssim T^{(1-\sigma)}b_j^2\|u\|^2_{X_T^s}\|u\|^{2\sigma}_{X_T^1}\sum_{k>j}2^{(j-k)((s-\frac{1}{2}(1-2\sigma)^2)-\delta)} \\
&\lesssim T^{(1-\sigma)}b_j^2\|u\|^2_{X_T^s}\|u\|^{2\sigma}_{X_T^1}
\end{aligned}
\tag{2.3.83}
$$

where we estimated $\|\tilde{P}_k(D_x^{2+\sigma-2\sigma^2}|u|^{2\sigma})\|_{L_x^{\frac{2}{2\sigma-1}}L_T^2}$ in essentially the same way as we did with the previous term.

Estimate for I_3^j

We have

$$
\begin{aligned}
[P_j, P_{<j-4}|u|^{2\sigma}]\partial_x u &= [P_j, P_{<j-4}|u|^{2\sigma}]\partial_x\tilde{P}_j u \\
&= 2^{-j}\int_{\mathbb{R}^2} K(y)\partial_x P_{<j-4}|u|^{2\sigma}(x+y_1)\partial_x\tilde{P}_j u(x+y_2)dy
\end{aligned}
\tag{2.3.84}
$$

for some kernel $K \in L^1$ with $\|K\|_{L^1} \lesssim 1$ (with a bound independent of j), see Lemma 2.2.1. Hence,

$$
\int_0^T\left|\int_{\mathbb{R}}\overline{u_j}[P_j, P_{<j-4}|u|^{2\sigma}]\partial_x\tilde{P}_j u\right| \lesssim 2^{-j}\sup_{y\in\mathbb{R}^2}\int_0^T\int_{\mathbb{R}}|\partial_x P_{<j-4}|u|^{2\sigma}(x+y_1)||\partial_x\tilde{P}_j u(x+y_2)||u_j|.
\tag{2.3.85}
$$

This is estimated analogously to I_j^1. Indeed, we obtain by Cauchy Schwarz, Bernstein's inequality and Proposition 2.2.7,

$$\begin{aligned}
2^{2js} \int_0^T & \left| \int_{\mathbb{R}} \overline{u_j} [P_j, P_{<j-4}|u|^{2\sigma}] \partial_x \tilde{P}_j u \right| \\
&\lesssim 2^{2js} \|\tilde{P}_j D_x^{\frac{3}{4\sigma}-\frac{1}{2}+c_1\varepsilon} u\|_{L_x^\infty L_T^2}^{2\sigma} \|\tilde{P}_j u\|_{L_T^2 L_x^2}^{2(1-\sigma)} \|u\|_{L_x^2 L_T^\infty}^{2\sigma-1} \|D_x^{\sigma-\frac{1}{2}-c_2\varepsilon} u\|_{L_x^2 L_T^\infty} \qquad (2.3.86)\\
&\lesssim T^{(1-\sigma)} \|u\|_{Y_T^1}^{2\sigma} \|\tilde{P}_j u\|_{X_T^s}^2 \\
&\lesssim T^{(1-\sigma)} b_j^2 \|u\|_{Y_T^1}^{2\sigma} \|u\|_{X_T^s}^2,
\end{aligned}$$

where c_1, c_2 are positive constants depending on σ, s. The second line follows from the fact that $\frac{3}{4\sigma} - \frac{1}{2} < \sigma - \frac{1}{2}$ as long as $\sigma > \frac{\sqrt{3}}{2}$.

Hence, we obtain

$$\|P_j u\|_{L_T^\infty H_x^s} \lesssim a_j \|u_0\|_{H_x^s} + T^{\frac{1-\sigma}{2}} b_j \|u\|_{X_T^1}^\sigma \|u\|_{X_T^s}, \qquad (2.3.87)$$

thus completing the proof of the $L_T^\infty H_x^s$ estimate. □

Proof of Proposition 2.3.6

We combine the energy estimate and the Y^s estimate to obtain

$$\|P_j u\|_{X_T^s} \lesssim_{\|u\|_{S_T^1}} a_j \|u_0\|_{H_x^s} + T^{\frac{1}{2}} b_j (1 + \|u\|_{S_T^1}^{4\sigma}) \|u\|_{X_T^s} + T^{\frac{1-\sigma}{2}} b_j \|u\|_{X_T^1}^\sigma \|u\|_{X_T^s}. \qquad (2.3.88)$$

This proves part a) of Proposition 2.3.6.

Now we move to part b). Let us first assume $T \ll 1$ (but independent of ε). There are two components to consider. For high frequency, square summing over $j > 0$ shows

$$\|P_{>0} u\|_{X_T^s} \lesssim_{\|u\|_{S_T^1}} \|u_0\|_{H_x^s} + T^{\frac{1}{2}} (1 + \|u\|_{S_T^1}^{4\sigma}) \|u\|_{X_T^s} + T^{\frac{1-\sigma}{2}} \|u\|_{X_T^1}^\sigma \|u\|_{X_T^s}. \qquad (2.3.89)$$

On the other hand, directly applying the maximal function/Strichartz estimates in Proposition 2.2.3 and Proposition 2.2.4 and Bernstein's inequality to $P_{\leq 0} u$, we easily obtain

$$\|P_{\leq 0} u\|_{X_T^s} \lesssim \|u_0\|_{L_x^2} + \|P_{\leq 0}(|u|^{2\sigma} u_x)\|_{L_T^1 L_x^2} \lesssim \|u_0\|_{L_x^2} + T \|u\|_{S_T^1}^{2\sigma+1}. \qquad (2.3.90)$$

From the above bounds, we see that the X_T^s norm of u converges to the H_x^1 norm of the initial data as $T \to 0^+$. Let us now make the bootstrap assumption $\|u\|_{X_T^1} \leq \varepsilon^{\frac{1}{2}}$. We then obtain from the above estimates,

$$\|u\|_{X_T^s} \lesssim_{\|u\|_{X_T^1}} \|u_0\|_{H_x^s} \leq \varepsilon \qquad (2.3.91)$$

where $T \ll 1$ (but independent of ε) and $1 \le s \le \frac{3}{2}$. To obtain the estimate for $T \sim 1$, we iterate the above procedure $O(T^{-1})$ many times (after suitable translating the initial data). This proves part b) of Proposition 2.3.6.

Next, we turn to the proof of Proposition 2.3.7. We proceed in a similar manner to Proposition 2.3.6, and prove separate estimates for the Y_T^0 and $L_T^\infty L_x^2$ components of the X_T^0 norm. For this purpose, we have the following two lemmas:

Lemma 2.3.19. (Y_T^0 estimate) Let v, σ, T, w, g and a be as in Proposition 2.3.7. Then we have the Y_T^0 estimate,

$$\|v\|_{Y_T^0} \lesssim \|v_0\|_{L_x^2} + T^{\frac{1}{2}}(1 + \|w\|_{X_T^1}^{4\sigma})\|v\|_{X_T^0} + T^{1-\sigma}\|g\|_Z\|a\|_{X_T^1}\|v\|_{X_T^0}. \tag{2.3.92}$$

Lemma 2.3.20. ($L_T^\infty L_x^2$ estimate) Let v, σ, T, w, g and a be as in Proposition 2.3.7. Then we have the estimate,

$$\|P_j v\|_{l_j^2 L_T^\infty L_x^2}^2 \lesssim \|v_0\|_{L_x^2}^2 + T^{1-\sigma}\|g\|_Z\|a\|_{X_T^1}\|v\|_{X_T^0}^2 + T^{1-\sigma}\|w\|_{X_T^1}^{2\sigma}\|v\|_{X_T^0}^2. \tag{2.3.93}$$

We begin with Lemma 2.3.19. The proof is almost the same as Lemma 2.3.10 with a couple of small differences. As in (2.3.28), we consider a similar paradifferential truncation of (2.3.21),

$$(i\partial_t + \partial_x^2)v_j = iP_{<j-4}(\chi_j(|w|^2)|w|^{2\sigma})\partial_x v_j + iP_{<j-4}(\varphi_j(|w|^2)|w|^{2\sigma})\partial_x v_j + f_j + g_j \tag{2.3.94}$$

where φ_j and χ_j are as in (2.3.28) and

$$f_j := iP_j(P_{\ge j-4}|w|^{2\sigma}\partial_x v) + i[P_j, P_{<j-4}|w|^{2\sigma}]\partial_x v, \tag{2.3.95}$$

$$g_j := 2P_j(\partial_x a \mathrm{Re}(gv)). \tag{2.3.96}$$

Analogously to the proof of Proposition 2.3.6, we define

$$\Psi_j(x) = -\frac{1}{2}P_{<j-4}\partial_x^{-1}[\chi_j(|w|^2)|w|^{2\sigma}] \tag{2.3.97}$$

and consider the new variable

$$\tilde{v}_j := v_j e^{i\Psi_j}. \tag{2.3.98}$$

By direct computation, $\tilde{v}_j$ solves the equation,

$$\begin{cases} (i\partial_t + \partial_x^2)\tilde{v}_j = ie^{i\Psi_j}P_{<j-4}[\varphi_j(|w|^2)|w|^{2\sigma}]\partial_x v_j + (-\partial_t\Psi_j + i\partial_x^2\Psi_j - (\partial_x\Psi_j)^2)\tilde{v}_j \\ \qquad\qquad +2e^{i\Psi_j}P_j(\partial_x a\mathrm{Re}(gv)) + e^{i\Psi_j}f_j, \\ \tilde{v}_j(0) = e^{i\Psi_j}v_j(0). \end{cases} \tag{2.3.99}$$

Now, Proposition 2.2.3, Proposition 2.2.4 and a similar argument to Proposition 2.3.6 yields the estimate

$$\begin{aligned}\|v\|_{Y_T^0} \lesssim_T \|v_0\|_{L_x^2} + T^{\frac{1}{2}}[1+\|w\|_{X_T^1}]^{4\sigma}\|v\|_{X_T^0} \\ + \left(\sum_{j>0} \|\langle D_x\rangle^{\sigma-1}P_j(g\partial_x av)\|_{L_T^1L_x^2}^2\right)^{\frac{1}{2}}.\end{aligned} \tag{2.3.100}$$

It remains to control the last term. Indeed, we have by Bernstein and Sobolev embedding,

$$\begin{aligned}\|\langle D_x\rangle^{\sigma-1}P_j(g\partial_x av)\|_{L_T^1L_x^2} &\lesssim 2^{j(\sigma-1)}\|P_j(g\partial_x av)\|_{L_T^1L_x^2} \\ &\lesssim 2^{j(\sigma-1)}\|P_{<j-4}(\partial_x ag)\tilde{P}_j v\|_{L_T^1L_x^2} + \|P_j(P_{\geq j-4}(\partial_x ag)v)\|_{L_T^1L_x^{\frac{2}{3-2\sigma}}}.\end{aligned} \tag{2.3.101}$$

For the first term, we have by Bernstein's inequality,

$$\begin{aligned}2^{j(\sigma-1)}\|P_{<j-4}(\partial_x ag)\tilde{P}_j v\|_{L_T^1L_x^2} &\lesssim T^{\frac{3}{4}}\|\partial_x a\|_{L_T^\infty L_x^2}\|g\|_{L_T^\infty L_x^\infty}\|\tilde{P}_j D_x^{\sigma-1}v\|_{L_T^4L_x^\infty} \\ &\lesssim T^{\frac{3}{4}}\|a\|_{X_T^1}\|g\|_Z\|\tilde{P}_j D_x^{\sigma-1}v\|_{L_T^4L_x^\infty}.\end{aligned} \tag{2.3.102}$$

For the second term, we have by the usual Littlewood-Paley trichotomy,

$$\begin{aligned}\|P_j(P_{\geq j-4}(\partial_x ag)v)\|_{L_T^1L_x^{\frac{2}{3-2\sigma}}} &\lesssim \|P_j(\tilde{P}_j(\partial_x ag)P_{<j}v)\|_{L_T^1L_x^{\frac{2}{3-2\sigma}}} + \sum_{k\geq j}\|P_j(\tilde{P}_k(\partial_x ag)\tilde{P}_k v)\|_{L_T^1L_x^{\frac{2}{3-2\sigma}}} \\ &:= K_1^j + K_2^j.\end{aligned} \tag{2.3.103}$$

To estimate K_1^j, we have

$$\begin{aligned}\|P_j(\tilde{P}_j(\partial_x ag)P_{<j}v)\|_{L_T^1L_x^{\frac{2}{3-2\sigma}}} &\lesssim \|\tilde{P}_j(\partial_x ag)\|_{L_T^2L_x^2}\|P_{<j}v\|_{L_T^2L_x^{\frac{1}{1-\sigma}}} \\ &\lesssim \|D_x^{(1-\sigma+\varepsilon)(2\sigma-1)}\tilde{P}_j(g\partial_x a)\|_{L_T^2L_x^2}\|P_{<j}v\|_{L_T^2L_x^2}^{2(1-\sigma)}\|P_{<j}D_x^{\sigma-1-\varepsilon}v\|_{L_T^2L_x^\infty}^{2\sigma-1} \\ &\lesssim T^{1-\sigma}\|D_x^{(1-\sigma+\varepsilon)(2\sigma-1)}\tilde{P}_j(g\partial_x a)\|_{L_T^2L_x^2}\|v\|_{X_T^0}\end{aligned} \tag{2.3.104}$$

where in the last line we used the fact that by Sobolev embedding,

$$\|P_{<j}D_x^{\sigma-1-\varepsilon}v\|_{L_T^2L_x^\infty} \lesssim \|v\|_{L_T^\infty L_x^2} + \|P_{>0}v\|_{X_T^0} \lesssim \|v\|_{X_T^0} \tag{2.3.105}$$

as well as $\|P_{<j}v\|_{L_T^2L_x^2} \lesssim T^{\frac{1}{2}}\|v\|_{X_T^0}$. Now, setting $\alpha = (1-\sigma+\varepsilon)(2\sigma-1)$, we have by Bernstein's inequality, and a simple application of the Littlewood-Paley trichotomy,

$$\begin{aligned}\|D_x^\alpha\tilde{P}_j(g\partial_x a)\|_{L_T^2L_x^2} &\lesssim 2^{-j\varepsilon}\|D_x^{\alpha+\varepsilon}\tilde{P}_j(g\partial_x a)\|_{L_T^2L_x^2} \\ &\lesssim 2^{-j\varepsilon}\|D_x^{\alpha+\varepsilon}\partial_x a\|_{L_x^{\frac{1}{1-\sigma}}L_T^2}\|g\|_{L_x^{\frac{2}{2\sigma-1}}L_T^\infty} + 2^{-j\varepsilon}\|\partial_x a\|_{L_T^\infty L_x^2}\|D_x^{\alpha+\varepsilon}g\|_{L_T^2L_x^\infty}.\end{aligned} \tag{2.3.106}$$

Next, by interpolating $\|D_x^{\alpha+\varepsilon}\partial_x a\|_{L_x^{\frac{1}{1-\sigma}}L_T^2}$ between $D_x^{\frac{\alpha+2\varepsilon}{2\sigma-1}}a$ in $L_x^\infty L_T^2$ and $\partial_x a$ in $L_x^2 L_T^2$, for ε small enough, $\|D_x^{\alpha+\varepsilon}\partial_x a\|_{L_x^{\frac{1}{1-\sigma}}L_T^2} \lesssim \|a\|_{X_T^1}$ as long as $\sigma > \frac{3}{4}$ (because this corresponds to when $\frac{\alpha}{2\sigma-1} < \sigma - \frac{1}{2}$). Furthermore, clearly $\|D_x^{\alpha+\varepsilon}g\|_{L_T^2 L_x^\infty} \lesssim \|g\|_Z$. Hence,

$$\|D_x^\alpha \tilde{P}_j(g\partial_x a)\|_{L_T^2 L_x^2} \lesssim 2^{-j\varepsilon}\|g\|_Z\|a\|_{X_T^1}. \tag{2.3.107}$$

It is easy to see that a similar analysis works for K_2^j. Hence, we ultimately deduce that

$$K_1^j + K_2^j \lesssim 2^{-j\varepsilon}T^{1-\sigma}\|g\|_Z\|a\|_{X_T^1}\|v\|_{X_T^0}. \tag{2.3.108}$$

Square summing now gives

$$\left(\sum_{j>0}\|\langle D_x\rangle^{\sigma-1}P_j(g\partial_x av)\|_{L_T^1 L_x^2}^2\right)^{\frac{1}{2}} \lesssim T^{1-\sigma}\|g\|_Z\|a\|_{X_T^1}\|v\|_{X_T^0}. \tag{2.3.109}$$

□

Next, we turn to the energy type $L_T^\infty L_x^2$ estimate in Lemma 2.3.20. First, it is straightforward to verify by a simple energy estimate that $P_{\le 0}v$ is controlled in $L_T^\infty L_x^2$ by the right hand side of (2.3.93). Hence, let us restrict to controlling $P_{>0}v$.

Proof. Let $j > 0$. Projecting (2.3.21) onto frequency 2^j, multiplying by $-i\overline{P_j v}$, taking real part and integrating from 0 to T gives

$$\begin{aligned}\|P_j v\|_{L_T^\infty L_x^2}^2 &\lesssim \|P_j v_0\|_{L_x^2}^2 + \int_0^T\left|\int_{\mathbb{R}} P_j(g\partial_x av)\overline{v_j} + P_j(\overline{g}\partial_x a\overline{v})\overline{v_j}\right| + \int_0^T\left|\int_{\mathbb{R}} P_j(|w|^{2\sigma}\partial_x v)\overline{v_j}\right| \\ &:= \|P_j v_0\|_{L_x^2}^2 + I_1^j + I_2^j.\end{aligned} \tag{2.3.110}$$

Estimate for I_1^j

For simplicity, we show how to deal with the first term,

$$\int_{\mathbb{R}} P_j(g\partial_x av)\overline{v}_j \tag{2.3.111}$$

as the other term (involving the complex conjugate of gv) is essentially identical.

We have by the Littlewood-Paley trichotomy,

$$\int_{\mathbb{R}} P_j(g\partial_x av)\overline{v_j} = \int_{\mathbb{R}} P_j(P_{\ge j-4}(g\partial_x a)v)\overline{v_j} + \int_{\mathbb{R}} \tilde{P}_{<j}(g\partial_x a)\tilde{P}_j v\overline{\tilde{P}_j v}. \tag{2.3.112}$$

We expand the first term as

$$P_j(P_{\geq j-4}(g\partial_x a)v) = P_j(\tilde{P}_j(g\partial_x a)\tilde{P}_{<j}v) + \sum_{k\geq j} P_j(\tilde{P}_k(g\partial_x a)\tilde{P}_k v). \tag{2.3.113}$$

We obtain by Bernstein's inequality, Hölder and a simple application of the Littlewood-Paley trichotomy,

$$\begin{aligned}
&\int_0^T \left|\int_{\mathbb{R}} P_j(\tilde{P}_j(g\partial_x a)\tilde{P}_{<j}v)\overline{v_j}\right| \\
&\lesssim \|\tilde{P}_j D_x^{\frac{3}{4\sigma}-\frac{1}{2}+\varepsilon}(g\partial_x a)\|_{L_x^{\frac{2}{2\sigma-1}}L_T^2} \|\tilde{P}_j D_x^{\frac{3}{4\sigma}-\frac{1}{2}}v\|^{2\sigma-1}_{L_x^\infty L_T^2} \|\tilde{P}_j v\|^{2(1-\sigma)}_{L_x^2 L_T^2} \|\tilde{P}_{<j}\langle D_x\rangle^{\sigma-\frac{3}{2}-\varepsilon}v\|_{L_x^2 L_T^\infty} \\
&\lesssim 2^{-j\varepsilon}T^{1-\sigma}\|\tilde{P}_j D_x^{\frac{3}{4\sigma}-\frac{1}{2}+2\varepsilon}(g\partial_x a)\|_{L_x^{\frac{2}{2\sigma-1}}L_T^2}\|v\|^2_{X_T^0} \\
&\lesssim 2^{-j\varepsilon}T^{1-\sigma}(\|D_x^{\frac{3}{4\sigma}-\frac{1}{2}+3\varepsilon}g\|_{L_x^\infty L_T^\infty}\|\partial_x a\|_{L_x^{\frac{2}{2\sigma-1}}L_T^2} + \|g\|_{L_x^{\frac{2}{2\sigma-1}}L_T^\infty}\|D_x^{\frac{3}{4\sigma}-\frac{1}{2}+2\varepsilon}\partial_x a\|_{L_x^\infty L_T^2})\|v\|^2_{X_T^0} \\
&\lesssim 2^{-j\varepsilon}T^{1-\sigma}\|g\|_Z\|a\|_{X_T^1}\|v\|^2_{X_T^0}
\end{aligned} \tag{2.3.114}$$

where in the last line, we used the assumption $\sigma > \frac{\sqrt{3}}{2}$. The second term in (2.3.113) is similarly estimated by $2^{-j\varepsilon}T^{1-\sigma}\|g\|_Z\|a\|_{X_T^1}\|v\|^2_{X_T^0}$. Hence,

$$\|P_j(P_{\geq j-4}(g\partial_x a)v)\overline{v_j}\|_{L_T^1 L_x^1} \lesssim 2^{-j\varepsilon}T^{1-\sigma}\|g\|_Z\|a\|_{X_T^1}\|v\|^2_{X_T^0}. \tag{2.3.115}$$

For the remaining term, we have

$$\begin{aligned}
-g\partial_x a &= gD_x Ha \\
&= D_x^{\frac{3}{2}-\sigma+\varepsilon}(gD_x^{\sigma-\frac{1}{2}-\varepsilon}Ha) - D_x^{\frac{3}{2}-\sigma+\varepsilon}gD_x^{\sigma-\frac{1}{2}-\varepsilon}Ha \\
&\quad - D_x^{\frac{3}{2}-\sigma+\varepsilon}(gD_x^{\sigma-\frac{1}{2}-\varepsilon}Ha) + D_x^{\frac{3}{2}-\sigma+\varepsilon}gD_x^{\sigma-\frac{1}{2}-\varepsilon}Ha + gD_x Ha.
\end{aligned} \tag{2.3.116}$$

Now, we estimate each term, thinking of the second line as a single term for which we will apply fractional Leibniz. For the first term in (2.3.116), we have by Hölder and Bernstein inequalities,

$$\begin{aligned}
&\int_0^T \left|\int_{\mathbb{R}} \tilde{P}_{<j}D_x^{\frac{3}{2}-\sigma+\varepsilon}(gD_x^{\sigma-\frac{1}{2}-\varepsilon}Ha)\tilde{P}_j\overline{v}\tilde{P}_j v\right| \\
&\quad\lesssim \|\tilde{P}_{<j}D_x^{\frac{3}{2}-\sigma+\varepsilon}(gD_x^{\sigma-\frac{1}{2}-\varepsilon}Ha)|\tilde{P}_j v|^{2(1-\sigma)}\|_{L_x^1 L_T^{\frac{1}{1-\sigma}}}\|\tilde{P}_j v\|^{2\sigma}_{L_x^\infty L_T^2} \\
&\quad\lesssim \|\tilde{P}_{<j}D_x^{\frac{3}{2}-\sigma+\varepsilon}(gD_x^{\sigma-\frac{1}{2}-\varepsilon}Ha)\|_{L_x^{\frac{1}{\sigma}}L_T^\infty}\|\tilde{P}_j v\|^{2(1-\sigma)}_{L_x^2 L_T^2}\|\tilde{P}_j v\|^{2\sigma}_{L_x^\infty L_T^2} \\
&\quad\lesssim T^{1-\sigma}\|\tilde{P}_{<j}(gD_x^{\sigma-\frac{1}{2}-\varepsilon}Ha)\|_{L_x^{\frac{1}{\sigma}}L_T^\infty}\|\tilde{P}_j v\|^2_{X_T^0} \\
&\quad\lesssim T^{1-\sigma}\|g\|_{L_x^{\frac{2}{2\sigma-1}}L_T^\infty}\|D_x^{\sigma-\frac{1}{2}-\varepsilon}Ha\|_{L_x^2 L_T^\infty}\|\tilde{P}_j v\|^2_{X_T^0},
\end{aligned} \tag{2.3.117}$$

where going from the second to the third line uses the fact that $\sigma > \frac{\sqrt{3}}{2}$.

Next, we estimate the second term in (2.3.116),

$$\begin{aligned}\int_0^T \left| \int_{\mathbb{R}} \tilde{P}_{<j}(D_x^{\frac{3}{2}-\sigma+\varepsilon} g D_x^{\sigma-\frac{1}{2}-\varepsilon} Ha)\tilde{P}_j\overline{v}\tilde{P}_j v \right| &\lesssim \|\tilde{P}_j v\|_{L_T^\infty L_x^2}^2 \|D_x^{\frac{3}{2}-\sigma+\varepsilon} g\|_{L_T^2 L_x^\infty} \|D_x^{\sigma-\frac{1}{2}-\varepsilon} Ha\|_{L_T^2 L_x^\infty} \\ &\lesssim T^{\frac{1}{2}} \|g\|_Z \|a\|_{X_T^1} \|\tilde{P}_j v\|_{L_T^\infty L_x^2}^2.\end{aligned} \tag{2.3.118}$$

Using Sobolev embedding and the fractional Leibniz rule, the third term is estimated analogously to the second term.

Combining the estimates and square summing then shows

$$\|I_1^j\|_{l_j^1(\mathbb{N})} \lesssim T^{1-\sigma} \|g\|_Z \|a\|_{X_T^1} \|v\|_{X_T^0}^2. \tag{2.3.119}$$

Estimate for I_2^j. A similar argument to Lemma 2.3.11 shows that

$$\|I_2^j\|_{l_j^1(\mathbb{N})} \lesssim T^{1-\sigma} \||w|^{2\sigma-1}\|_Z \|w\|_{X_T^1} \|v\|_{X_T^0}^2. \tag{2.3.120}$$

We now use the fact that for $\sigma > \frac{\sqrt{3}}{2}$, we have

$$\||w|^{2\sigma-1}\|_Z \lesssim \|w\|_{X_T^1}^{2\sigma-1}. \tag{2.3.121}$$

To see (2.3.121), first note that the $L_x^{\frac{2}{2\sigma-1}} L_T^\infty$ component is controlled by

$$\||w|^{2\sigma-1}\|_{L_x^{\frac{2}{2\sigma-1}} L_T^\infty} \lesssim \|w\|_{L_T^2 L_x^\infty}^{2\sigma-1} \lesssim \|w\|_{X_T^1}^{2\sigma-1}. \tag{2.3.122}$$

For the $L_T^\infty W_x^{\frac{3}{4\sigma}-\frac{1}{2}+\varepsilon,\infty}$ component, we have by Corollary 2.2.11, Sobolev embedding, and the fact that $\frac{(\frac{3}{4\sigma}-\frac{1}{2})}{2\sigma-1} < \frac{1}{2}$,

$$\|D_x^{\frac{3}{4\sigma}-\frac{1}{2}+\varepsilon} |w|^{2\sigma-1}\|_{L_T^\infty L_x^\infty} \lesssim \|w\|_{L_T^\infty H_x^1}^{2\sigma-1} \lesssim \|w\|_{X_T^1}^{2\sigma-1}. \tag{2.3.123}$$

This easily gives

$$\||w|^{2\sigma-1}\|_{L_T^\infty W_x^{\frac{3}{4\sigma}-\frac{1}{2}+\varepsilon,\infty}} \lesssim \|w\|_{X_T^1}^{2\sigma-1}. \tag{2.3.124}$$

Finally, for the $L_T^4 W_x^{\frac{3}{2}-\sigma+\varepsilon,\infty}$ component, we have by Corollary 2.2.11 and the fact that $\frac{\frac{3}{2}-\sigma}{2\sigma-1} < \sigma$,

$$\|D_x^{\frac{3}{2}-\sigma+\varepsilon} |w|^{2\sigma-1}\|_{L_T^4 L_x^\infty} \lesssim \|w\|_{L_T^4 W_x^{\sigma,\infty}}^{2\sigma-1} \lesssim \|w\|_{X_T^1}^{2\sigma-1} \tag{2.3.125}$$

which clearly gives

$$\||w|^{2\sigma-1}\|_{L_T^4 W_x^{\frac{3}{2}-\sigma+\varepsilon}} \lesssim \|w\|_{X_T^1}^{2\sigma-1}. \tag{2.3.126}$$

Combining the above three estimates gives (2.3.121).

Combining (2.3.121) and (2.3.120) gives

$$\|I_2^j\|_{l_j^1(\mathbb{N})} \lesssim T^{1-\sigma}\|w\|_{X_T^1}^{2\sigma}\|v\|_{X_T^0}^2. \tag{2.3.127}$$

Combining the above estimates for I_j^1 and I_j^2 completes the proof of Lemma 2.3.20.

Proof of Proposition 2.3.7. Now we complete the proof of Proposition 2.3.7.

Proof. Combining Lemma 2.3.19 and Lemma 2.3.20 with an argument similar to what was done in Proposition 2.3.6 gives for $T \sim 1$ and $\|g\|_Z, \|w\|_{X_T^1}, \|a\|_{X_T^1} \ll 1$,

$$\|v\|_{X_T^0} \lesssim \|v_0\|_{L_x^2}. \tag{2.3.128}$$

□

2.4 Well-posedness at low regularity

In this section, we aim to prove local well-posedness in H_x^s for $s \in [1, \frac{3}{2}]$ and $\sigma > \frac{\sqrt{3}}{2}$ assuming the conclusion of Theorem 2.1.2 when $\frac{3}{2} < s < 4\sigma$, which will be justified in a later section when we prove high-regularity estimates. Given the estimates established in the previous section, the scheme to prove well-posedness is relatively standard. We essentially follow the approach of [240]. See also the recent preprint [173] for a more detailed overview.

Frequency envelope bounds

Proposition 2.4.1. Let $\frac{\sqrt{3}}{2} < \sigma < 1$ and let u be as in Proposition 2.3.6. If a_j is an admissible frequency envelope for u_0 in H_x^s, then a_j is an admissible frequency envelope for u in X_T^s.

Indeed, let b_j be a X_T^s frequency envelope for the solution u. Obviously $b_0 \lesssim a_0$, so let us consider $j > 0$. By Proposition 2.3.6 a), we have

$$\|P_j u\|_{X_T^s} \lesssim_T a_j\|u_0\|_{H_x^s} + T^{\frac{1}{2}} b_j(1 + \|u\|_{S_T^1}^{4\sigma})\|u\|_{X_T^s} + T^{\frac{1-\sigma}{2}} b_j \|u\|_{X_T^1}^{\sigma}\|u\|_{X_T^s}. \tag{2.4.1}$$

Hence, by definition we have

$$b_j \lesssim a_j(1 + \|u_0\|_{H_x^s}\|u\|_{X_T^s}^{-1}) + T^{\frac{1-\sigma}{2}} b_j \|u\|_{X_T^1}^{\sigma} + T^{\frac{1}{2}} b_j (1 + \|u\|_{X_T^1}^{4\sigma}). \tag{2.4.2}$$

For T small enough, it follows from Proposition 2.3.6 that

$$b_j \lesssim a_j. \tag{2.4.3}$$

Iterating this procedure $O(T^{-1})$ many times shows that this is true for $T \lesssim 1$. This completes the proof.

Existence of H^s solutions

Now, we construct local H^s solutions to (gDNLS) for $1 \le s \le \frac{3}{2}$ as limits of more regular solutions.

Indeed, let $u_0 \in H^s$. Let $u^{(n)}$ be the globally well-posed $C_{loc}(\mathbb{R}; H_x^3)$ solution (to be constructed in a later section) to the equation,

$$\begin{cases} (i\partial_t + \partial_x^2)u^{(n)} = i|u^{(n)}|^{2\sigma}\partial_x u^{(n)}, \\ u_0^{(n)} = P_{<n}u_0. \end{cases} \tag{2.4.4}$$

Let $n > m$. We see that $v^{(m,n)} := u^{(n)} - u^{(m)}$ satisfies the equation

$$\begin{cases} (i\partial_t + \partial_x^2)v^{(m,n)} = i|u^{(n)}|^{2\sigma}\partial_x v^{(m,n)} + iG^{(n,m)}\partial_x u^{(m)} v^{(m,n)}, \\ v^{(m,n)}(0) = P_{m\le \cdot < n}u_0, \end{cases} \tag{2.4.5}$$

where

$$G^{(n,m)} := \frac{(|u^{(n)}|^{2\sigma} - |u^{(m)}|^{2\sigma})}{u^{(n)} - u^{(m)}}. \tag{2.4.6}$$

Using Corollary 2.2.11, Sobolev embedding, the fact that $\sigma > \frac{\sqrt{3}}{2}$ and Proposition 2.3.6, one easily verifies that $G^{(n,m)}$ satisfies the conditions of Proposition 2.3.7 with $\|G^{(n,m)}\|_Z \lesssim_{\|u_0\|_{H_x^s}} 1$ (with the implicit constant independent of n and m). One likewise checks using Proposition 2.3.6 that $u^{(n)}$ satisfies $\|u^{(n)}\|_{X_T^1} \lesssim_{\|u_0\|_{H_x^s}} 1$ uniformly in n. Hence, by Proposition 2.3.7, we obtain for T small enough (depending on the size of the H_x^s norm of u_0),

$$\|v^{(m,n)}\|_{X_T^0} \lesssim \|P_{m\le \cdot < n}u_0\|_{L_x^2}. \tag{2.4.7}$$

Hence, $u^{(n)}$ is Cauchy in X^0_T and thus converges to some $u \in X^0_T$. We show that in fact $u^{(n)} \to u$ in X^s_T.

To see this, we let a^n_j and a_j be admissable frequency envelopes for $P_{<n}u_0$ and u_0 respectively, in H^s_x. Clearly $(a^n_j) \to (a_j)$ in $l^2_j(\mathbb{N}_0)$ as $n \to \infty$. Now let $\varepsilon > 0$. Then thanks to Proposition 2.4.1, we have

$$\|P_{>j}u^{(n)}\|_{X^s_T} \lesssim \|(a^n_j)_{N>j}\|_{l^2_N(\mathbb{N})}\|u_0\|_{H^s_x}. \tag{2.4.8}$$

Hence, for $n \geq n_0(\varepsilon)$ large enough, we obtain the bound,

$$\|P_{>j}u^{(n)}\|_{X^s_T} \lesssim (\varepsilon + \|(a_j)_{N>j}\|_{l^2_N(\mathbb{N})})\|u_0\|_{H^s_x} \tag{2.4.9}$$

where the implicit constant is independent of j and n. Hence, there is $j = j(\varepsilon)$ such that for every $n > n_0$, we have

$$\|P_{>j}u^{(n)}\|_{X^s_T} \lesssim \varepsilon. \tag{2.4.10}$$

On the other hand, since $u^{(n)}$ converges in X^0_T, it follows that for $m, n > n_0$ large enough that

$$\|u^{(n)} - u^{(m)}\|_{X^s_T} \lesssim 2^{js}\|u^{(n)} - u^{(m)}\|_{X^0_T} + \|P_{\geq j}u^{(n)}\|_{X^s_T} + \|P_{\geq j}u^{(m)}\|_{X^s_T} \lesssim \varepsilon. \tag{2.4.11}$$

Hence, $u^{(n)}$ is Cauchy in X^s_T and thus converges to u. It is clear at this regularity that u solves the equation (gDNLS) in the sense of distributions. This shows existence.

Uniqueness and Lipschitz dependence in X^0

Here, we aim to show that solutions in X^1_T (and thus, also in X^s_T for $s > 1$) are unique and that they satisfy a weak Lipschitz type bound in X^0_T. For this, consider the difference of two solutions u^1 and u^2, $v := u^1 - u^2$. We see that v solves the equation,

$$\begin{cases} (i\partial_t + \partial_x^2)v = i|u^1|^{2\sigma}\partial_x v + iG\partial_x u^2 v, \\ v(0) = u^1(0) - u^2(0), \end{cases} \tag{2.4.12}$$

where

$$G = \frac{|u^1|^{2\sigma} - |u^2|^{2\sigma}}{u^1 - u^2}. \tag{2.4.13}$$

We see that Proposition 2.3.7 applies, and we obtain the weak Lipschitz bound

$$\|u^1 - u^2\|_{X^0_T} \lesssim \|u^1(0) - u^2(0)\|_{L^2_x}. \tag{2.4.14}$$

In particular, this shows uniqueness.

Continuous dependence in H^s

Here, we aim to show that the solution map is continuous in H^s. Specifically, we show that for each $R > 0$, there is $T = T(R) > 0$ such that the solution map from $\{u_0 : \|u_0\|_{H^s} < R\}$ to the corresponding X^s_T space is continuous. By rescaling the data and restricting to small enough time, we may assume without loss of generality that the conditions of Proposition 2.4.1 are satisfied.

Now, let $u_0^{(n)}$ be a sequence in H^s_x converging to u_0 in H^s_x. Let a_j and $a_j^{(n)}$ be the associated frequency envelopes for u_0 and $u_0^{(n)}$ given by (2.3.7). We have $(a_j^{(n)}) \to (a_j)$ in l^2. Now, let $\varepsilon > 0$. Let $N = N(\varepsilon)$ be such that $\|a^{(n)}_{j>N}\|_{l^2_j} \lesssim \varepsilon$. Using Proposition 2.4.1, we have $\|P_{>N}u^{(n)}\|_{X^s_T} \lesssim \varepsilon$ for all n. On the other hand, using the Lipschitz dependence at low frequency, we have

$$\|P_{<N}(u^{(n)} - u)\|_{X^s_T} \lesssim 2^{sN}\|u_0^{(n)} - u_0\|_{L^2}. \tag{2.4.15}$$

Now, for $n(N)$ large enough, we have

$$\|P_{<N}u^{(n)} - P_{<N}u\|_{X^s_T} \lesssim \varepsilon. \tag{2.4.16}$$

Hence, for such n, we have

$$\|u^{(n)} - u\|_{X^s_T} \lesssim \|P_{<N}(u^{(n)} - u)\|_{X^s_T} + \|P_{\geq N}u^{(n)}\|_{X^s_T} + \|P_{\geq N}u\|_{X^s_T} \lesssim \varepsilon. \tag{2.4.17}$$

It follows that

$$\limsup_{n\to\infty} \|u^{(n)} - u\|_{X^s_T} \lesssim \varepsilon. \tag{2.4.18}$$

Taking $\varepsilon \to 0$ then yields

$$\lim_{n\to\infty} \|u^{(n)} - u\|_{X^s_T} = 0 \tag{2.4.19}$$

as desired. This completes the proof of continuous dependence and also concludes the local well-posedness portion of the proof of Theorem 2.1.2 when $s \leq \frac{3}{2}$. □

Further discussion of the proofs

We now provide a brief discussion on how one can, in principle, go below the H^1_x well-posedness threshold, as well as justify some of the choices made in the proof.

It is instructive to discuss a version of this gauge transformation method which was successfully implemented in Tao's article [312] which established local well-posedness of the

Benjamin-Ono equation,

$$\begin{cases} u_t + Hu_{xx} = uu_x, \\ u(0) = u_0, \end{cases} \tag{2.4.20}$$

in H^1_x. The idea in Tao's paper was to do a type of gauge transformation by defining essentially,

$$w = P_{+hi}(e^{-iF}) \tag{2.4.21}$$

where $F(t,x)$ is a suitable spatial primitive of $u(t,x)$ and P_{+hi} is a projection onto large positive frequencies. Then one proves a priori H^2_x estimates for w (which can be translated into H^1_x estimates for u). While the coefficient u in the nonlinearity in Benjamin-Ono is only of linear order (and so one might at first naïvely suspect that this equation behaves similarly to (gDNLS) when $\sigma = \frac{1}{2}$), the spatial primitive F still essentially solves a linear Schrödinger equation (up to a perturbative error). A refinement of this gauge transformation idea appeared in [176] in which L^2_x well-posedness (among other results) for Benjamin-Ono was proven. Loosely speaking, in this latter paper, the authors performed a gauge transformation on each frequency scale to remove the leading order paradifferential part of the nonlinearity and then performed a quadratic normal form correction to remove the milder terms in the nonlinearity. Our so-called partial gauge transformation is more analogous to what was done in that paper. Specifically, the analogue of F in our proof is essentially the family of functions Φ_j as defined in (2.3.29), which in addition to the frequency localization scale, takes into account the pointwise size of u relative to the frequency scale. However, in our case, there is no obvious cancellation arising in the term $(i\partial_t\Phi_j + \partial_x^2\Phi_j)$, which forces us to estimate each term $\partial_t\Phi_j$ and $\partial_x^2\Phi_j$ separately. This is one of the major sources for the losses in our low regularity estimates.

This issue actually also adds technical difficulty when trying to lower the local well-posedness threshold below H^1_x. For instance, when estimating $\partial_t\Phi_j$ in Proposition 2.3.6, there are expressions essentially of the form

$$P_{<j}\partial_x^{-1}(gv_1v_2) \tag{2.4.22}$$

that we bound in $L^1_T L^\infty_x$, where g is some bounded function and v_1 and v_2 are linear expressions in u_x or $\overline{u_x}$. Unfortunately, in these expressions, it doesn't seem that typically the output frequency of the product gv_1v_2 is comparable to the frequencies of the individual terms v_1 and v_2, and so the ∂_x^{-1} can't be "distributed" amongst these factors to obtain expressions with lower order derivatives in place of u_x. One workaround to this issue could be to place any factors of u_x arising in such an expression in an appropriate maximal function/smoothing space as in Proposition 2.2.4. Proceeding this way will likely lead to losses worse than the $1-\sigma$ derivatives already observed in the current low regularity estimates. However, this should work in principle to lower the well-posedness threshold below H_x^1 when σ is close to 1. We decided not to do this for the sake of simplicity, as our preliminary calculations suggested that the dependence of the well-posedness threshold on σ would be rather complicated when $s<1$, at least without introducing some new tools.

2.5 High regularity estimates

In this section, we aim to prove a priori H_x^{2s}-type bounds for a global solution u to a family of regularizations of (gDNLS),

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\eta P_{<k}|v|^{2\sigma}u_x, \\ u(0) = P_{<k}u_0, \end{cases} \tag{2.5.1}$$

where $k\in\mathbb{N}$, $v\in C^2(\mathbb{R};H_x^\infty)$, $2s$ is in the range $2-\sigma<2s<4\sigma$, η is a suitable time-dependent cutoff function which is equal to 1 on the unit time interval $[-1,1]$ and supported within $(-2,2)$, and $u_0\in H_x^{2s}$ has sufficiently small norm. The key difficulty here is to obtain estimates independent of the regularization parameter k. As mentioned earlier, this is somewhat subtle because the nonlinearity is too rough to directly obtain an energy estimate by simply applying D_x^{2s} to the equation. Our overarching idea, morally, is to instead obtain suitable estimates for time derivatives, $D_t^s u$, of order $s<2\sigma$ for solutions to (2.5.1). This is one of the key technical reasons for truncating the nonlinearity with the time-dependent cutoff η and working with global in-time solutions to (2.5.1). For small enough data, one expects to be able to construct a solution u to this equation on the time interval $[-2,2]$, and then extend it to a global solution using the fact that u should solve the linear Schrödinger equation for $|t|>2$. The idea of truncating the nonlinearity with a time-dependent cutoff in order to obtain global in time solutions (to facilitate use of Fourier analysis in the time variable) is not a new idea. See for instance, [59] and [60].

Before outlining our strategy in more detail, we give an overview of the functional setting and relevant notation for this problem.

Function spaces and notation

Here, we fix some basic notation and describe the function spaces used in our construction of solutions at high regularity.

We will use S_k, $S_{<k}$ and $S_{\geq k}$ to denote the temporal variants of the spatial Littlewood-Paley projections $P_k, P_{<k}$ and $P_{\geq k}$ as defined in Section 2.2. We write $\phi(2^{-j}\xi)$ to denote the spatial Fourier multiplier for P_j and $\psi(2^{-k}\tau)$ to denote the temporal Fourier multiplier for S_k.

We will also need to sometimes distinguish between a compact time interval and the whole space in our estimates. For this purpose, let us denote for a Banach space X, $L^p_t X := L^p(\mathbb{R}; X)$ (that is, we use a lowercase t to emphasize when the underlying time interval is $\mathbb{R}$). For $T > 0$, we use $L^p_T X := L^p([-T, T]; X)$ when we want to emphasize that the time interval is compact.

Next, for the range of $2s \in (2 - \sigma, 4\sigma)$ we are considering, the smoothing and maximal function type norms from the low regularity estimates are not needed. We modify our function spaces accordingly and only use standard L^2_x based Sobolev spaces and standard Strichartz spaces (see below). Since both spatial and temporal regularity will be relevant in our analysis, we make the convention from here on that a real number s will correspond to the Sobolev regularity of a function in the time variable. In light of the scaling of the linear Schrödinger equation, it is natural to use $2s$ to denote the corresponding spatial regularity. With this in mind, for $s \geq 0$ and $T > 0$, we denote the relevant Strichartz type space by $\mathcal{S}^{2s}_T := L^4_T W^{2s,\infty}_x \cap L^\infty_T H^{2s}_x$. We also define the energy type space $\mathcal{X}^{2s}_T$ by the norm,

$$\|u\|_{\mathcal{X}^{2s}_T} := \|P_{\leq 0} u\|_{L^\infty_T H^{2s}_x} + \left(\sum_{j>0} \|P_j u\|^2_{L^\infty_T H^{2s}_x} \right)^{\frac{1}{2}}. \tag{2.5.2}$$

Clearly this controls the $C([-T, T]; H^{2s}_x)$ norm. The reason we opt for this slightly stronger norm (as opposed to just $\|u\|_{L^\infty_T H^{2s}_x}$) is because it will be slightly more convenient for proving frequency envelope bounds. Furthermore, we have the trivial embedding

$$X^{2s}_T \subseteq \mathcal{X}^{2s}_T. \tag{2.5.3}$$

Finally, since estimates for time derivatives will play a key role in our analysis, it will also be convenient to introduce the auxiliary norm

$$\|u\|_{Z^s_{p,q}} := \|\langle D_t \rangle^s u\|_{L^p_t L^q_x} + \|\langle D_x \rangle^{2s} u\|_{L^p_t L^q_x}. \tag{2.5.4}$$

When $q = 2$, we will simply abbreviate this by Z^s_p.

The reader should keep in mind that although we will often time-localize u (or the nonlinearity) to be compactly supported in time, some mild care must be taken in the estimates when nonlocal operators such as D^s_t are involved. This is especially relevant when comparing $L_t X$ and $L_T X$ type norms.

A frequency localized H^{2s}_x bound

The key result for this section is the following frequency localized H^{2s}_x a priori bound for (2.5.1).

Proposition 2.5.1. Let $2-\sigma < 2s < 4\sigma$, $T = 2$ and $u_0 \in H^{2s}_x$. Suppose that $u \in C^2(\mathbb{R}; H^\infty_x)$ solves (2.5.1). Furthermore, let a_j be a H^{2s}_x frequency envelope for u_0 and let b^1_j and b^2_j be $\mathcal{X}^{2s}_T$ frequency envelopes for u and v, respectively. Let $b_j := \max\{b^1_j, b^2_j\}$. Furthermore, let $0 < \varepsilon \ll 1$ and assume that for each $0 < \delta \ll 1$

$$\|v\|_{\mathcal{S}^{1+\delta}_T} + \|(i\partial_t + \partial_x^2)v\|_{Z^{s-1+\delta}_\infty \cap \mathcal{S}^\delta_T} \lesssim_\delta \varepsilon. \tag{2.5.5}$$

Then $P_j u$ satisfies the estimate,

$$\begin{aligned}\|P_j u\|^2_{\mathcal{X}^{2s}_T} \lesssim a_j^2 \|u_0\|^2_{H^{2s}_x} + b_j^2 \varepsilon^{2\sigma}(\|u\|^2_{\mathcal{X}^{2s}_T} + \|u\|^2_{\mathcal{S}^1_T}) + b_j^2 \varepsilon^{2\sigma-1} \|u\|_{\mathcal{S}^1_T} \|u\|_{\mathcal{X}^{2s}_T} \|v\|_{\mathcal{X}^{2s}_T} \\ + b_j^2 \varepsilon^{4\sigma-2} \|u\|^2_{\mathcal{S}^1_T} \|v\|^2_{\mathcal{X}^{2s}_T}.\end{aligned} \tag{2.5.6}$$

Furthermore, by square summing, we also have

$$\|u\|^2_{\mathcal{X}^{2s}_T} \lesssim \|u_0\|^2_{H^{2s}_x} + \varepsilon^{2\sigma}(\|u\|^2_{\mathcal{X}^{2s}_T} + \|u\|^2_{\mathcal{S}^1_T}) + \varepsilon^{2\sigma-1}\|u\|_{\mathcal{S}^1_T}\|u\|_{\mathcal{X}^{2s}_T}\|v\|_{\mathcal{X}^{2s}_T} + \varepsilon^{4\sigma-2}\|u\|^2_{\mathcal{S}^1_T}\|v\|^2_{\mathcal{X}^{2s}_T}. \tag{2.5.7}$$

Remark 2.5.2. Crucially, it should be noted that the implied constant in the bound above does not depend on the regularization parameter k.

Remark 2.5.3. The reader should carefully observe the restriction $T = 2$ and not $T \leq 2$ in Proposition 2.5.1. This is because η is localized in time to a unit scale. More work is required to show that we have suitable bounds for $T \leq 2$. This will be studied further in Section 2.6.

Next, we give a brief outline for how we will obtain such an estimate. As mentioned above, to minimize the number of derivatives which fall on the rough part of the inhomogeneous term, $|v|^{2\sigma}$, we will prove what is essentially an energy type estimate for $D^s_t u$ instead of $D_x^{2s}u$ and use the bounds for $D_t^s u$ to estimate $D_x^{2s}u$. This is consistent with the scaling symmetry of (gDNLS). There is one technical caveat however. Namely, one expects to be able to convert estimates for $D_t^s u$ to estimates for $D_x^{2s}u$ when the time frequency τ of a solution u to (2.5.1) is close to $-\xi^2$ where ξ is the spatial frequency (i.e. in the so-called low modulation region). However, this is not guaranteed due to the presence of the inhomogeneous term in the equation. Therefore, we need a suitable way of controlling $D_x^{2s}u$ for the portion of u which has space-time Fourier support far away from the characteristic hypersurface $\tau = -\xi^2$. In other words, we also need an estimate for u in the so-called high modulation region.

With this in mind, we split our analysis into two parts. First, we prove an elliptic type estimate in the high modulation region for solutions to (2.5.1) which will allow us to suitably control $D_x^{2s}u$ in terms of the portion of $D_x^{2s}u$ localized near the characteristic hypersurface, as well as a lower order term stemming from the nonlinearity. To control $D_x^{2s}u$ in the low modulation region, we essentially obtain an energy type estimate for $D_t^s u$ (the benefit being that we only have to differentiate the nonlinearity s times in the time variable as opposed to $2s$ times in the spatial variable). When u is localized near the characteristic hypersurface, this is precisely the regime in which we expect to be able to suitably control $D_x^{2s}u$ by $D_t^s u$. Proposition 2.5.1 will then follow from combining the low and high modulation analysis.

The high modulation estimate

We begin with the high modulation estimate, Lemma 2.5.4. This will be useful for estimating the portion of a (time-localized) solution to (2.5.1) which has space-time Fourier support away from the characteristic hypersurface. This can also be thought of as an elliptic space-time estimate.

Lemma 2.5.4. Let $u_0 \in H_x^\infty$ and suppose $u \in C^1(\mathbb{R}; H_x^\infty)$ solves the equation,

$$\begin{cases} (i\partial_t + \partial_x^2)u = f, \\ u(0) = u_0. \end{cases} \tag{2.5.8}$$

Let $0 \leq s \leq 1$, $j, k > 0$, $p, q \in [1, \infty]$ and suppose $|k - 2j| > 4$. Then $P_j S_k u$ satisfies the estimate,

$$\|P_j S_k \langle D_x\rangle^{2s} u\|_{L_t^p L_x^q} + \|P_j S_k \langle D_t\rangle^s u\|_{L_t^p L_x^q} \lesssim \|\tilde{P}_j \tilde{S}_k \langle D_t\rangle^{s-1} f\|_{L_t^p L_x^q}. \tag{2.5.9}$$

The result also holds for $k = 0$, when S_0 is replaced by $S_{\leq 0}$.

Proof. We prove the estimate for $\langle D_x\rangle^{2s}u$. The estimate for $\langle D_t\rangle^s u$ is similar. Notice that

$$\begin{aligned}[\mathcal{F}_{t,x}(\langle D_x\rangle^{2s}S_kP_ju)](\tau,\xi) &= \langle\xi\rangle^{2s}\psi(2^{-k}\tau)\phi(2^{-j}\xi)[\mathcal{F}_{t,x}(\tilde{S}_k\tilde{P}_ju)](\tau,\xi)\\ &= -\frac{\langle\xi\rangle^{2s}}{\tau+\xi^2}\psi(2^{-k}\tau)\phi(2^{-j}\xi)[\mathcal{F}_{t,x}\tilde{S}_k\tilde{P}_j(i\partial_t+\partial_x^2)u](\tau,\xi).\end{aligned} \tag{2.5.10}$$

Hence, by Young's inequality and (2.5.8), we have (using that $\psi(2^{-k}\tau)\phi(2^{-j}\xi)$ is supported away from $\tau+\xi^2=0$),

$$\begin{aligned}\|\langle D_x\rangle^{2s}S_kP_ju\|_{L_t^pL_x^q} &\lesssim \|\mathcal{F}_{t,x}^{-1}[\frac{\langle\xi\rangle^{2s}}{\tau+\xi^2}\psi(2^{-k}\tau)\phi(2^{-j}\xi)]\|_{L_t^1L_x^1}\|(i\partial_t+\partial_x^2)\tilde{S}_k\tilde{P}_ju\|_{L_t^pL_x^q}\\ &\lesssim \|\mathcal{F}_{t,x}^{-1}[\frac{\langle\xi\rangle^{2s}}{\tau+\xi^2}\psi(2^{-k}\tau)\phi(2^{-j}\xi)]\|_{L_t^1L_x^1}\|\tilde{S}_k\tilde{P}_jf\|_{L_t^pL_x^q}.\end{aligned} \tag{2.5.11}$$

It remains then to show that

$$\|\mathcal{F}_{t,x}^{-1}[\frac{\langle\xi\rangle^{2s}}{\tau+\xi^2}\psi(2^{-k}\tau)\phi(2^{-j}\xi)]\|_{L_t^1L_x^1}\lesssim 2^{-k(1-s)}. \tag{2.5.12}$$

A simple change of variables shows that

$$\|\mathcal{F}_{t,x}^{-1}[\frac{\langle\xi\rangle^{2s}}{\tau+\xi^2}\psi(2^{-k}\tau)\phi(2^{-j}\xi)]\|_{L_t^1L_x^1} = \|\mathcal{F}_{t,x}^{-1}[\frac{\langle 2^j\xi\rangle^{2s}}{2^k\tau+2^{2j}\xi^2}\psi(\tau)\phi(\xi)]\|_{L_t^1L_x^1}. \tag{2.5.13}$$

Then we have

$$\frac{\langle 2^j\xi\rangle^{2s}}{2^k\tau+2^{2j}\xi^2}\psi(\tau)\phi(\xi) = 2^{k(s-1)}\frac{(2^{-k}+2^{2j-k}\xi^2)^s}{\tau+2^{2j-k}\xi^2}\psi(\tau)\phi(\xi) := 2^{k(s-1)}F_{j,k}(\tau,\xi). \tag{2.5.14}$$

It is easy to see that for multi-indices $0\le|\alpha|\le 3$,

$$|\partial_{\tau,\xi}^\alpha F_{j,k}|\lesssim 1 \tag{2.5.15}$$

so that (since $\phi\psi$ is supported on $[-2,2]\times[-2,2]$)

$$\|\partial_{\tau,\xi}^\alpha F_{j,k}\|_{L_{\tau,\xi}^1}\lesssim 1 \tag{2.5.16}$$

with bound independent of j and k. It follows that

$$\|\mathcal{F}_{t,x}^{-1}[\frac{\langle 2^j\xi\rangle^{2s}}{2^k\tau+2^{2j}\xi^2}\psi(\tau)\phi(\xi)]\|_{L_t^1L_x^1}\lesssim 2^{k(s-1)}\|(1+|x|+|t|)^{-3}\|_{L_t^1L_x^1}\lesssim 2^{k(s-1)} \tag{2.5.17}$$

which is what we wanted to show. The case for $\langle D_t\rangle^s u$ is similar. $\square$

From this lemma, we obtain a very useful corollary which will allow us to control derivatives of u in the high modulation region with convenience and reduce matters to proving a suitable low modulation bound.

Corollary 2.5.5. Let $u \in C^2(\mathbb{R}; H_x^\infty)$, and let the notation be as in Lemma 2.5.4. Then for every $\delta > 0$ and $j > 0$, we have

a) If $0 \leq s < 1$,

$$\|P_j\langle D_x\rangle^{2s}u\|_{L_t^pL_x^q} + \|P_j\langle D_t\rangle^s u\|_{L_t^pL_x^q} \lesssim_\delta \|\tilde{S}_{2j}P_j\langle D_x\rangle^{2s}u\|_{L_t^pL_x^q} + \|\tilde{P}_j\langle D_t\rangle^{s-1+\delta}f\|_{L_t^pL_x^q} \quad (2.5.18)$$

and

b) If $1 \leq s < 2\sigma$,

$$\|P_j\langle D_x\rangle^{2s}u\|_{L_t^pL_x^2} + \|P_j\partial_t\langle D_t\rangle^{s-1}u\|_{L_t^pL_x^2} \lesssim_\delta \|\tilde{S}_{2j}P_j\langle D_x\rangle^{2s}u\|_{L_t^pL_x^2} + \|\tilde{P}_jf\|_{Z_{p,2}^{s-1+\delta}} \quad (2.5.19)$$

where $\tilde{S}_{2j} = S_{[2j-4,2j+4]}$.

Proof. For a), this follows from the Bernstein type estimate

$$\|D_x^{2s}\tilde{S}_{2j}P_ju\|_{L_t^pL_x^q} \sim \|D_t^s\tilde{S}_{2j}P_ju\|_{L_t^pL_x^q}$$

and from Lemma 2.5.4 by summing over $k > 0, |k-2j| > 4$ (which is where the requirement of having $\delta > 0$ comes in to play). Then b) follows from part a) with u replaced by $\partial_t u$ and s replaced by $s-1$, and then by expanding $\partial_t D_x^{2s-2}P_ju = i\partial_x^2 D_x^{2s-2}P_ju - iD_x^{2s-2}P_jf$. $\square$

Remark 2.5.6. We remark that in part b), if f takes the form of $f = i\eta P_{<k}|u|^{2\sigma}u_x$ as in (2.5.1) then if δ is sufficiently small, we expect to be able to control the last term on the right as long as $2s-2 < 2\sigma$ which is satisfied automatically, because $2s < 4\sigma < 2\sigma+2$ in the range $\frac{1}{2} < \sigma < 1$. If we were looking at the case $\sigma > 1$, this would present a new limiting threshold for which we expect to obtain estimates for u, c.f. [324].

In light of the above remark, one should observe at this point that the high modulation estimate above essentially reduces proving Proposition 2.5.1 to obtaining an estimate for the $L_T^\infty H_x^{2s}$ norm of a solution u to (2.5.1) in the low modulation region, as well as controlling an essentially perturbative source term stemming from the nonlinearity in (2.5.1). With this in mind, we now turn to the low modulation estimate, which is essentially the heart of the matter.

Low modulation estimates

Next we prove suitable bounds for the $L_T^\infty H_x^{2s}$ norm of a solution u to (2.5.1) in the low modulation region. Specifically, we prove the following energy type bound to control the portion of u which is localized near the characteristic hypersurface.

Lemma 2.5.7. Let $u_0 \in H_x^{2s}$ and suppose that $u \in C^2(\mathbb{R}; H_x^\infty)$ solves (2.5.1). Let $T = 2$, $2-\sigma < 2s < 4\sigma$, a_j be an admissible H_x^{2s} frequency envelope for u_0, and b_j^1, b_j^2 be $\mathcal{X}_T^{2s}$ frequency envelopes for u and v, respectively. Take $b_j := \max\{b_j^1, b_j^2\}$. Let $0 < \varepsilon \ll 1$ and suppose v satisfies the estimates,

$$\|v\|_{\mathcal{S}_T^{1+\delta}} + \|(i\partial_t + \partial_x^2)v\|_{Z_\infty^{s-1+\delta} \cap \mathcal{S}_T^\delta} \lesssim_\delta \varepsilon \tag{2.5.20}$$

for each $0 < \delta \ll 1$. Then for every $j \geq 0$, we have

$$\begin{aligned} \|\tilde{S}_{2j} P_j D_x^{2s} u\|_{L_T^\infty L_x^2}^2 \lesssim_\delta\, & a_j^2 \|u_0\|_{H_x^{2s}}^2 + b_j^2 \varepsilon^{2\sigma}(\|u\|_{\mathcal{X}_T^{2s}}^2 + \|u\|_{\mathcal{S}_T^1}^2) + b_j^2 \varepsilon^{2\sigma-1} \|u\|_{\mathcal{S}_T^1} \|u\|_{\mathcal{X}_T^{2s}} \|v\|_{\mathcal{X}_T^{2s}} \\ & + b_j^2 \varepsilon^{4\sigma-2} \|u\|_{\mathcal{S}_T^1}^2 \|v\|_{\mathcal{X}_T^{2s}}^2. \end{aligned} \tag{2.5.21}$$

Remark 2.5.8. As a brief but important remark, it should be noted that for $\alpha \geq 0$ there is no need to distinguish between $\|u\|_{L_t^\infty H_x^\alpha}$ and $\|u\|_{L_T^\infty H_x^\alpha}$. This is because outside of $[-2, 2]$, u solves a linear Schrödinger equation, and so the H_x^α norms are constant on both $(-\infty, -2]$ and $[2, \infty)$.

It will also be convenient to introduce the notation $\tilde{v} := \tilde{\eta} v$ where $\tilde{\eta}$ is a time-dependent cutoff supported in $(-2, 2)$ which is equal to 1 on the support of η. For notational convenience, we also write $|v|_{<k}^{2\sigma}$ to denote $P_{<k}|v|^{2\sigma}$. Now, we begin with the proof of the energy type bound in Lemma 2.5.7.

Proof. Note that we can write $\eta |v|_{<k}^{2\sigma} = \eta |\tilde{v}|_{<k}^{2\sigma}$. Next, we apply $\tilde{S}_{2j} P_j := S_{[2j-4, 2j+4]} P_j$ to the equation and see that $\tilde{S}_{2j} P_j u$ solves the equation,

$$(i\partial_t + \partial_x^2)\tilde{S}_{2j} P_j u = i\tilde{S}_{2j} P_j (\eta |\tilde{v}|_{<k}^{2\sigma} u_x), \tag{2.5.22}$$

with initial data $(\tilde{S}_{2j} P_j u)(0)$. Next, we do a paradifferential expansion of the "nonlinear" term $i\tilde{S}_{2j} P_j (\eta |\tilde{v}|_{<k}^{2\sigma} u_x)$, in both the space and time variable, which splits this term into five interactions. Indeed, first by commuting the spatial projection P_j, we have

$$\begin{aligned} \tilde{S}_{2j} P_j (i\eta |\tilde{v}|_{<k}^{2\sigma} u_x) = &\, \tilde{S}_{2j}(i\eta P_{<j-4} |\tilde{v}|_{<k}^{2\sigma} \partial_x P_j u) + \tilde{S}_{2j}(i\eta [P_j, P_{<j-4} |\tilde{v}|_{<k}^{2\sigma}] \partial_x u) \\ & + \tilde{S}_{2j} P_j (i\eta P_{\geq j-4} |\tilde{v}|_{<k}^{2\sigma} \partial_x u). \end{aligned} \tag{2.5.23}$$

Then by commuting the temporal projection $\tilde{S}_{2j}$ in the first term, we obtain

$$\begin{aligned}\tilde{S}_{2j}P_j(i\eta|\tilde{v}|^{2\sigma}_{<k}u_x) &= S_{<2j-8}(i\eta P_{<j-4}|\tilde{v}|^{2\sigma}_{<k})\partial_x P_j\tilde{S}_{2j}u + [\tilde{S}_{2j}, S_{<2j-8}(i\eta P_{<j-4}|\tilde{v}|^{2\sigma}_{<k})]\partial_x P_j u \\ &+ \tilde{S}_{2j}(S_{\geq 2j-8}(i\eta P_{<j-4}|\tilde{v}|^{2\sigma}_{<k})\partial_x P_j u) + \tilde{S}_{2j}(i\eta[P_j, P_{<j-4}|\tilde{v}|^{2\sigma}_{<k}]\partial_x u) \\ &+ i\tilde{S}_{2j}P_j(\eta P_{\geq j-4}|\tilde{v}|^{2\sigma}_{<k}\partial_x u).\end{aligned} \tag{2.5.24}$$

We label these terms in the order they appear above as $A_1, \dots, A_5$.

We make a brief remark about each of the above interactions before proceeding with the estimates. The first term, A_1, which corresponds to the low-high interaction (in spatial frequency) between the coefficient $i\eta|\tilde{v}|^{2\sigma}_{<k}$ and $\partial_x u$ reacts well to a standard energy type estimate for $P_j\tilde{S}_{2j}u$ since the single spatial derivative ∂_x on $P_j\tilde{S}_{2j}u$ can be integrated by parts onto the coefficient $i\eta|\tilde{v}|^{2\sigma}_{<k}$. The terms A_2, A_3 and A_4 are expected to be treated perturbatively. These in a very loose sense correspond to more balanced frequency interactions for which (space or time) derivatives can be distributed somewhat evenly between the terms $\partial_x u$ and $i\eta|\tilde{v}|^{2\sigma}_{<k}$. The most serious issue comes from A_5, which is the situation in which the coefficient $i\eta|\tilde{v}|^{2\sigma}_{<k}$ is at high spatial frequency compared to $\partial_x u$. Some care must be taken here to ensure that this term is not "differentiated" $2s$ times in the spatial variable, but instead "differentiated" at most only s times in the time variable.

Now, we continue with the proof. We begin with a standard energy type estimate. Indeed, multiplying (2.5.22) by $-i2^{4js}\overline{\tilde{S}_{2j}P_j u}$, taking real part and integrating over $\mathbb{R}$ in the spatial variable and from 0 to $\overline{T}$ with $|\overline{T}| \leq 2$ gives,

$$\|D_x^{2s}\tilde{S}_{2j}P_j u\|^2_{L^\infty_T L^2_x} \lesssim 2^{4js}\|(\tilde{S}_{2j}P_j u)(0)\|^2_{L^2_x} + \sum_{k=1}^{5} I_j^k \tag{2.5.25}$$

where

$$I_j^k := 2^{4js}\int_{-T}^{T}\left|\mathrm{Re}\int_{\mathbb{R}} -iA_k\overline{\tilde{S}_{2j}P_j u}\right|. \tag{2.5.26}$$

Now, we estimate each term. We need to deal with both the initial data term given by $2^{4js}\|(\tilde{S}_{2j}P_j u)(0)\|^2_{L^2_x}$ and the I_j^k terms for $k = 1, ..., 5$. First we deal with the latter terms.

Estimate for I_j^1

We integrate by parts and use Bernstein's inequality to obtain

$$\begin{aligned}
I_j^1 &= 2^{4js}\int_{-T}^{T}\Big|\mathrm{Re}\int_{\mathbb{R}} S_{<2j-8}(\eta P_{<j-4}|\tilde{v}|^{2\sigma}_{<k})\partial_x\tilde{S}_{2j}P_ju\overline{\tilde{S}_{2j}P_ju}\Big| \\
&\lesssim 2^{4js}\int_{-T}^{T}\Big|\mathrm{Re}\int_{\mathbb{R}} S_{<2j-8}\partial_x(\eta P_{<j-4}|\tilde{v}|^{2\sigma}_{<k})|\tilde{S}_{2j}P_ju|^2\Big| \\
&\lesssim 2^{4js}\|v\|^{2\sigma-1}_{L_T^\infty L_x^\infty}\|v_x\|_{L_T^1L_x^\infty}\|P_ju\|^2_{L_T^\infty L_x^2} \qquad (2.5.27)\\
&\lesssim \|v\|^{2\sigma}_{\mathcal{S}_T^1}\|D_x^{2s}P_ju\|^2_{L_T^\infty L_x^2} \\
&\lesssim b_j^2\|v\|^{2\sigma}_{\mathcal{S}_T^1}\|u\|^2_{\mathcal{X}_T^{2s}} \\
&\lesssim b_j^2\varepsilon^{2\sigma}\|u\|^2_{\mathcal{X}_T^{2s}}.
\end{aligned}$$

Estimate for I_j^2

As mentioned above, this term can be treated perturbatively. For simplicity, we denote $g := i\eta P_{<j-4}|\tilde{v}|^{2\sigma}_{<k}$. Then Lemma 2.2.1 gives

$$[\tilde{S}_{2j}, S_{<2j-8}(i\eta P_{<j-4}|\tilde{v}|^{2\sigma}_{<k})]\partial_xP_ju = 2^{-2j}\int_{\mathbb{R}^2}K(s)[\partial_tS_{<2j-8}g](t+s_1,x)[\partial_xP_ju](t+s_2,x)ds \qquad (2.5.28)$$

for some $K \in L^1(\mathbb{R}^2)$. Hölder's inequality, Minkowski's inequality, Bernstein's inequality and the fact that $\|P_ju\|_{L_t^\infty L_x^2} = \|P_ju\|_{L_T^\infty L_x^2}$ then gives

$$\begin{aligned}
I_j^2 &\lesssim 2^{-2j}2^{4js}\int_{\mathbb{R}^2}|K(s)|\int_{-T}^{T}\int_{\mathbb{R}}|[\partial_tS_{<2j-8}g](t+s_1,x)[\partial_xP_ju](t+s_2,x)||(\tilde{S}_{2j}P_ju)(t,x)|dxdtds \\
&\lesssim 2^{-j}2^{4js}\|\partial_tS_{<2j-8}g\|_{L_t^2L_x^\infty}\|P_ju\|^2_{L_T^\infty L_x^2} \\
&\lesssim 2^{(\varepsilon_0-1)j}2^{4js}\|\partial_tS_{<2j-8}g\|_{L_t^2L_x^{\frac{1}{\varepsilon_0}}}\|P_ju\|^2_{L_T^\infty L_x^2} \\
&\lesssim \|D_t^{\frac{1}{2}+\frac{\varepsilon_0}{2}}(\eta P_{<j-4}|\tilde{v}|^{2\sigma}_{<k})\|_{L_t^2L_x^{\frac{1}{\varepsilon_0}}}\|P_jD_x^{2s}u\|^2_{L_T^\infty L_x^2},
\end{aligned} \qquad (2.5.29)$$

where $\varepsilon_0 < \delta$ is some small positive constant. From the fractional Leibniz rule and then the vector valued Moser bound Proposition 2.2.7, Sobolev embedding and then Corollary 2.5.5, we obtain

$$\begin{aligned}
\|D_t^{\frac{1}{2}+\frac{\varepsilon_0}{2}}(\eta P_{<j-4}|\tilde{v}|^{2\sigma}_{<k})\|_{L_t^2L_x^{\frac{1}{\varepsilon_0}}} &\lesssim \|\tilde{v}\|^{2\sigma}_{\mathcal{S}_T^1} + \|\tilde{v}\|^{2\sigma-1}_{L_t^\infty L_x^\infty}\|D_t^{\frac{1}{2}+\frac{\varepsilon_0}{2}}\tilde{v}\|_{L_t^4L_x^{\frac{1}{\varepsilon_0}}} \\
&\lesssim \varepsilon^{2\sigma}.
\end{aligned} \qquad (2.5.30)$$

Hence,

$$I_j^2 \lesssim b_j^2 \varepsilon^{2\sigma} \|u\|_{\mathcal{X}_T^{2s}}^2. \tag{2.5.31}$$

Estimate for I_j^3

This term can also be dealt with perturbatively. Indeed, we can use Hölder and then Bernstein's inequality to shift a factor of $D_t^{\frac{1}{2}}$ onto the rough part of the nonlinearity,

$$\begin{aligned} I_j^3 &\lesssim 2^{4js} \|\tilde{S}_{2j}(S_{\geq 2j-8}(\eta P_{<j-4}|\tilde{v}|_{<k}^{2\sigma})\partial_x P_j u)\|_{L_t^2 L_x^2} \|P_j u\|_{L_T^2 L_x^2} \\ &\lesssim 2^j \|S_{\geq 2j-8}(\eta P_{<j-4}|\tilde{v}|_{<k}^{2\sigma})\|_{L_t^2 L_x^\infty} \|P_j D_x^{2s} u\|_{L_T^\infty L_x^2}^2 \\ &\lesssim \|S_{\geq 2j-8} D_t^{\frac{1}{2}}(\eta P_{<j-4}|\tilde{v}|_{<k}^{2\sigma})\|_{L_t^2 L_x^\infty} \|P_j D_x^{2s} u\|_{L_T^\infty L_x^2}^2 \\ &\lesssim b_j^2 \|D_t^{\frac{1}{2}+\frac{\varepsilon_0}{2}}(\eta P_{<j-4}|\tilde{v}|_{<k}^{2\sigma})\|_{L_t^2 L_x^{\frac{1}{\varepsilon_0}}} \|u\|_{\mathcal{X}_T^{2s}}^2. \end{aligned} \tag{2.5.32}$$

By a similar argument to the estimate for I_j^2, we then obtain,

$$I_j^3 \lesssim b_j^2 \varepsilon^{2\sigma} \|u\|_{\mathcal{X}_T^{2s}}^2. \tag{2.5.33}$$

Estimate for I_j^4

This term is also straightforward to deal with directly. The estimate is somewhat analogous to I_j^2. We have by Lemma 2.2.1,

$$[P_j, P_{<j-4}|\tilde{v}|_{<k}^{2\sigma}]\partial_x u = 2^{-j} \int_{\mathbb{R}^2} K(y)[P_{<j-4}\partial_x|\tilde{v}|_{<k}^{2\sigma}](x+y_1)[\tilde{P}_j \partial_x u](x+y_2)dy \tag{2.5.34}$$

for some integrable kernel $K \in L^1(\mathbb{R}^2)$. Hence, by Minkowski's inequality, Hölder's inequality and Bernstein's inequality,

$$\begin{aligned} I_j^4 &\lesssim \|\partial_x|\tilde{v}|_{<k}^{2\sigma}\|_{L_T^1 L_x^\infty} \|D_x^{2s}\tilde{P}_j u\|_{L_T^\infty L_x^2}^2 \\ &\lesssim b_j^2 \varepsilon^{2\sigma} \|u\|_{\mathcal{X}_T^{2s}}^2. \end{aligned} \tag{2.5.35}$$

Estimate for I_j^5

As remarked on earlier, this is the most troublesome term to deal with since the rough coefficient $|\tilde{v}|_{<k}^{2\sigma}$ is at high spatial frequency. To deal with this, first write $w = \eta u$. We expand using the Littlewood-Paley trichotomy,

$$\tilde{S}_{2j} P_j(\eta P_{\geq j-4}|\tilde{v}|_{<k}^{2\sigma}\partial_x u) = \sum_{m \geq j} \tilde{S}_{2j} P_j(\tilde{P}_m|\tilde{v}|_{<k}^{2\sigma}\partial_x \tilde{P}_m w) + \tilde{S}_{2j} P_j(\tilde{P}_j|\tilde{v}|_{<k}^{2\sigma}\partial_x \tilde{P}_{<j} w). \tag{2.5.36}$$

The first term above, where the frequency interactions between $\partial_x w$ and $|\tilde{v}|^{2\sigma}_{<k}$ are balanced, is relatively straightforward to estimate. Indeed,

$$\begin{aligned}
2^{4js}\int_{-T}^{T}|\int_{\mathbb{R}}&\overline{\tilde{S}_{2j}P_j u}\sum_{m\geq j}\tilde{S}_{2j}P_j(\tilde{P}_m|\tilde{v}|^{2\sigma}_{<k}\partial_x\tilde{P}_m w)| \\
&\lesssim \|D_x^{2s}\tilde{S}_{2j}P_j u\|_{L_T^\infty L_x^2}2^{2js}\|\sum_{m\geq j}\tilde{S}_{2j}P_j(\tilde{P}_m|\tilde{v}|^{2\sigma}_{<k}\partial_x\tilde{P}_m w)\|_{L_T^1 L_x^2} \\
&\lesssim \|D_x^{2s}\tilde{S}_{2j}P_j u\|_{L_T^\infty L_x^2}\sum_{m\geq j}2^{2js}\|\tilde{P}_m|\tilde{v}|^{2\sigma}_{<k}\partial_x\tilde{P}_m w\|_{L_t^1 L_x^2} \\
&\lesssim b_j\|u\|_{\mathcal{X}_T^{2s}}\sum_{m\geq j}2^{2js}\|\tilde{P}_m|\tilde{v}|^{2\sigma}_{<k}\partial_x\tilde{P}_m w\|_{L_t^1 L_x^2} \\
&\lesssim b_j\|\partial_x|\tilde{v}|^{2\sigma}_{<k}\|_{L_T^1 L_x^\infty}\|u\|_{\mathcal{X}_T^{2s}}\sum_{m\geq j}2^{2(j-m)s}\|\tilde{P}_m D_x^{2s}u\|_{L_T^\infty L_x^2} \\
&\lesssim b_j\varepsilon^{2\sigma}\|u\|^2_{\mathcal{X}_T^{2s}}\sum_{m\geq j}2^{2(j-m)s}b_m \\
&\lesssim b_j^2\varepsilon^{2\sigma}\|u\|^2_{\mathcal{X}_T^{2s}}
\end{aligned} \tag{2.5.37}$$

where in the last line we used the slowly varying property of b_j.

For the second term in (2.5.36), we distribute the temporal projection to obtain

$$\tilde{S}_{2j}P_j(\tilde{P}_j|\tilde{v}|^{2\sigma}_{<k}\partial_x\tilde{P}_{<j}w) = \tilde{S}_{2j}P_j(\tilde{P}_j|\tilde{v}|^{2\sigma}_{<k}\partial_x\tilde{P}_{<j}S_{\geq 2j-8}w) + \tilde{S}_{2j}P_j(\tilde{P}_j\tilde{S}_{2j}|\tilde{v}|^{2\sigma}_{<k}\partial_x P_{<j}S_{<2j-8}w). \tag{2.5.38}$$

For the first term in (2.5.38), we use Bernstein's inequality and then Corollary 2.5.5, which yields

$$\begin{aligned}
2^{2js}\|\tilde{S}_{2j}P_j(\tilde{P}_j|\tilde{v}|^{2\sigma}_{<k}\partial_x\tilde{P}_{<j}S_{\geq 2j-8}w)\|_{L_T^1 L_x^2} &\lesssim 2^{-j\varepsilon_0}\|D_x^{1+\varepsilon_0}|v|^{2\sigma}\|_{L_T^1 L_x^\infty}\|S_{\geq 2j-8}D_t^s w\|_{L_t^\infty L_x^2} \\
&\lesssim 2^{-j\varepsilon_0}\|D_x^{1+\varepsilon_0}|v|^{2\sigma}\|_{L_T^1 L_x^\infty}\|P_{\leq 0}S_{\geq 2j-8}D_t^s w\|_{L_t^\infty L_x^2} \\
&+ 2^{-j\varepsilon_0}\|D_x^{1+\varepsilon_0}|v|^{2\sigma}\|_{L_T^1 L_x^\infty}\left(\sum_{m>0}\|P_m D_t^s w\|^2_{L_t^\infty L_x^2}\right)^{\frac{1}{2}} \\
&\lesssim 2^{-j\varepsilon_0}\|D_x^{1+\varepsilon_0}|v|^{2\sigma}\|_{L_T^1 L_x^\infty}(\|u\|_{\mathcal{X}_T^{2s}} + \|g\|_{Z_\infty^{s-1+\delta}})
\end{aligned} \tag{2.5.39}$$

where $g := (i\partial_t + \partial_x^2)w$ and $0 < \varepsilon_0 \ll \delta$ is some small positive constant. If ε_0 is small enough, then Corollary 2.2.11 gives $\|D_x^{1+\varepsilon_0}|v|^{2\sigma}\|_{L_T^1 L_x^\infty} \lesssim \|v\|^{2\sigma}_{\mathcal{S}_T^{1+\delta}} \lesssim \varepsilon^{2\sigma}$. Then finally by taking $2^{-j\varepsilon_0} \lesssim b_j$, it follows that

$$2^{2js}\|\tilde{S}_{2j}P_j(\tilde{P}_j|\tilde{v}|^{2\sigma}_{<k}\partial_x\tilde{P}_{<j}S_{\geq 2j-8}w)\|_{L^1_TL^2_x} \lesssim b_j\varepsilon^{2\sigma}(\|u\|_{\mathcal{X}^{2s}_T} + \|g\|_{Z^{s-1+\delta}_\infty}). \tag{2.5.40}$$

Now we look at controlling the second term in (2.5.38). We use Bernstein's inequality and the fact that $w = \eta u$ is time-localized to obtain

$$\begin{aligned} 2^{2js}\|\tilde{S}_{2j}P_j(\tilde{P}_j\tilde{S}_{2j}|\tilde{v}|^{2\sigma}_{<k}\partial_xP_{<j}S_{<2j-8}w)\|_{L^1_TL^2_x} &\lesssim 2^{2js}\|\tilde{P}_j\tilde{S}_{2j}|\tilde{v}|^{2\sigma}_{<k}\partial_xP_{<j}S_{<2j-8}w\|_{L^1_tL^2_x} \\ &\lesssim \|u\|_{\mathcal{S}^1_T}\|D^s_t\tilde{P}_j\tilde{S}_{2j}|\tilde{v}|^{2\sigma}\|_{L^2_tL^2_x}. \end{aligned} \tag{2.5.41}$$

Here we crucially ensured that the time derivative D^s_t, rather than the spatial derivative D^{2s}_x fell on the rough part of the nonlinearity.

To control $\|D^s_t\tilde{P}_j\tilde{S}_{2j}|\tilde{v}|^{2\sigma}\|_{L^2_tL^2_x}$ we will need the following low modulation Moser type estimate.

Lemma 2.5.9. Given the conditions of Lemma 2.5.7, the following estimate holds:

$$\|\tilde{P}_j\tilde{S}_{2j}D^s_t|\tilde{v}|^{2\sigma}\|_{L^2_tL^2_x} \lesssim b_j\varepsilon^{2\sigma-1}(\varepsilon + \|v\|_{\mathcal{X}^{2s}_T}). \tag{2.5.42}$$

We will postpone the proof of this technical lemma until the end of the section.

Combining Lemma 2.5.9 and the estimate (2.5.37) allows us to estimate I^5_j by

$$I^5_j \lesssim b^2_j\varepsilon^{2\sigma}(\|u\|^2_{\mathcal{X}^{2s}_T} + \|u\|^2_{\mathcal{S}^1_T} + \|g\|^2_{Z^{s-1+\delta}_\infty}) + b^2_j\varepsilon^{2\sigma-1}\|u\|_{\mathcal{S}^1_T}\|u\|_{\mathcal{X}^{2s}_T}\|v\|_{\mathcal{X}^{2s}_T}. \tag{2.5.43}$$

Finally, combining the estimates for $I^1_j, \ldots, I^5_j$ now yields

$$\begin{aligned} \|D^{2s}_x\tilde{S}_{2j}P_ju\|^2_{L^\infty_TL^2_x} \lesssim\ & 2^{4js}\|(\tilde{S}_{2j}P_ju)(0)\|^2_{L^2_x} + b^2_j\varepsilon^{2\sigma}(\|u\|^2_{\mathcal{X}^{2s}_T} + \|u\|^2_{S^1_T} + \|g\|^2_{Z^{s-1+\delta}_\infty}) \\ & + b^2_j\varepsilon^{2\sigma-1}\|u\|_{\mathcal{S}^1_T}\|u\|_{\mathcal{X}^{2s}_T}\|v\|_{\mathcal{X}^{2s}_T}. \end{aligned} \tag{2.5.44}$$

Next, we need to control $(\tilde{S}_{2j}P_ju)(0)$ in terms of P_ju_0. To accomplish this, we use the high modulation estimate Lemma 2.5.4. Namely,

$$\begin{aligned} 2^{2js}\|(\tilde{S}_{2j}P_ju)(0)\|_{L^2_x} &\lesssim \|D^{2s}_xP_ju_0\|_{L^2_x} + \|(1-\tilde{S}_{2j})P_jD^{2s}_xu\|_{L^\infty_tL^2_x} \\ &\lesssim \|D^{2s}_xP_ju_0\|_{L^2_x} + \|S_{\leq 0}P_jD^{2s}_xu\|_{L^\infty_tL^2_x} + \sum_{m>0,|m-2j|>4}\|P_jS_mD^{2s}_xu\|_{L^\infty_tL^2_x} \\ &\lesssim \|D^{2s}_xP_ju_0\|_{L^2_x} + \|\langle D_t\rangle^{s-1+\delta}\tilde{P}_j(\eta|\tilde{v}|^{2\sigma}_{<k}u_x)\|_{L^\infty_tL^2_x}. \end{aligned} \tag{2.5.45}$$

In light of (2.5.44) and (2.5.45), to complete the proof of Lemma 2.5.7 it remains to estimate the latter term on the right hand side of (2.5.45) as well as $\|g\|_{Z_\infty^{s-1+\delta}}$. This is done in the following lemma.

Lemma 2.5.10. Let $s, \sigma, T, u_0, u, a_j$ and b_j be as in Proposition 2.5.1. Let v also be as in Proposition 2.5.1, but with (2.5.5) replaced by the weaker assumption that for all $0 < \varepsilon \ll 1$ and $0 < \delta \ll 1$,

$$\|v\|_{\mathcal{S}_T^{1+\delta}} + \|(i\partial_t + \partial_x^2)v\|_{Z_\infty^{s-\frac{3}{2}+\delta}} \lesssim_\delta \varepsilon. \tag{2.5.46}$$

Then we have

$$\|\tilde{P}_j(\eta|\tilde{v}|^{2\sigma}_{<k}u_x)\|_{Z_\infty^{s-1+\delta}} \lesssim b_j\varepsilon^{2\sigma}(\|u\|_{\mathcal{S}_T^1} + \|u\|_{\mathcal{X}_T^{2s-1+c\delta}}) + b_j\varepsilon^{2\sigma-1}\|u\|_{\mathcal{S}_T^1}\|v\|_{\mathcal{X}_T^{2s-1+c\delta}} \tag{2.5.47}$$

and

$$\|(i\partial_t + \partial_x^2)w\|_{Z_\infty^{s-1+\delta}} := \|g\|_{Z_\infty^{s-1+\delta}} \lesssim \|u\|_{\mathcal{X}_T^{2s-1+c\delta}} + \|u\|_{\mathcal{S}_T^1} + \|u\|_{\mathcal{S}_T^1}\|v\|_{\mathcal{X}_T^{2s-1+c\delta}}, \tag{2.5.48}$$

for some constant $c > 0$.

Remark 2.5.11. The reader may wonder why we estimate the full $Z_\infty^{s-1+\delta}$ norm in the above lemma. Although the argument up until this point only requires us to estimate the component of the $Z_\infty^{s-1+\delta}$ norm involving the time derivative, we will need to also estimate the component involving spatial derivatives in the next section when we establish well-posedness for the full equation in $\mathcal{X}_T^{2s}$.

Proof. We begin with (2.5.47). For the purpose of not having to track all the factors of δ that appear throughout the proof, we will denote by $c > 0$ some positive constant which is allowed to grow from line to line. First we study the component of the $Z_\infty^{s-1+\delta}$ norm which involves the time derivative. By considering separately temporal frequencies larger than 2^{2j} and smaller than 2^{2j}, we obtain (using the vector valued Bernstein inequality),

$$\begin{aligned}\|\tilde{P}_j\langle D_t\rangle^{s-1+\delta}(\eta|\tilde{v}|^{2\sigma}_{<k}u_x)\|_{L_t^\infty L_x^2} &\lesssim 2^{-2j\delta}\|\tilde{P}_j\langle D_x\rangle^{2s-2+c\delta}(\eta|\tilde{v}|^{2\sigma}_{<k}u_x)\|_{L_t^\infty L_x^2} \\ &\quad + 2^{-2j\delta}\|\tilde{P}_jS_{>2j}\langle D_t\rangle^{s-1+c\delta}(\eta|\tilde{v}|^{2\sigma}_{<k}u_x)\|_{L_t^\infty L_x^2}.\end{aligned} \tag{2.5.49}$$

Hence,

$$\begin{aligned}\|\tilde{P}_j(\eta|\tilde{v}|^{2\sigma}_{<k}u_x)\|_{Z_\infty^{s-1+\delta}} &\lesssim 2^{-2j\delta}\|\tilde{P}_j\langle D_x\rangle^{2s-2+c\delta}(\eta|\tilde{v}|^{2\sigma}_{<k}u_x)\|_{L_t^\infty L_x^2} \\ &\quad + 2^{-2j\delta}\|\tilde{P}_jS_{>2j}\langle D_t\rangle^{s-1+c\delta}(\eta|\tilde{v}|^{2\sigma}_{<k}u_x)\|_{L_t^\infty L_x^2}.\end{aligned} \tag{2.5.50}$$

We now look at the first term in (2.5.50). The bound

$$2^{-2j\delta}\|\tilde{P}_j\langle D_x\rangle^{2s-2+c\delta}(\eta|\tilde{v}|^{2\sigma}_{<k}u_x)\|_{L_t^\infty L_x^2} \lesssim b_j\varepsilon^{2\sigma}\|u\|_{\mathcal{X}_T^{2s-1+c\delta}} + b_j\varepsilon^{2\sigma-1}\|u\|_{\mathcal{S}_T^1}\|v\|_{\mathcal{X}_T^{2s-1+c\delta}} \tag{2.5.51}$$

is a straightforward consequence of $2^{-2j\delta} \lesssim b_j$ and the fractional Leibniz rule if $2s-2<1$. If $2s-2 \geq 1$, then for the homogeneous component, we have

$$\begin{aligned}\|D_x^{2s-2+c\delta}(i\eta P_{<k}|v|^{2\sigma}u_x)\|_{L_t^\infty L_x^2} \lesssim \|&D_x^{2s-3+c\delta}(i\eta P_{<k}|v|^{2\sigma}u_{xx})\|_{L_t^\infty L_x^2} \\ &+ \|\eta D_x^{2s-3+c\delta}(\mathrm{Re}P_{<k}(|v|^{2\sigma-2}\overline{v}v_x)u_x)\|_{L_t^\infty L_x^2}.\end{aligned} \tag{2.5.52}$$

By the fractional Leibniz rule and Sobolev embedding, clearly the first term above can be controlled by $\varepsilon^{2\sigma}\|u\|_{\mathcal{X}_T^{2s-1+c\delta}}$. Using the fact that $2s-3<2\sigma-1$ and applying the fractional Leibniz rule, Corollary 2.2.11 (when $D_x^{2s-3+c\delta}$ falls on $|v|^{2\sigma-2}\overline{v}$) and interpolation, we can control the second term by

$$\varepsilon^{2\sigma}\|u\|_{\mathcal{X}_T^{2s-1+c\delta}} + \varepsilon^{2\sigma-1}\|u\|_{\mathcal{S}_T^1}\|v\|_{\mathcal{X}_T^{2s-1+c\delta}} \tag{2.5.53}$$

to obtain the desired bound (2.5.51).

Now, to estimate the second term on the right hand side of (2.5.50), we use that $2^{-2j\delta} \lesssim b_j$ and estimate

$$\begin{aligned}2^{-2j\delta}\|\tilde{P}_jS_{>2j}\langle D_t\rangle^{s-1+c\delta}(\eta|\tilde{v}|^{2\sigma}_{<k}u_x)\|_{L_t^\infty L_x^2} &\lesssim b_j\|\tilde{P}_jS_{>2j}\langle D_t\rangle^{s-1+c\delta}(\eta|\tilde{v}|^{2\sigma}_{<k}u_x)\|_{L_t^\infty L_x^2} \\ &\lesssim b_j\sum_{m\geq 2j}\|\tilde{P}_jS_m\langle D_t\rangle^{s-1+c\delta}(\eta|\tilde{v}|^{2\sigma}_{<k}u_x)\|_{L_t^\infty L_x^2} \\ &\lesssim b_j\sum_{m\geq 2j}\|\tilde{P}_jS_m\langle D_t\rangle^{s-1+c\delta}(S_{<m-4}(\eta|\tilde{v}|^{2\sigma}_{<k})\tilde{S}_mu_x)\|_{L_t^\infty L_x^2} \\ &+ b_j\sum_{m\geq 2j}\|\tilde{P}_jS_m\langle D_t\rangle^{s-1+c\delta}(S_{\geq m-4}(\eta|\tilde{v}|^{2\sigma}_{<k})u_x)\|_{L_t^\infty L_x^2}.\end{aligned} \tag{2.5.54}$$

For the first term in (2.5.54), we have by Bernstein's inequality,

$$\begin{aligned}&b_j\sum_{m\geq 2j}\|\tilde{P}_jS_m\langle D_t\rangle^{s-1+c\delta}(S_{<m-4}(\eta|\tilde{v}|^{2\sigma}_{<k})\tilde{S}_mu_x)\|_{L_t^\infty L_x^2} \\ &\lesssim b_j\sum_{m\geq 2j}\|\tilde{P}_jS_m\langle D_t\rangle^{s-\frac{1}{2}+c\delta}(S_{<m-4}(\eta|\tilde{v}|^{2\sigma}_{<k})\tilde{S}_mu)\|_{L_t^\infty L_x^2} \\ &+ b_j\sum_{m\geq 2j}\|\tilde{P}_jS_m\langle D_t\rangle^{s-1+c\delta}(S_{<m-4}(\eta\partial_x|\tilde{v}|^{2\sigma}_{<k})\tilde{S}_mu)\|_{L_t^\infty L_x^2}.\end{aligned} \tag{2.5.55}$$

Using Bernstein's inequality and Corollary 2.5.5, we may control the first term by

$$\begin{aligned}
b_j \sum_{m\geq 2j} \|\tilde{P}_j S_m \langle D_t\rangle^{s-\frac{1}{2}+c\delta}(S_{<m-4}(\eta|\tilde{v}|^{2\sigma}_{<k})\tilde{S}_m u)\|_{L^\infty_t L^2_x} &\lesssim b_j \|v\|^{2\sigma}_{\mathcal{S}^1_T} \|\langle D_t\rangle^{s-\frac{1}{2}+c\delta} u\|_{L^\infty_t L^2_x} \\
&\lesssim b_j \varepsilon^{2\sigma} \|\langle D_t\rangle^{s-\frac{1}{2}+c\delta} u\|_{L^\infty_t L^2_x} \\
&\lesssim b_j \varepsilon^{2\sigma} (\|u\|_{\mathcal{X}^{2s-1+c\delta}_T} + \|\eta|v|^{2\sigma}_{<k} u_x\|_{Z^{s-\frac{3}{2}+c\delta}_\infty}).
\end{aligned} \tag{2.5.56}$$

For the second term in (2.5.55), we obtain also

$$\begin{aligned}
b_j \sum_{m\geq 2j} \|\tilde{P}_j S_m \langle D_t\rangle^{s-1+c\delta}(S_{<m-4}(\eta\partial_x|\tilde{v}|^{2\sigma}_{<k})\tilde{S}_m u)\|_{L^\infty_t L^2_x} &\lesssim b_j \||\tilde{v}|^{2\sigma}\|_{L^\infty_t H^1_x} \|\langle D_t\rangle^{s-1+c\delta} u\|_{L^\infty_t L^\infty_x} \\
&\lesssim b_j \||\tilde{v}|^{2\sigma}\|_{L^\infty_t H^1_x} \|\langle D_t\rangle^{s-1+c\delta}\langle D_x\rangle^{\frac{1}{2}+\delta} u\|_{L^\infty_t L^2_x} \\
&\lesssim b_j \varepsilon^{2\sigma} (\|\langle D_t\rangle^{s-\frac{1}{2}+c\delta} u\|_{L^\infty_t L^2_x} + \|\langle D_x\rangle^{s-\frac{1}{2}+c\delta} u\|_{L^\infty_t L^2_x}) \\
&\lesssim b_j \varepsilon^{2\sigma} (\|u\|_{\mathcal{X}^{2s-1+c\delta}_T} + \|\eta|v|^{2\sigma}_{<k} u_x\|_{Z^{s-\frac{3}{2}+c\delta}_\infty}).
\end{aligned} \tag{2.5.57}$$

For the second term in (2.5.54), we obtain

$$\begin{aligned}
b_j \sum_{m\geq 2j} \|\tilde{P}_j S_m \langle D_t\rangle^{s-1+c\delta}(S_{\geq m-4}(\eta|\tilde{v}|^{2\sigma}_{<k}) u_x)\|_{L^\infty_t L^2_x} &\lesssim b_j \|u_x\|_{L^\infty_T L^2_x} \|\langle D_t\rangle^{s-1+c\delta}(\eta|\tilde{v}|^{2\sigma}_{<k})\|_{L^\infty_t L^\infty_x} \\
&\lesssim b_j \|u\|_{\mathcal{S}^1_T} \|\langle D_t\rangle^{s-1+c\delta}(\eta|\tilde{v}|^{2\sigma}_{<k})\|_{L^\infty_t L^\infty_x}.
\end{aligned} \tag{2.5.58}$$

We have by Sobolev embedding, the fractional Leibniz rule and the Moser bound Proposition 2.2.8,

$$\begin{aligned}
\|\langle D_t\rangle^{s-1+c\delta}(\eta|\tilde{v}|^{2\sigma}_{<k})\|_{L^\infty_t L^\infty_x} &\lesssim \|\langle D_t\rangle^{s-1+c\delta}(\eta|\tilde{v}|^{2\sigma}_{<k})\|_{L^\infty_x L^{\frac{1}{\delta}}_t} \\
&\lesssim \||\tilde{v}|^{2\sigma}\|_{L^\infty_x L^{\frac{2}{\delta}}_t} + \|\langle D_t\rangle^{s-1+c\delta}|\tilde{v}|^{2\sigma}_{<k}\|_{L^\infty_x L^{\frac{2}{\delta}}_t} \\
&\lesssim \||\tilde{v}|^{2\sigma}\|_{L^{\frac{2}{\delta}}_t L^\infty_x} + \||\tilde{v}|^{2\sigma-1}\|_{L^\infty_x L^{\frac{4}{\delta}}_t} \|\langle D_t\rangle^{s-1+c\delta}\tilde{v}\|_{L^\infty_x L^{\frac{4}{\delta}}_t} \\
&\lesssim \varepsilon^{2\sigma} + \||\tilde{v}|^{2\sigma-1}\|_{L^{\frac{4}{\delta}}_t L^\infty_x} \|\langle D_t\rangle^{s-1+c\delta}\tilde{v}\|_{L^{\frac{4}{\delta}}_t L^\infty_x} \\
&\lesssim \varepsilon^{2\sigma} + \|v\|^{2\sigma-1}_{\mathcal{S}^1_T} \|\langle D_t\rangle^{s-1+c\delta}\tilde{v}\|_{L^{\frac{4}{\delta}}_t L^\infty_x}.
\end{aligned} \tag{2.5.59}$$

Now, notice that by Corollary 2.5.5,

$$\begin{aligned}
\|\langle D_t\rangle^{s-1+c\delta}\tilde{v}\|_{L_t^{\frac{4}{\delta}}L_x^\infty} &\lesssim \sum_{j\geq 0}\|\langle D_t\rangle^{s-1+c\delta}P_j\tilde{v}\|_{L_t^{\frac{4}{\delta}}L_x^\infty}\\
&\lesssim \sum_{j\geq 0}\|\langle D_t\rangle^{s-1+c\delta}\langle D_x\rangle^{\frac{1}{2}}P_j\tilde{v}\|_{L_t^{\frac{4}{\delta}}L_x^2}\\
&\lesssim \sum_{j\geq 0}\|\langle D_x\rangle^{s-\frac{1}{2}+c\delta}P_j\tilde{v}\|_{L_t^{\frac{4}{\delta}}L_x^2}+\sum_{j\geq 0}\|\langle D_t\rangle^{s-\frac{1}{2}+c\delta}P_j\tilde{v}\|_{L_t^{\frac{4}{\delta}}L_x^2}\\
&\lesssim \|\tilde{v}\|_{\mathcal{X}_T^{2s-1+c\delta}}+\sum_{j\geq 0}\left(\|\langle D_x\rangle^{2s-1+c\delta}P_j\tilde{v}\|_{L_t^{\frac{4}{\delta}}L_x^2}+\|\tilde{P}_j\langle D_t\rangle^{s-\frac{3}{2}+c\delta}(i\partial_t+\partial_x^2)\tilde{v}\|_{L_t^{\frac{4}{\delta}}L_x^2}\right)\\
&\lesssim \|v\|_{\mathcal{X}_T^{2s-1+c\delta}}+\sum_{j\geq 0}\|\tilde{P}_j\langle D_t\rangle^{s-\frac{3}{2}+c\delta}(i\partial_t+\partial_x^2)\tilde{v}\|_{L_t^{\frac{4}{\delta}}L_x^2}.
\end{aligned} \tag{2.5.60}$$

To control the latter term above, there are two cases. If $s-\frac{3}{2}<0$, then this term can be easily controlled by ε by commuting $(i\partial_t+\partial_x^2)$ with $\tilde{\eta}$ and applying Hölder's inequality. If $s-\frac{3}{2}\geq 0$, then we have (after possibly enlarging $c\delta$)

$$\begin{aligned}
&\sum_{j\geq 0}\|\tilde{P}_j\langle D_t\rangle^{s-\frac{3}{2}+c\delta}(i\partial_t+\partial_x^2)\tilde{v}\|_{L_t^{\frac{4}{\delta}}L_x^2}\\
&\qquad\lesssim \|\langle D_t\rangle^{s-\frac{3}{2}+c\delta}(\eta(i\partial_t+\partial_x^2)v)\|_{L_t^{\frac{4}{\delta}}L_x^2}+\|\langle D_t\rangle^{s-\frac{3}{2}+c\delta}(\partial_t\tilde{\eta}v)\|_{L_t^{\frac{4}{\delta}}L_x^2}\\
&\qquad\lesssim \sum_{k\geq 0}\|\langle D_t\rangle^{s-\frac{3}{2}+c\delta}S_k(\eta(i\partial_t+\partial_x^2)v)\|_{L_t^{\frac{4}{\delta}}L_x^2}+\|\langle D_t\rangle^{s-\frac{3}{2}+c\delta}S_k(\partial_t\tilde{\eta}v)\|_{L_t^{\frac{4}{\delta}}L_x^2}.
\end{aligned} \tag{2.5.61}$$

By doing a paraproduct expansion of

$$S_k(\eta(i\partial_t+\partial_x^2)v)=S_k(S_{<k-4}\eta(i\partial_t+\partial_x^2)\tilde{S}_kv)+S_k(S_{\geq k-4}\eta(i\partial_t+\partial_x^2)v),$$

using Bernstein and Hölder's inequality, summing over k, and possibly enlarging the factor of $c\delta$, we obtain

$$\sum_{k\geq 0}\|\langle D_t\rangle^{s-\frac{3}{2}+c\delta}S_k(\eta(i\partial_t+\partial_x^2)v)\|_{L_t^{\frac{4}{\delta}}L_x^2}\lesssim \|(i\partial_t+\partial_x^2)v\|_{Z_\infty^{s-\frac{3}{2}+c\delta}}\lesssim\varepsilon. \tag{2.5.62}$$

A similar argument involving a paraproduct expansion of $S_k(\partial_t\eta v)$ can be used to show

$$\|\langle D_t\rangle^{s-\frac{3}{2}+c\delta}S_k(\partial_t\eta v)\|_{L_t^{\frac{4}{\delta}}L_x^2}\lesssim\varepsilon. \tag{2.5.63}$$

Therefore, the second term in (2.5.54) can be controlled by

$$b_j\varepsilon^{2\sigma}\|u\|_{\mathcal{S}^1_T} + b_j\varepsilon^{2\sigma-1}\|u\|_{\mathcal{S}^1_T}\|v\|_{\mathcal{X}_T^{2s-1+c\delta}}. \tag{2.5.64}$$

Combining this and (2.5.55) with (2.5.51) yields the estimate,

$$\begin{aligned}&\|\tilde{P}_j(\eta|\tilde{v}|^{2\sigma}_{<k}u_x)\|_{Z_\infty^{s-1+\delta}} \\ &\quad\lesssim b_j\varepsilon^{2\sigma}(\|u\|_{\mathcal{X}_T^{2s-1+c\delta}} + \|u\|_{\mathcal{S}^1_T} + \|\eta|v|^{2\sigma}_{<k}u_x\|_{Z_\infty^{s-\frac{3}{2}+c\delta}}) + b_j\varepsilon^{2\sigma-1}\|u\|_{\mathcal{S}^1_T}\|v\|_{\mathcal{X}_T^{2s-1+c\delta}}.\end{aligned} \tag{2.5.65}$$

By square summing (2.5.65) and applying (2.5.65) with $s-1$ replaced by $s-\frac{3}{2}$, we obtain

$$\|\eta|v|^{2\sigma}_{<k}u_x\|_{Z_\infty^{s-\frac{3}{2}+c\delta}} \lesssim \varepsilon^{2\sigma}(\|u\|_{\mathcal{X}_T^{2s-1+c\delta}} + \|u\|_{\mathcal{S}^1_T} + \|\eta|v|^{2\sigma}_{<k}u_x\|_{Z_\infty^{s-2+c\delta}}) + \varepsilon^{2\sigma-1}\|u\|_{\mathcal{S}^1_T}\|v\|_{\mathcal{X}_T^{2s-1+c\delta}} \tag{2.5.66}$$

and since $s<2$, it follows that if δ is small enough, then

$$\|\eta|v|^{2\sigma}_{<k}u_x\|_{Z_\infty^{s-2+c\delta}} \lesssim \varepsilon^{2\sigma}\|u\|_{\mathcal{X}_T^{2s-1+c\delta}}. \tag{2.5.67}$$

Therefore, the bound

$$\|\tilde{P}_j(\eta|\tilde{v}|^{2\sigma}_{<k}u_x)\|_{Z_\infty^{s-1+\delta}} \lesssim b_j\varepsilon^{2\sigma}(\|u\|_{\mathcal{X}_T^{2s-1+c\delta}} + \|u\|_{\mathcal{S}^1_T}) + b_j\varepsilon^{2\sigma-1}\|u\|_{\mathcal{S}^1_T}\|v\|_{\mathcal{X}_T^{2s-1+c\delta}} \tag{2.5.68}$$

follows.

For the estimate (2.5.48), we have

$$\|g\|_{Z_\infty^{s-1+\delta}} \lesssim \|\partial_t\eta u\|_{Z_\infty^{s-1+\delta}} + \|\eta P_{<k}|v|^{2\sigma}u_x\|_{Z_\infty^{s-1+\delta}}. \tag{2.5.69}$$

The first term above is controlled using Corollary 2.5.5 by

$$\begin{aligned}\|\partial_t\eta u\|_{Z_\infty^{s-1+\delta}} &\lesssim \|u\|_{\mathcal{X}_T^{2s-1+c\delta}} + \|\langle D_t\rangle^{s-1+\delta}(\partial_t\eta u)\|_{L_t^\infty L_x^2} \\ &\lesssim \|u\|_{\mathcal{X}_T^{2s-1+c\delta}} + \|\partial_t^2\eta u\|_{Z_\infty^{s-2+c\delta}} + \|\partial_t\eta(\eta P_{<k}|v|^{2\sigma}u_x)\|_{Z_\infty^{s-2+c\delta}} \\ &\lesssim \|u\|_{\mathcal{X}_T^{2s-1+c\delta}}\end{aligned} \tag{2.5.70}$$

where in the last line, we used that $s<2$. The second term in (2.5.69) can be estimated by square summing (2.5.68). This completes the proof of Lemma 2.5.10. □

Finally, we complete the proof of Lemma 2.5.7. This simply follows by combining Lemma 2.5.10 with the estimates (2.5.44) and (2.5.45). □

Proof of Proposition 2.5.1

Finally, we prove the main estimate of the section, Proposition 2.5.1.

Proof. Let $0 < \delta \ll 1$. From Corollary 2.5.5, we have

$$\|P_j u\|^2_{L^\infty_T H^{2s}_x} \lesssim_\delta \|\tilde{S}_{2j} P_j u\|^2_{L^\infty_T H^{2s}_x} + \|\tilde{P}_j(\eta|\tilde{v}|^{2\sigma}_{<k} u_x)\|^2_{Z^{s-1+\delta}_\infty}. \tag{2.5.71}$$

By Lemma 2.5.7, we have

$$\begin{aligned}\|\tilde{S}_{2j} P_j u\|^2_{L^\infty_T H^{2s}_x} \lesssim_\delta\ & a_j^2 \|u_0\|^2_{H^{2s}_x} + b_j^2 \varepsilon^{2\sigma}(\|u\|^2_{\mathcal{X}^{2s}_T} + \|u\|^2_{\mathcal{S}^1_T}) + b_j^2 \varepsilon^{2\sigma-1}\|u\|_{\mathcal{S}^1_T}\|u\|_{\mathcal{X}^{2s}_T}\|v\|_{\mathcal{X}^{2s}_T} \\ & + b_j^2 \varepsilon^{4\sigma-2}\|u\|^2_{\mathcal{S}^1_T}\|v\|^2_{\mathcal{X}^{2s}_T}.\end{aligned} \tag{2.5.72}$$

Furthermore, by Lemma 2.5.10, we have

$$\begin{aligned}\|\tilde{P}_j(\eta|\tilde{v}|^{2\sigma}_{<k} u_x)\|^2_{Z^{s-1+\delta}_\infty} &\lesssim b_j^2 \varepsilon^{4\sigma}(\|u\|^2_{\mathcal{X}^{2s}_T} + \|u\|^2_{\mathcal{S}^1_T}) + b_j^2 \varepsilon^{4\sigma-2}\|u\|^2_{\mathcal{S}^1_T}\|v\|^2_{\mathcal{X}^{2s}_T} \\ &\lesssim b_j^2 \varepsilon^{2\sigma}(\|u\|^2_{\mathcal{X}^{2s}_T} + \|u\|^2_{\mathcal{S}^1_T}) + b_j^2 \varepsilon^{4\sigma-2}\|u\|^2_{\mathcal{S}^1_T}\|v\|^2_{\mathcal{X}^{2s}_T}.\end{aligned} \tag{2.5.73}$$

This completes the proof. □

Proof of Lemma 2.5.9

It remains to prove the technical estimate Lemma 2.5.9. This will follow from the slightly more general estimate:

Lemma 2.5.12. Let $T = 2$, $\frac{1}{2} < \sigma < 1$ and u be a $C^2(\mathbb{R}; H^\infty_x)$ solution to the inhomogeneous Schrödinger equation,

$$(i\partial_t + \partial_x^2)u = f \tag{2.5.74}$$

supported in the time interval $[-2, 2]$. Furthermore, let b_j be an admissible $\mathcal{X}^{2s}_T$ frequency envelope for u (here we don't assume that the formula is necessarily given explicitly by (2.2.4)). Then for $j > 0$ we have,

a) If $0 < s < 1$, then

$$\|\tilde{P}_j \tilde{S}_{2j} D^s_t(|u|^{2\sigma})\|_{L^2_t L^2_x} \lesssim b_j \|u\|^{2\sigma-1}_{\mathcal{S}^1_T}(\|u\|_{\mathcal{X}^{2s}_T} + \|f\|_{\mathcal{S}^0_T}). \tag{2.5.75}$$

b) If $1 \le s < 2\sigma$ and $0 < \delta \ll 1$, then

$$\|\tilde{P}_j \tilde{S}_{2j} D^s_t(|u|^{2\sigma})\|_{L^2_t L^2_x} \lesssim_\delta b_j \Lambda(\|u\|_{\mathcal{X}^{2s}_T} + \|f\|_{\mathcal{S}^0_T} + \|f\|_{Z^{s-1+\delta}_\infty}) \tag{2.5.76}$$

where

$$\Lambda := (\|u\|_{\mathcal{S}_T^{1+\delta}} + \|f\|_{\mathcal{S}_T^{\delta}})^{2\sigma-1}\Lambda_0 \tag{2.5.77}$$

and Λ_0 is some polynomial in $\|u\|_{\mathcal{S}_T^{1+\delta}} + \|f\|_{\mathcal{S}_T^{\delta}}$.

Remark 2.5.13. We only prove the above estimate for $\tilde{P}_j\tilde{S}_{2j}D_t^s(|u|^{2\sigma})$ in $L_t^2L_x^2$. Although the estimate is almost certainly true for a suitable range of $p \geq 1$ in $L_t^pL_x^2$, we do not pursue this here, so as not to further complicate the argument (specifically, the proof of b)).

Remark 2.5.14. We do not claim that the factors of $\|f\|_{\mathcal{S}_T^0}$, $\|f\|_{Z_\infty^{s-1+\delta}}$ and Λ that appear in the estimate are in any way optimal (in fact, in many instances in the below estimates, they arise in relatively crude ways). We opted not to carefully optimize the inequality because it will not affect the range of s for which Lemma 2.5.7 holds, and also because the current form of Lemma 2.5.12 can be more easily applied to establish Proposition 2.5.1.

Proof. a) For notational convenience, we will sometimes write $F(z) = |z|^{2\sigma-2}\overline{z}$ and $P_{<j}u = u_{<j}$. Now, for each $j > 0$, we write

$$\begin{aligned} D_t^s\tilde{S}_{2j}P_j|u|^{2\sigma} &= D_t^s\tilde{S}_{2j}P_j|P_{<j}u|^{2\sigma} - D_t^s\tilde{S}_{2j}P_j(|P_{<j}u|^{2\sigma} - |u|^{2\sigma}) \\ &= D_t^s\tilde{S}_{2j}P_j|P_{<j}u|^{2\sigma} + 2\sigma\mathrm{Re}\int_0^1 P_j\tilde{S}_{2j}D_t^s(F(y(\theta))P_{\geq j}u)d\theta \end{aligned} \tag{2.5.78}$$

where

$$y(\theta) := \theta u + (1-\theta)P_{<j}u. \tag{2.5.79}$$

For the first term, interpolating gives

$$\|D_t^s\tilde{S}_{2j}P_j|P_{<j}u|^{2\sigma}\|_{L_t^2L_x^2} \lesssim \|\tilde{S}_{2j}P_j|P_{<j}u|^{2\sigma}\|_{L_t^2L_x^2}^{1-s}\|\tilde{S}_{2j}P_j(F(u_{<j})P_{<j}u_t)\|_{L_t^2L_x^2}^s. \tag{2.5.80}$$

By expanding u_t in the second factor, we obtain

$$\begin{aligned} \|\tilde{S}_{2j}P_j(F(u_{<j})P_{<j}u_t)\|_{L_t^2L_x^2} \lesssim\ & \|\tilde{S}_{2j}P_j(F(u_{<j})P_{<j}u_{xx})\|_{L_t^2L_x^2} \\ &+ \|\tilde{S}_{2j}P_j(F(u_{<j})P_{<j}f)\|_{L_t^2L_x^2}. \end{aligned} \tag{2.5.81}$$

We expand the first term in (2.5.81) using the Littlewood-Paley trichotomy. Then Bernstein's inequality and Corollary 2.2.11 yields

$$\begin{aligned} &\|\tilde{S}_{2j}P_j(F(u_{<j})P_{<j}u_{xx})\|_{L_t^2L_x^2} \\ &\lesssim \|P_{<j-4}F(u_{<j})\tilde{P}_ju_{xx}\|_{L_t^2L_x^2} + \|P_{\geq j-4}F(u_{<j})P_{<j}u_{xx}\|_{L_t^2L_x^2} \\ &\lesssim b_j2^{2j(1-s)}\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|u\|_{\mathcal{X}_T^{2s}} + 2^{2j(1-s)}2^{-\delta j}\|D_x^\delta F(u_{<j})\|_{L_t^\infty L_x^\infty}\|D_x^{2s}u\|_{L_t^\infty L_x^2} \\ &\lesssim b_j2^{2j(1-s)}\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|u\|_{\mathcal{X}_T^{2s}}. \end{aligned} \tag{2.5.82}$$

For the second term in (2.5.81), we obtain (by taking $2^{2j(s-1)} \lesssim b_j$)

$$\|\tilde{S}_{2j}P_j(F(u_{<j})P_{<j}f)\|_{L_t^2L_x^2} \lesssim b_j 2^{2j(1-s)}\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|f\|_{L_t^\infty L_x^2} \tag{2.5.83}$$

and so by Bernstein, the estimate (2.5.80) becomes

$$\begin{aligned}
&\|D_t^s\tilde{S}_{2j}P_j|P_{<j}u|^{2\sigma}\|_{L_t^2L_x^2} \\
&\quad\lesssim 2^{2js(1-s)}[b_j\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|u\|_{\mathcal{X}_T^{2s}} + b_j\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|f\|_{L_t^\infty L_x^2}]^s\|\tilde{S}_{2j}P_j|P_{<j}u|^{2\sigma}\|_{L_t^2L_x^2}^{1-s} \\
&\quad\lesssim [b_j\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|u\|_{\mathcal{X}_T^{2s}} + b_j\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|f\|_{L_t^\infty L_x^2}]^s\|D_t^s\tilde{S}_{2j}P_j|P_{<j}u|^{2\sigma}\|_{L_t^2L_x^2}^{1-s}.
\end{aligned} \tag{2.5.84}$$

Hence,

$$\|D_t^s\tilde{S}_{2j}P_j|P_{<j}u|^{2\sigma}\|_{L_t^2L_x^2} \lesssim b_j\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|u\|_{\mathcal{X}_T^{2s}} + b_j\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|f\|_{L_t^\infty L_x^2}. \tag{2.5.85}$$

For the second term in (2.5.78), using that $2^{2js}\|P_{\geq j}u\|_{L_t^2L_x^2} \lesssim \|D_x^{2s}P_{\geq j}u\|_{L_t^2L_x^2}$ and Corollary 2.2.11 leads to the estimate,

$$\begin{aligned}
&\|P_j\tilde{S}_{2j}D_t^s(F(y(\theta))P_{\geq j}u)\|_{L_t^2L_x^2} \\
&\lesssim 2^{2js}\|P_j(F(y(\theta))P_{\geq j}u)\|_{L_t^2L_x^2} \\
&\lesssim b_j\|\langle D_x\rangle^\delta F(y(\theta))\|_{L_t^\infty L_x^\infty}\|u\|_{\mathcal{X}_T^{2s}} \\
&\lesssim b_j\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|u\|_{\mathcal{X}_T^{2s}}.
\end{aligned} \tag{2.5.86}$$

Hence, by Minkowski's inequality,

$$2\sigma\|\mathrm{Re}\int_0^1 P_j\tilde{S}_{2j}D_t^s(F(y(\theta))P_{\geq j}u)d\theta\|_{L_t^2L_x^2} \lesssim b_j\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|u\|_{\mathcal{X}_T^{2s}}. \tag{2.5.87}$$

Combining everything shows that

$$\|D_t^s\tilde{S}_{2j}P_j|u|^{2\sigma}\|_{L_t^2L_x^2} \lesssim b_j\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|u\|_{\mathcal{X}_T^{2s}} + b_j\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|f\|_{L_t^\infty L_x^2}. \tag{2.5.88}$$

This proves part a).

Next, we prove part b). By commuting through the temporal projection, we obtain

$$\begin{aligned}
\|\tilde{S}_{2j}P_jD_t^s(|u|^{2\sigma})\|_{L_t^2L_x^2} &\lesssim \|\tilde{S}_{2j}D_t^{s-1}(\tilde{S}_{<2j}(|u|^{2\sigma-2}\overline{u})\partial_t\tilde{S}_{2j}u)\|_{L_t^2L_x^2} \\
&\quad+ \|\tilde{S}_{2j}D_t^{s-1}(\tilde{S}_{\geq 2j}(|u|^{2\sigma-2}\overline{u})\partial_tu)\|_{L_t^2L_x^2}.
\end{aligned} \tag{2.5.89}$$

The first term in (2.5.89) can be estimated by Bernstein's inequality to obtain

$$\|\tilde{S}_{2j}D_t^{s-1}(\tilde{S}_{<2j}(|u|^{2\sigma-2}\overline{u})\partial_t\tilde{S}_{2j}u)\|_{L_t^2L_x^2} \lesssim \|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|D_t^{s-1}\partial_t\tilde{S}_{2j}u\|_{L_t^2L_x^2}. \tag{2.5.90}$$

Then writing

$$\|D_t^{s-1}\partial_t\tilde{S}_{2j}u\|_{L_t^2L_x^2} \sim \|D_t^{s-1}\partial_t\tilde{S}_{2j}P_{\leq 0}u\|_{L_t^2L_x^2} + \left(\sum_{k>0}\|D_t^{s-1}\partial_t\tilde{S}_{2j}P_k u\|_{L_t^2L_x^2}^2\right)^{\frac{1}{2}} \tag{2.5.91}$$

and requiring $b_j \geq 2^{-j\delta}$, applying Lemma 2.5.4 and then square summing over k yields

$$\begin{aligned}\|D_t^{s-1}\partial_t\tilde{S}_{2j}u\|_{L_t^2L_x^2} &\lesssim \|D_x^{2s}\tilde{S}_{2j}P_j u\|_{L_t^2L_x^2} + 2^{-j\delta}\|\langle D_t\rangle^{s-1+\delta}\tilde{S}_{2j}f\|_{L_t^2L_x^2} \\ &\lesssim b_j\|u\|_{\mathcal{X}_T^{2s}} + b_j\|f\|_{Z_\infty^{s-1+\delta}}.\end{aligned} \tag{2.5.92}$$

To estimate the second term in (2.5.89), we have two cases:

If $1 \leq s \leq \sigma + \frac{1}{2}$, we obtain from the equation,

$$\begin{aligned}&\|\tilde{S}_{2j}D_t^{s-1}(\tilde{S}_{\geq 2j}(|u|^{2\sigma-2}\overline{u})\partial_t u)\|_{L_t^2L_x^2} \\ &\lesssim \|\tilde{S}_{2j}D_t^{s-1}(\tilde{S}_{\geq 2j}(|u|^{2\sigma-2}\overline{u})\partial_x^2 u)\|_{L_t^2L_x^2} + \|\tilde{S}_{2j}D_t^{s-1}(\tilde{S}_{\geq 2j}(|u|^{2\sigma-2}\overline{u})f)\|_{L_t^2L_x^2}.\end{aligned} \tag{2.5.93}$$

By Hölder and Bernstein's inequality, Sobolev embedding and Corollary 2.2.11, the first term can be estimated by

$$\begin{aligned}\|\tilde{S}_{2j}D_t^{s-1}(\tilde{S}_{\geq 2j}(|u|^{2\sigma-2}\overline{u})\partial_x^2 u)\|_{L_t^2L_x^2} &\lesssim 2^{-j\delta}\|D_t^{s-1+\frac{\delta}{2}}(|u|^{2\sigma-2}\overline{u})\|_{L_t^{\frac{1}{s-1}}L_x^{\frac{1}{s-1}}}\|\partial_x^2 u\|_{L_t^\infty L_x^{\frac{2}{3-2s}}} \\ &\lesssim 2^{-j\delta}\|\langle D_t\rangle^{\frac{s-1+\delta}{2\sigma-1}}u\|_{L_t^{\frac{2\sigma-1}{s-1}}L_x^{\frac{2\sigma-1}{s-1}}}^{2\sigma-1}\|\partial_x^2 u\|_{L_t^\infty L_x^{\frac{2}{3-2s}}} \\ &\lesssim 2^{-j\delta}\|\langle D_t\rangle^{\frac{s-1+\delta}{2\sigma-1}}u\|_{L_t^{\frac{2\sigma-1}{s-1}}L_x^{\frac{2\sigma-1}{s-1}}}^{2\sigma-1}\|u\|_{L_t^\infty H_x^{s+1}}.\end{aligned} \tag{2.5.94}$$

Applying Corollary 2.5.5 gives

$$\|\langle D_t\rangle^{\frac{s-1+\delta}{2\sigma-1}}u\|_{L_t^{\frac{2\sigma-1}{s-1}}L_x^{\frac{2\sigma-1}{s-1}}}^{2\sigma-1} \lesssim \|\langle D_x\rangle^{\frac{2s-2+4\delta}{2\sigma-1}}u\|_{L_t^{\frac{2\sigma-1}{s-1}}L_x^{\frac{2\sigma-1}{s-1}}}^{2\sigma-1} + \|\langle D_t\rangle^{\frac{s-1+2\delta}{2\sigma-1}-1}f\|_{L_t^{\frac{2\sigma-1}{s-1}}L_x^{\frac{2\sigma-1}{s-1}}}^{2\sigma-1}. \tag{2.5.95}$$

Since $\frac{2s-2}{2\sigma-1} \leq 1 - (\frac{1}{2} - \frac{s-1}{2\sigma-1})$, when $s \leq \sigma + \frac{1}{2}$, we have by Sobolev embedding in the spatial variable,

$$\|\langle D_x\rangle^{\frac{2s-2+4\delta}{2\sigma-1}}u\|_{L_t^{\frac{2\sigma-1}{s-1}}L_x^{\frac{2\sigma-1}{s-1}}}^{2\sigma-1}\|u\|_{L_t^\infty H_x^{s+1}} \lesssim \|u\|_{\mathcal{S}_T^{1+c\delta}}^{2\sigma-1}\|u\|_{L_t^\infty H_x^{2s}} \tag{2.5.96}$$

for some fixed constant $c > 0$.

Next, applying Sobolev embedding in the time variable, and using the inequality $\|g\|_{L^p_x L^q_t} \lesssim \|g\|_{L^q_t L^p_x}$ when $p \geq q$, we also obtain

$$\begin{aligned} \|\langle D_t\rangle^{\frac{s-1+2\delta}{2\sigma-1}-1} f\|_{L_t^{\frac{2\sigma-1}{s-1}} L_x^{\frac{2\sigma-1}{s-1}}} &\lesssim \|f\|_{L_x^{\frac{2\sigma-1}{s-1}} L_t^2} \lesssim \|f\|_{L_t^2 L_x^{\frac{2\sigma-1}{s-1}}} \\ &\lesssim \|f\|_{\mathcal{S}^0_T} \end{aligned} \tag{2.5.97}$$

and so, the first term in (2.5.93) can be controlled by (after possibly relabelling δ),

$$2^{-j\delta}(\|u\|_{\mathcal{S}_T^{1+c\delta}} + \|f\|_{\mathcal{S}^0_T})^{2\sigma-1}\|u\|_{L_t^\infty H_x^{2s}} \lesssim b_j \Lambda \|u\|_{L_t^\infty H_x^{2s}} \lesssim b_j\Lambda\|u\|_{\mathcal{X}_T^{2s}}. \tag{2.5.98}$$

For the second term in (2.5.93), we simply have by Bernstein, and Corollary 2.2.11 and Corollary 2.5.5,

$$\begin{aligned} \|\tilde{S}_{2j} D_t^{s-1}(\tilde{S}_{\geq 2j}(|u|^{2\sigma-2}\overline{u})f)\|_{L^2_t L^2_x} &\lesssim 2^{-j\delta}\|D_t^{s-1+\frac{\delta}{2}}(|u|^{2\sigma-2}\overline{u})\|_{L_t^{\frac{4}{2\sigma-1}} L_x^{\frac{4}{2\sigma-1}}}\|f\|_{L_t^4 L_x^{\frac{4}{3-2\sigma}}} \\ &\lesssim 2^{-j\delta}\|\langle D_t\rangle^{\frac{1}{2}+\frac{\delta}{2}} u\|^{2\sigma-1}_{L^4_t L^4_x}\|f\|_{\mathcal{S}^0_T} \\ &\lesssim 2^{-j\delta}(\|u\|^{2\sigma-1}_{\mathcal{S}_T^{1+\delta}} + \|f\|^{2\sigma-1}_{\mathcal{S}^0_T})\|f\|_{\mathcal{S}^0_T} \\ &\lesssim b_j\Lambda\|f\|_{\mathcal{S}^0_T}. \end{aligned} \tag{2.5.99}$$

This handles the case $1 \leq s \leq \sigma + \frac{1}{2}$.

Next, suppose $\sigma + \frac{1}{2} < s < 2\sigma$. By Bernstein's inequality,

$$\begin{aligned} \|\tilde{S}_{2j} D_t^{s-1}(\tilde{S}_{\geq 2j}(|u|^{2\sigma-2}\overline{u})\partial_t u)\|_{L^2_t L^2_x} &\lesssim 2^{-j\delta}\|D_t^{s-1+\frac{\delta}{2}}(|u|^{2\sigma-2}\overline{u})\|_{L_t^{\frac{2}{2\sigma-1}} L_x^{\frac{2}{2\sigma-1}}}\|\partial_t u\|_{L_t^{\frac{1}{1-\sigma}} L_x^{\frac{1}{1-\sigma}}} \\ &\lesssim b_j\|D_t^{s-1+\frac{\delta}{2}}(|u|^{2\sigma-2}\overline{u})\|_{L_t^{\frac{2}{2\sigma-1}} L_x^{\frac{2}{2\sigma-1}}}\|\partial_t u\|_{L_t^{\frac{1}{1-\sigma}} L_x^{\frac{1}{1-\sigma}}}. \end{aligned} \tag{2.5.100}$$

Using Corollary 2.2.11 and then Corollary 2.5.5, we estimate,

$$\begin{aligned} \|D_t^{s-1+\frac{\delta}{2}}(|u|^{2\sigma-2}\overline{u})\|_{L_t^{\frac{2}{2\sigma-1}} L_x^{\frac{2}{2\sigma-1}}} &\lesssim \|\langle D_t\rangle^{\frac{s-1+\delta}{2\sigma-1}} u\|^{2\sigma-1}_{L^2_t L^2_x} \\ &\lesssim \|P_{\leq 0}\langle D_t\rangle^{\frac{s-1+\delta}{2\sigma-1}} u\|^{2\sigma-1}_{L^2_t L^2_x} + \left(\sum_{j>0}\|\langle D_t\rangle^{\frac{s-1+\delta}{2\sigma-1}} P_j u\|^2_{L^2_t L^2_x}\right)^{\frac{1}{2}(2\sigma-1)} \\ &\lesssim \|\langle D_x\rangle^{\frac{2s-2+2\delta}{2\sigma-1}} u\|^{2\sigma-1}_{L^2_t L^2_x} + \|f\|^{2\sigma-1}_{\mathcal{S}^0_T}. \end{aligned} \tag{2.5.101}$$

Furthermore, we have by Sobolev embedding and the equation,

$$\|\partial_t u\|_{L_t^{\frac{1}{1-\sigma}} L_x^{\frac{1}{1-\sigma}}} \lesssim \|\langle D_x\rangle^{\sigma+\frac{3}{2}} u\|_{L_t^\infty L_x^2} + \|\langle D_x\rangle^{\sigma-\frac{1}{2}} f\|_{L_t^\infty L_x^2}. \tag{2.5.102}$$

Hence, we obtain

$$\begin{aligned} &b_j \|D_t^{s-1+\frac{\delta}{2}}(|u|^{2\sigma-2}\overline{u})\|_{L_t^{\frac{2}{2\sigma-1}} L_x^{\frac{2}{2\sigma-1}}} \|\partial_t u\|_{L_t^{\frac{1}{1-\sigma}} L_x^{\frac{1}{1-\sigma}}} \\ &\lesssim b_j(\|\langle D_x\rangle^{\sigma+\frac{3}{2}} u\|_{L_t^\infty L_x^2} + \|\langle D_x\rangle^{\sigma-\frac{1}{2}} f\|_{L_t^\infty L_x^2}) \|\langle D_x\rangle^{\frac{2s-2+2\delta}{2\sigma-1}} u\|_{L_t^2 L_x^2}^{2\sigma-1} \\ &+ \Lambda b_j(\|u\|_{L_t^\infty H_x^{2s}} + \|f\|_{Z_\infty^{s-1+\delta}}). \end{aligned} \tag{2.5.103}$$

To control the first term, interpolating each factor between $L_t^\infty H_x^{2s}$ and $L_t^\infty H_x^1$ shows that

$$\|\langle D_x\rangle^{\frac{2s-2+2\delta}{2\sigma-1}} u\|_{L_t^2 L_x^2}^{2\sigma-1} \|\langle D_x\rangle^{\sigma+\frac{3}{2}} u\|_{L_t^\infty L_x^2} \lesssim \|u\|_{\mathcal{S}_T^{1+\delta}}^{2\sigma-1} \|u\|_{L_t^\infty H_x^{2s}}. \tag{2.5.104}$$

For the second term, interpolating the $\langle D_x\rangle^{\frac{2s-2+2\delta}{2\sigma-1}} u$ factor between $L_t^\infty H_x^1$ and $L_t^\infty H_x^{2s}$ and the $\langle D_x\rangle^{\sigma-\frac{1}{2}} f$ factor between $L_t^\infty L_x^2$ and $L_t^\infty H_x^{2s-2+\delta}$ and using that $s > \sigma + \frac{1}{2}$ leads to

$$\|\langle D_x\rangle^{\frac{2s-2+2\delta}{2\sigma-1}} u\|_{L_t^2 L_x^2}^{2\sigma-1} \|\langle D_x\rangle^{\sigma-\frac{1}{2}} f\|_{L_t^\infty L_x^2} \lesssim \Lambda(\|u\|_{L_t^\infty H_x^{2s}} + \|f\|_{Z_\infty^{s-1+\delta}}). \tag{2.5.105}$$

Now, collecting all of the estimates and using that $\|u\|_{L_t^\infty H_x^{2s}} \lesssim \|u\|_{\mathcal{X}_T^{2s}}$ completes the proof. □

Finally, we use Lemma 2.5.12 to establish Lemma 2.5.9.

Proof. First, it is straightforward to verify that b_j^2 is a $\mathcal{X}_T^{2s}$ frequency envelope for $\tilde{v}$ in the sense that b_j^2 satisfies property (2.2.3) and is slowly varying. Next, we expand

$$(i\partial_t + \partial_x^2)\tilde{v} = i\partial_t \tilde{\eta} v + \tilde{\eta}(i\partial_t + \partial_x^2)v := f. \tag{2.5.106}$$

Using an argument similar to what was done to estimate (2.5.61) and applying Corollary 2.5.5, it is straightforward to verify $\|f\|_{\mathcal{S}_T^{\frac{\delta}{2}}} + \|f\|_{Z_\infty^{s-1+\delta}} \lesssim \varepsilon + \|v\|_{\mathcal{X}_T^{2s}}$, and so the conclusion immediately follows from Lemma 2.5.12. □

2.6 Well-posedness at high regularity

In this section, we aim to prove Theorem 2.1.1. We begin by studying a suitable regularized equation.

Well-posedness of a regularized equation

Since there is an apparent limit to the possible regularity of solutions to (gDNLS), we construct H^{2s}_x solutions as limits of smooth solutions to an appropriate regularized approximate equation. Like in the previous section η will denote a time-dependent cutoff with $\eta = 1$ on $[-1, 1]$ with support in $(-2, 2)$. To construct the requisite solutions, we need the following lemma:

Lemma 2.6.1. Let $2 - \sigma < 2s < 4\sigma$. Let $2s \geq \alpha > \max\{2 - \sigma, 2s - 1\}$. Then there is an $\varepsilon > 0$ such that for every $u_0 \in H^{2s}_x$ with $\|u_0\|_{H^\alpha_x} \leq \varepsilon$ and for all $j > 0$, the regularized equation

$$\begin{cases} (i\partial_t + \partial_x^2)u = i\eta P_{<j}|u|^{2\sigma}\partial_x u, \\ u(0) = P_{<j}u_0, \end{cases} \tag{2.6.1}$$

admits a global solution $u \in C^2(\mathbb{R}; H^\infty_x)$. Moreover, we have the following bounds for $T = 2$,

$$\begin{aligned} &\|u\|_{\mathcal{X}^\alpha_T \cap \mathcal{S}^{1+\delta}_T} \lesssim \varepsilon, \\ &\|(i\partial_t + \partial_x^2)u\|_{\mathcal{S}^\delta_T \cap Z^{s-1+\delta}_\infty} \lesssim \varepsilon, \end{aligned} \tag{2.6.2}$$

where the implicit constant in the above inequality is independent of the parameter j and where $0 < \delta \ll 1$ is any small positive constant.

Remark 2.6.2. The smallness assumption on the H^α_x norm of u_0 will turn out to be inconsequential (by L^2_x subcriticality for (gDNLS)). This assumption is made for convenience to guarantee (2.6.2).

Let us now construct solutions to (2.6.1). The first step is to construct solutions to an appropriate linear equation. For this, we have the following lemma.

Lemma 2.6.3. Let $\eta = \eta(t)$ be a smooth time-dependent cutoff with $\eta = 1$ on $[-1, 1]$ and with support in $(-2, 2)$. Let $T > 0$ and $v \in L^{2\sigma}_T L^\infty_x$. Let $u_0 \in H^{2s}_x$. Then for each $j > 0$, there exists a unique solution $w \in C([-T, T]; H^\infty_x)$ solving the equation

$$\begin{cases} \partial_t w = i\partial_x^2 w + \eta P_{<j}|v|^{2\sigma}\partial_x w, \\ w(0) = P_{<j}u_0. \end{cases} \tag{2.6.3}$$

Proof. First, observe that for each $n > j$ a simple (iterated) application of the contraction mapping theorem in the closed subspace of $C([-T, T]; L^2_x)$ consisting of functions whose spatial Fourier transform is supported on $[-2^n, 2^n]$ gives rise to a solution $w^{(n)} \in C([-T, T]; H^\infty_x)$

to the following regularized linear equation,

$$\begin{cases} \partial_t w^{(n)} = i\partial_x^2 w^{(n)} + \eta P_{\leq n}(P_{<j}|v|^{2\sigma}\partial_x w^{(n)}), \\ w^{(n)}(0) = P_{<j}u_0. \end{cases} \tag{2.6.4}$$

We show that the sequence $w^{(n)}$ converges as $n \to \infty$ to some $w \in C([-T,T];H_x^\infty)$ which solves (2.6.3). This follows in two stages, but is standard. First, for each integer $k \geq 0$, a standard energy estimate and Bernstein's inequality shows that $w^{(n)}$ satisfies the bound

$$\|w^{(n)}\|_{C([-T,T];H_x^k)} \lesssim \exp(2^{j(k+1)}\|v\|_{L_T^{2\sigma}L_x^\infty}^{2\sigma})\|P_{<j}u_0\|_{H_x^k} \tag{2.6.5}$$

where importantly, the bound is independent of n (but can depend on j). Furthermore, a simple energy estimate in L_x^2 for differences of solutions $w^{(n)} - w^{(m)}$ to (2.6.4) shows that the sequence $w^{(n)}$ is Cauchy in $C([-T,T];L_x^2)$ and thus converges to some $w \in C([-T,T];L_x^2)$. Interpolating against (2.6.5) shows that in fact $w^{(n)}$ converges to some w in $C([-T,T];H_x^\infty)$ and that w solves (2.6.3) in the sense of distributions, and furthermore that w satisfies the bound (2.6.5) for each $k \geq 0$. □

The next step in the proof of Lemma 2.6.1 is to construct the corresponding $C^2(\mathbb{R};H_x^\infty)$ solution to (2.6.1). For this purpose, consider the following iteration scheme,

$$\begin{cases} (i\partial_t + \partial_x^2)u^{(n+1)} = i\eta P_{<j}|u^{(n)}|^{2\sigma}\partial_x u^{(n+1)}, \\ u^{(n+1)}(0) = P_{<j}u_0, \end{cases} \tag{2.6.6}$$

with the initialization $u^{(0)} = 0$. Thanks to Lemma 2.6.3 it follows that for each n, there is a solution $u^{(n+1)} \in C([-2,2];H_x^\infty)$ to the above equation. In particular, $u^{(n+1)}$ can be extended globally in time because for $|t| > 2$, $u^{(n+1)}$ solves the linear Schrödinger equation.

Next, we have the following lemma concerning the convergence of this iteration scheme, from which Lemma 2.6.1 is immediate.

Lemma 2.6.4. Let $2-\sigma < 2s < 4\sigma$. Let $2s \geq \alpha > \max\{2-\sigma, 2s-1\}$. Let $u_0 \in H_x^{2s}$ and let $u^{(n+1)}$ be the corresponding $C(\mathbb{R};H_x^\infty)$ solution to (2.6.6). Then there is $\varepsilon > 0$ independent of j such that if $\|u_0\|_{H_x^\alpha} \leq \varepsilon$, then $u^{(n)}$ converges to some $u \in C(\mathbb{R};H_x^\infty)$ solving (2.6.1). Furthermore, we have $u \in C^2(\mathbb{R};H_x^\infty)$ and the bounds

$$\begin{aligned} &\|u\|_{\mathcal{X}_T^\alpha \cap \mathcal{S}_T^{1+\delta}} \lesssim \varepsilon, \\ &\|(i\partial_t + \partial_x^2)u\|_{\mathcal{S}_T^\delta \cap Z_\infty^{s-1+\delta}} \lesssim \varepsilon. \end{aligned} \tag{2.6.7}$$

Proof. We begin by showing that $u^{(n+1)}$ satisfies the bounds

$$\begin{aligned} &\|u^{(n+1)}\|_{\mathcal{X}_T^\alpha \cap \mathcal{S}_T^{1+\delta}} \lesssim \varepsilon, \\ &\|(i\partial_t + \partial_x^2)u^{(n+1)}\|_{\mathcal{S}_T^\delta \cap Z_\infty^{s-1+\delta}} \lesssim \varepsilon, \end{aligned} \tag{2.6.8}$$

for $T = 2$ uniformly in n. Given the initialization $u^{(0)} = 0$, we may make the inductive hypothesis that (2.6.8) holds with $n+1$ replaced by n. Now, we prove the above two bounds for $u^{(n+1)}$.

We begin by showing $\|u^{(n+1)}\|_{\mathcal{X}_T^\alpha \cap \mathcal{S}_T^{1+\delta}} \lesssim \varepsilon$. Indeed, it follows from the modification of the low regularity bounds outlined in Lemma 2.3.18 that for $2s \geq \alpha > 2 - \sigma$,

$$\|u^{(n+1)}\|_{\mathcal{X}_T^\alpha \cap \mathcal{S}_T^{1+\delta}} \lesssim \|u^{(n+1)}\|_{\mathcal{X}_T^\alpha}. \tag{2.6.9}$$

Then Proposition 2.5.1 and the inductive hypothesis gives

$$\begin{aligned} \|u^{(n+1)}\|_{\mathcal{X}_T^\alpha}^2 \lesssim \|u_0\|_{H_x^\alpha}^2 &+ \varepsilon^{2\sigma}(\|u^{(n+1)}\|_{\mathcal{X}_T^\alpha}^2 + \|u^{(n+1)}\|_{\mathcal{S}_T^1}^2) + \varepsilon^{2\sigma-1}\|u^{(n+1)}\|_{\mathcal{S}_T^1}\|u^{(n+1)}\|_{\mathcal{X}_T^\alpha}\|u^{(n)}\|_{\mathcal{X}_T^\alpha} \\ &+ \varepsilon^{4\sigma-2}\|u^{(n+1)}\|_{\mathcal{S}_T^1}^2\|u^{(n)}\|_{\mathcal{X}_T^\alpha}^2, \end{aligned} \tag{2.6.10}$$

and so,

$$\|u^{(n+1)}\|_{\mathcal{X}_T^\alpha}^2 \lesssim \|u_0\|_{H_x^\alpha}^2 + \varepsilon^{2\sigma}\|u^{(n+1)}\|_{\mathcal{X}_T^\alpha}^2. \tag{2.6.11}$$

From this, we deduce

$$\|u^{(n+1)}\|_{\mathcal{X}_T^\alpha} \lesssim \varepsilon. \tag{2.6.12}$$

Next, we aim to verify the bound,

$$\|(i\partial_t + \partial_x^2)u^{(n+1)}\|_{\mathcal{S}_T^\delta \cap Z_\infty^{s-1+\delta}} \lesssim \varepsilon. \tag{2.6.13}$$

For this, we use the equation,

$$(i\partial_t + \partial_x^2)u^{(n+1)} = i\eta P_{<j}|u^{(n)}|^{2\sigma}\partial_x u^{(n+1)}. \tag{2.6.14}$$

From Lemma 2.5.10 and (2.6.9), we have

$$\|i\eta P_{<j}|u^{(n)}|^{2\sigma}\partial_x u^{(n+1)}\|_{\mathcal{S}_T^\delta \cap Z_\infty^{s-1+\delta}} \lesssim \varepsilon^{2\sigma}\|u^{(n+1)}\|_{\mathcal{X}_T^\alpha} \lesssim \varepsilon. \tag{2.6.15}$$

This verifies the uniform in n bound (2.6.8).

Next, we show that that $u^{(n)}$ converges to $u \in C(\mathbb{R}; L^2_x)$. Clearly it suffices to show (by the localization properties of η) that $u^{(n)}$ converges to $u \in C([-2,2]; L^2_x)$.

We begin by estimating the L^2_x norm of $u^{(n+1)}(t) - u^{(n)}(t)$ for $|t| \le 2$. Indeed, we see that $u^{(n+1)} - u^{(n)}$ satisfies the equation,

$$(i\partial_t + \partial_x^2)(u^{(n+1)} - u^{(n)}) = i\eta P_{<j}|u^{(n)}|^{2\sigma}\partial_x(u^{(n+1)} - u^{(n)}) + i\eta P_{<j}(|u^{(n)}|^{2\sigma} - |u^{(n-1)}|^{2\sigma})\partial_x u^{(n)}, \tag{2.6.16}$$

with $(u^{(n+1)} - u^{(n)})(0) = 0$. A simple energy estimate shows that for each $-2 \le T \le 2$

$$\begin{aligned} &\|u^{(n+1)} - u^{(n)}\|^2_{L^\infty_T L^2_x} \lesssim \|u^{(n)}\|_{S^1_T}\|u^{(n)}\|^{2\sigma-1}_{L^\infty_T L^\infty_x}\|u^{(n+1)} - u^{(n)}\|^2_{L^\infty_T L^2_x} \\ &\quad + \|u^{(n)}\|_{S^1_T}(\|u^{(n)}\|^{2\sigma-1}_{L^\infty_T L^\infty_x} + \|u^{(n-1)}\|^{2\sigma-1}_{L^\infty_T L^\infty_x})\|u^{(n)} - u^{(n-1)}\|_{L^\infty_T L^2_x}\|u^{(n+1)} - u^{(n)}\|_{L^\infty_T L^2_x} \end{aligned} \tag{2.6.17}$$

where all the implicit constants are independent of j. Using (2.6.8) and Cauchy Schwarz, we obtain

$$\|u^{(n+1)} - u^{(n)}\|^2_{L^\infty_T L^2_x} \le \frac{1}{4}\|u^{(n+1)} - u^{(n)}\|^2_{L^\infty_T L^2_x} + \frac{1}{4}\|u^{(n)} - u^{(n-1)}\|^2_{L^\infty_T L^2_x}. \tag{2.6.18}$$

From this, one obtains

$$\|u^{(n+1)} - u^{(n)}\|^2_{L^\infty_T L^2_x} \le \frac{1}{2}\|u^{(n)} - u^{(n-1)}\|^2_{L^\infty_T L^2_x}. \tag{2.6.19}$$

Hence, we see that $u^{(n)}$ converges to u in $C([-2,2]; L^2_x)$. By a simple energy estimate, and Bernstein's inequality, it is straightforward to verify that for each integer $k \ge 0$, we have the uniform (in n) bound

$$\|u^{(n+1)}\|_{C([-2,2];H^k_x)} \lesssim \exp(2^{j(k+1)}\|u^{(n)}\|^{2\sigma}_{L^{2\sigma}_T L^\infty_x})\|P_{<j}u_0\|_{H^k_x} \lesssim_j \|u_0\|_{H^{2s}_x}. \tag{2.6.20}$$

Hence, by interpolating against (2.6.20), we see that $u^{(n)}$ converges to u in $C([-2,2]; H^\infty_x)$. By differentiating the equation in time, we find $u \in C^2([-2,2]; H^\infty_x)$.

It remains to show (2.6.7). Since $u^{(n)} \to u$ in $C([-2,2]; H^\infty_x)$, the $\mathcal{X}^\alpha_T \cap \mathcal{S}^{1+\delta}_T$ bound follows immediately from (2.6.8). For the remaining estimate, we may clearly control

$$(i\partial_t + \partial_x^2)u = i\eta P_{<j}|u|^{2\sigma}\partial_x u \tag{2.6.21}$$

in $\mathcal{S}^\delta_T \cap Z^{s-1+\delta}_\infty$ by (after possibly slightly enlarging δ)

$$\|u\|^{2\sigma}_{\mathcal{X}^\alpha_T \cap \mathcal{S}^{1+\delta}_T} + \|i\eta P_{<j}|u|^{2\sigma}\partial_x u\|_{Z^{s-1+\delta}_\infty} \lesssim \varepsilon + \|iP_{<j}\eta|u|^{2\sigma}\partial_x u\|_{Z^{s-1+\delta}_\infty}. \tag{2.6.22}$$

From Lemma 2.5.10, we have

$$\|i\eta P_{<j}|u|^{2\sigma}\partial_x u\|_{Z_\infty^{s-\frac{3}{2}+\delta}} \lesssim \varepsilon. \tag{2.6.23}$$

Then applying Lemma 2.5.10 again, using (2.6.23) then gives

$$\|i\eta P_{<j}|u|^{2\sigma}\partial_x u\|_{Z_\infty^{s-1+\delta}} \lesssim \varepsilon. \tag{2.6.24}$$

This completes the proof. □

Remark 2.6.5. Note that at this point, we haven't said anything about the behavior of (2.6.1) as $j \to \infty$. For this, we will again need the uniform bounds from Proposition 2.5.1.

Well-posedness for the full equation

In this section, we prove the local well-posedness of (gDNLS) in H_x^{2s} for $2-\sigma < 2s < 4\sigma$.

Indeed, let $u_0 \in H_x^{2s}$ and let $2-\sigma < \alpha \le 2s$. By rescaling (recalling the problem is L_x^2 subcritical), we may assume without loss of generality that $\|u_0\|_{H_x^\alpha} \le \varepsilon$ for some $\varepsilon > 0$ sufficiently small, and construct the corresponding H_x^{2s} solution on the time interval $[-1,1]$. For $2-\sigma < 2s \le \frac{3}{2}$, we construct the solution in the Strichartz type space $\mathcal{X}_T^{2s} \cap \mathcal{S}_T^{1+\delta}$, where $0 < \delta \ll 1$ is any sufficiently small positive constant. When $s > \frac{3}{2}$, the extra $\mathcal{S}_T^{1+\delta}$ component is, of course, redundant, thanks to Sobolev embedding.

We will realize H_x^{2s} well-posed solutions as (restrictions to the interval $[-1,1]$ of) limits of smooth solutions to the regularized equation (2.6.1). To establish this, we have the following lemma.

Lemma 2.6.6. Let $2-\sigma < 2s < 4\sigma$. Let $2s \ge \alpha > \max\{2-\sigma, 2s-1\}$. Then there is an $\varepsilon > 0$ such that for every $u_0 \in H_x^{2s}$ with $\|u_0\|_{H_x^\alpha} \le \varepsilon$, the time-truncated equation,

$$\begin{cases} (i\partial_t + \partial_x^2)u = i\eta|u|^{2\sigma}\partial_x u, \\ u(0) = u_0, \end{cases} \tag{2.6.25}$$

admits a global solution $u \in C^2(\mathbb{R}; H_x^\infty)$. Moreover, we have the following bounds for $T=2$,

$$\begin{aligned} &\|u\|_{\mathcal{X}_T^\alpha \cap \mathcal{S}_T^{1+\delta}} \lesssim \varepsilon, \\ &\|(i\partial_t + \partial_x^2)u\|_{\mathcal{S}_T^\delta \cap Z_\infty^{s-1+\delta}} \lesssim \varepsilon, \end{aligned} \tag{2.6.26}$$

and also
$$\|u\|_{\mathcal{X}_T^{2s}}^2 \lesssim \frac{1}{1-C\varepsilon^{2\sigma}}\|u_0\|_{H_x^{2s}}^2, \tag{2.6.27}$$
where $C > 0$ is some universal constant.

Proof. If ε is small enough, thanks to Lemma 2.6.1, for each $j > 0$, there is a smooth solution $u^{(j)} \in C^2(\mathbb{R}; H_x^\infty)$ to the equation,
$$\begin{cases} (i\partial_t + \partial_x^2)u^{(j)} = i\eta P_{<j}|u^{(j)}|^{2\sigma}u_x^{(j)}, \\ u^{(j)}(0) = P_{<j}u_0, \end{cases} \tag{2.6.28}$$
satisfying
$$\|u^{(j)}\|_{\mathcal{X}_T^\alpha \cap \mathcal{S}_T^{1+\delta}} + \|(i\partial_t + \partial_x^2)u^{(j)}\|_{\mathcal{S}_T^\delta \cap Z_\infty^{s-1+\delta}} \lesssim \varepsilon \tag{2.6.29}$$
uniformly in j. Now, define for $k > j$, $v^{(k,j)} := u^{(k)} - u^{(j)}$. Then $v^{(k,j)}$ satisfies the equation,
$$(i\partial_t+\partial_x^2)v^{(k,j)} = i\eta P_{<k}|u^{(k)}|^{2\sigma}\partial_x v^{(k,j)} + i\eta P_{<k}(|u^{(k)}|^{2\sigma} - |u^{(j)}|^{2\sigma})\partial_x u^{(j)} + i\eta P_{j\leq\cdot<k}|u^{(j)}|^{2\sigma}\partial_x u^{(j)}, \tag{2.6.30}$$
with $v^{(k,j)}(0) = P_{j\leq\cdot<k}u_0$. Multiplying by $-i\overline{v^{(k,j)}}$ taking real part and integrating over $\mathbb{R}$ and from 0 to t with $|t| \leq T$ leads to the simple energy estimate
$$\begin{aligned}\|v^{(k,j)}\|_{L_T^\infty L_x^2}^2 \lesssim{}& \|P_{j\leq\cdot<k}u_0\|_{L_x^2}^2 + (\|u^{(j)}\|_{\mathcal{S}_T^1}^{2\sigma-1} + \|u^{(k)}\|_{\mathcal{S}_T^1}^{2\sigma-1})\|u^{(j)}\|_{\mathcal{S}_T^1}\|v^{(k,j)}\|_{L_T^\infty L_x^2}^2 \\ &+ \|u^{(k)}\|_{\mathcal{S}_T^1}^{2\sigma}\|v^{(k,j)}\|_{L_T^\infty L_x^2}^2 + \|P_{j\leq\cdot<k}|u^{(j)}|^{2\sigma}\|_{L_T^\infty L_x^2}\|u^{(j)}\|_{\mathcal{S}_T^1}\|v^{(k,j)}\|_{L_T^\infty L_x^2}.\end{aligned} \tag{2.6.31}$$
Using the uniform in j bound
$$\|u^{(j)}\|_{\mathcal{S}_T^{1+\delta}} \lesssim \varepsilon \tag{2.6.32}$$
from Lemma 2.6.1 and Cauchy Schwarz gives
$$\|v^{(k,j)}\|_{L_T^\infty L_x^2}^2 \lesssim \|P_{j\leq\cdot<k}u_0\|_{L_x^2}^2 + \|P_{j\leq\cdot<k}|u^{(j)}|^{2\sigma}\|_{L_T^\infty L_x^2}^2\|u^{(j)}\|_{\mathcal{S}_T^1}^2. \tag{2.6.33}$$
Furthermore,
$$\|P_{j\leq\cdot<k}|u^{(j)}|^{2\sigma}\|_{L_T^\infty L_x^2} \lesssim 2^{-j}\|u^{(j)}\|_{\mathcal{S}_T^1}^{2\sigma}. \tag{2.6.34}$$
Hence, the right hand side of (2.6.33) goes to zero as $j, k \to \infty$. Therefore, $u^{(j)}$ converges to some u in $C([-2,2]; L_x^2)$. On the other hand, thanks to the uniform (in k) bounds from the energy estimate Proposition 2.5.1, we obtain
$$\|P_j u^{(k)}\|_{\mathcal{X}_T^{2s}}^2 \lesssim a_j^2\|u_0\|_{H_x^{2s}}^2 + [b_j^{(k)}]^2\varepsilon^{2\sigma}\|u^{(k)}\|_{\mathcal{X}_T^{2s}}^2, \tag{2.6.35}$$

where $b_j^{(k)}$ is a $\mathcal{X}_T^{2s}$ frequency envelope for $u^{(k)}$. Using that $\|u^{(k)}\|_{S_T^{1+\delta}} \lesssim \varepsilon$, an argument similar to the low regularity well-posedness shows that for ε small enough, a_j is a $\mathcal{X}_T^{2s}$ frequency envelope for $u^{(k)}$. Analogously to the low regularity argument, this can be used to show that $u^{(k)} \to u$ in $\mathcal{X}_T^{2s}$ and that a_j is a $\mathcal{X}_T^{2s}$ frequency envelope for u and that u solves the time truncated equation,

$$\begin{cases} i\partial_t u + u_{xx} = i\eta|u|^{2\sigma}u_x, \\ u(0) = u_0, \end{cases} \tag{2.6.36}$$

in the sense of distributions. Moreover, by square summing over j and passing to the limit in (2.6.35), we obtain the uniform bound

$$\|u\|_{\mathcal{X}_T^{2s}}^2 \lesssim \frac{1}{1 - C\varepsilon^{2\sigma}}\|u_0\|_{H_x^{2s}}^2. \tag{2.6.37}$$

□

Next, we establish local well-posedness for the full equation (gDNLS).

For existence, we may rescale (using the L_x^2 subcriticality of the equation) to assume $u_0 \in H_x^{2s}$ has sufficiently small data. Then we may construct a $\mathcal{X}_T^{2s}$ solution to (gDNLS) on the time interval $[-1, 1]$ by applying Lemma 2.6.6 and restricting to $|t| \le 1$.

For uniqueness, we consider the difference of two H_x^{2s} solutions u_1, u_2 to (gDNLS) and obtain, by a standard energy estimate, the weak Lipschitz bound,

$$\|u_1 - u_2\|_{L_T^\infty L_x^2} \lesssim_{\|u_1\|_{\mathcal{S}_T^1}, \|u_2\|_{\mathcal{S}_T^1}} \|u_1(0) - u_2(0)\|_{L_x^2}. \tag{2.6.38}$$

for $T > 0$. Among other things, this shows uniqueness in $C([-1, 1]; H_x^{2s}) \cap \mathcal{S}_T^1$.

For continuous dependence, again assume without loss of generality that u_0 has sufficiently small H_x^{2s} norm. To show continuous dependence for the full equation (gDNLS), it clearly suffices (by restricting to $T \le 1$) to show that the data to solution map $u_0 \in H_x^{2s} \mapsto u \in \mathcal{X}_{T=2}^{2s} \cap \mathcal{S}_{T=2}^{1+\delta}$ for the time-truncated equation (2.6.36) is continuous. For this, let $u_0^n \in H_x^{2s}$ be a sequence of initial data converging to some u_0 in H_x^{2s}. Let u^n and u denote the corresponding $\mathcal{X}_{T=2}^{2s} \cap \mathcal{S}_{T=2}^{1+\delta}$ solutions to the time-truncated equation (2.6.36), respectively. From the frequency envelope bound (2.6.35) and an argument almost identical to the proof of continuous dependence at low regularity, it follows that

$$\lim_{n\to\infty} \|u^n - u\|_{\mathcal{X}_{T=2}^{2s} \cap \mathcal{S}_{T=2}^{1+\delta}} = 0. \tag{2.6.39}$$

We omit the details. This finally completes the proof of Theorem 2.1.1.

2.7 Global well-posedness

Here, we complete the proof of Theorem 2.1.2. That is, we show that for $\frac{\sqrt{3}}{2} < \sigma < 1$ and $1 \leq 2s < 4\sigma$, (gDNLS) is globally well-posed in H_x^{2s}. The proof of local well-posedness in H_x^{2s} for $1 \leq 2s \leq \frac{3}{2}$ and $\sigma > \frac{\sqrt{3}}{2}$ established in Section 2.4 relied on having global well-posedness when $\frac{3}{2} < 2s < 4\sigma$, so we establish this first. Ultimately, global well-posedness will follow from the conservation laws, which we use in the next lemma to establish uniform control of the H_x^1 norm of solutions:

Lemma 2.7.1. (H_x^1 norm remains bounded) Let $u_0 \in H_x^{2s}$, $1 \leq 2s < 4\sigma$ and $\frac{\sqrt{3}}{2} < \sigma < 1$. Let $T > 0$ be sufficiently small. If $2s \leq \frac{3}{2}$, suppose that there is a corresponding well-posed solution $u \in X_T^{2s}$ to (gDNLS). Likewise, if $4\sigma > 2s > \frac{3}{2}$, let $u \in \mathcal{X}_T^{2s}$ be the corresponding well-posed solution to (gDNLS). Then for $0 \leq |t| \leq T$, we have

$$\|u(t)\|_{H_x^1} \lesssim_{\|u_0\|_{H_x^1}} 1 \tag{2.7.1}$$

where the implied constant depends only on the size of $\|u_0\|_{H_x^1}$. In particular, the H_x^1 norm of u cannot blow up in finite time.

Remark 2.7.2. There is one small technical caveat to be aware of. Namely, in Lemma 2.7.1, it is assumed for $1 \leq 2s \leq \frac{3}{2}$ that the equation (gDNLS) is locally well-posed X_T^{2s}. As mentioned above, this will follow from the results proven in Section 2.4 once we have established global well-posedness in the range $\frac{3}{2} < 2s < 4\sigma$ (where we already have local well-posedness from Section 6).

Proof. Recall that we have the conserved mass and energy, respectively

$$M(u) := \frac{1}{2}\int_{\mathbb{R}} |u|^2 dx = M(u_0), \tag{2.7.2}$$

$$E(u) := \frac{1}{2}\int_{\mathbb{R}} |u_x|^2 dx + \frac{1}{2(1+\sigma)}\mathrm{Re}\int_{\mathbb{R}} i|u|^{2\sigma}\overline{u}u_x dx = E(u_0). \tag{2.7.3}$$

It is also straightforward to verify that any well-posed solution in $\mathcal{X}_T^{2s}$ (when $\frac{3}{2} < 2s < 4\sigma$) or X_T^{2s} (when $1 \leq 2s \leq \frac{3}{2}$) satisfies these conservation laws. By interpolation, we have the following lower bound for the energy (where C is some constant that may change from line to line)

$$\begin{aligned} E(u) &\geq \frac{1}{2}\|u_x\|_{L_x^2}^2 - C\|u\|_{L_x^{4\sigma+2}}^{2\sigma+1}\|u_x\|_{L_x^2} \\ &\geq \frac{1}{4}\|u_x\|_{L_x^2}^2 - C\|u\|_{L_x^2}^{\frac{1+\sigma}{1-\sigma}} \\ &\geq \frac{1}{4}\|u_x\|_{L_x^2}^2 - CM(u)^{\frac{1+\sigma}{2(1-\sigma)}}. \end{aligned} \tag{2.7.4}$$

Hence, for $0 \leq |t| \leq T$, we have

$$\|u(t)\|_{H_x^1}^2 \lesssim E(u_0) + M(u_0) + M(u_0)^{\frac{1+\sigma}{2(1-\sigma)}} \lesssim_{\|u_0\|_{H_x^1}} 1. \tag{2.7.5}$$

□

Corollary 2.7.3. Let $u_0 \in H_x^{2s}$, $0 < T^* < \infty$, $\frac{3}{2} < 2s < 4\sigma$ and $\frac{\sqrt{3}}{2} < \sigma < 1$. Suppose that for each $T < T^*$, there is a corresponding well-posed solution $u \in \mathcal{X}_T^{2s}$ with initial data u_0. Then for each $0 < \delta \ll 1$, we have

$$\limsup_{T \nearrow T^*} \|u\|_{\mathcal{S}_T^{1+\delta} \cap X_T^{2-\sigma+2\delta}} < \infty. \tag{2.7.6}$$

In particular, the $\mathcal{S}_T^{1+\delta} \cap X_T^{2-\sigma+2\delta}$ norm of a solution cannot blow up in finite time.

Proof. Lemma 2.7.1 shows that for all $0 < T < T^*$, the norm $\|u\|_{L_T^\infty H_x^1}$ is bounded by a constant depending on the initial data $\|u_0\|_{H_x^1}$. Therefore, iterating (after appropriately translating and rescaling the initial data) Proposition 2.3.6 shows that

$$\limsup_{T \nearrow T^*} \|u\|_{X_T^1} \lesssim_{\|u_0\|_{H_x^1}} 1. \tag{2.7.7}$$

By virtue of (2.7.7) and iterating Proposition 2.3.6, we find that

$$\limsup_{T \nearrow T^*} \|u\|_{X_T^{2-\sigma+2\delta}} < \infty. \tag{2.7.8}$$

It follows that

$$\limsup_{T \nearrow T^*} \|u\|_{\mathcal{S}_T^{1+\delta}} \leq \limsup_{T \nearrow T^*} \|u\|_{X_T^{2-\sigma+2\delta}} < \infty. \tag{2.7.9}$$

□

Next, we use Corollary 2.7.3 and Lemma 2.6.6 to establish global well-posedness in the high regularity regime $\frac{3}{2} < 2s < 4\sigma$. Indeed, for $u_0 \in H_x^{2s}$ let $T^* > 0$ be the maximal time for which there is a corresponding well-posed solution $u \in \mathcal{X}_T^{2s}$ for each $T < T^*$. If $T^* = \infty$, then we are done. We can therefore assume for the sake of contradiction that $T^* < \infty$. Then we have

$$\limsup_{T \nearrow T^*} \|u\|_{\mathcal{X}_T^{2s}} = \infty. \tag{2.7.10}$$

We show that this is impossible. By rescaling and translation, we may without loss of generality take $T^* = 1$.

We begin with the case $\frac{3}{2} < 2s < 2$. Set $\alpha = 2 - \sigma + 2\delta$ where δ is some small positive constant.

Let $0 < \varepsilon \ll 1$. Define now the rescaled solution $u_\lambda(t,x) = \lambda^{\frac{1}{2\sigma}} u(\lambda^2 t, \lambda x)$ to (gDNLS), where λ satisfies $k := \lambda^{-2} \in \mathbb{N}$ and where λ is small enough so that for each $T < \lambda^{-2}$,

$$\|u_\lambda\|_{L^\infty_{T<\lambda^{-2}} H^\alpha_x} \lesssim \lambda^{\frac{1}{2\sigma} - \frac{1}{2}} \|u\|_{L^\infty_{T<1} H^\alpha_x} \lesssim \varepsilon. \tag{2.7.11}$$

By assumption u_λ is a $\mathcal{X}_T^{2s}$ solution to (gDNLS) for $T < \lambda^{-2}$ with

$$\limsup_{T \nearrow \lambda^{-2}} \|u_\lambda\|_{\mathcal{X}_T^{2s}} = \infty. \tag{2.7.12}$$

Now, we iterate Lemma 2.6.6. We consider the initial value problem for each natural number $n < k$,

$$\begin{cases} (i\partial_t + \partial_x^2) w_n = i\eta |w_n|^{2\sigma} \partial_x w_n, \\ w_n(0) = u_\lambda(n). \end{cases} \tag{2.7.13}$$

By Lemma 2.6.6 by taking $\alpha = 2-\sigma+2\delta$, and (2.7.11) there is a global solution $w \in C(\mathbb{R}; H_x^{2s})$ to the above equation satisfying

$$\|w_n\|^2_{\mathcal{X}^{2s}_{T=2}} \lesssim \frac{1}{1 - C\varepsilon^{2\sigma}} \|u_\lambda(n)\|^2_{H_x^{2s}} \tag{2.7.14}$$

from which we deduce (by restricting w to times in $[-1, 1]$),

$$\|u_\lambda(n + \cdot)\|^2_{\mathcal{X}^{2s}_{T=1}} \lesssim \frac{1}{1 - C\varepsilon^{2\sigma}} \|u_\lambda(n)\|^2_{H_x^{2s}}. \tag{2.7.15}$$

Iterating this k times gives the bound

$$\|u_\lambda\|^2_{\mathcal{X}^{2s}_{T<\lambda^{-2}}} \lesssim \left(\frac{1}{1 - C\varepsilon^{2\sigma}} \right)^k \|u_\lambda(0)\|^2_{H_x^{2s}}. \tag{2.7.16}$$

This contradicts (2.7.12). Therefore $T^* = \infty$ and the $\mathcal{X}_T^{2s}$ norm cannot blow up in finite time when $\frac{3}{2} < 2s < 2$.

Next, we proceed with the case $2 \leq 2s < 4\sigma$. If $2 \leq 2s < 3$, then if we assume a maximal time of existence $T^* < \infty$ for a $\mathcal{X}_T^{2s}$ solution, then the previous case shows that for $\delta > 0$ sufficiently small,

$$\limsup_{T \nearrow T*} \|u\|_{\mathcal{X}_T^{2s-1+\delta}} < \infty. \tag{2.7.17}$$

Replacing α in the previous case with $\max\{2s - 1 + \delta, 2 - \sigma + 2\delta\}$ and repeating the proof verbatim shows once again that $T^* = \infty$. Iterating once more shows that in the case $3 \leq 2s < 4\sigma$, we also have the same conclusion. Thus, (gDNLS) is globally well-posed in H_x^{2s} when $\frac{3}{2} < 2s < 4\sigma$.

We finally turn to the last case. Namely, we show that (gDNLS) is globally well-posed when $1 \leq 2s \leq \frac{3}{2}$.

Indeed, at this point, we know from Section 4 and the previous two cases that we have a locally well-posed X_T^{2s} solution. Iterating the low regularity bounds Proposition 2.3.6 and using Lemma 2.7.1 shows that such a solution can be continued for all time. This finally completes the proof of Theorem 2.1.2.

Chapter 3

No pure capillary solitary water waves exist in 2D

3.1 Introduction

Solitary water waves are localized disturbances of a fluid surface which travel at constant speed and with a fixed profile. Such waves were first observed by Russell in the mid–19th century [293], and are fundamental features of many water wave models. The objective of this chapter is to settle the existence/non-existence problem for the full irrotational water wave system in 2D, with the physical parameters being gravity, surface tension, and the fluid depth. Five of the six combinations have already been dealt with, and the results are summarized in Table 3.1 – it is our intent to fill in the missing case. The results in this chapter are based on the article [177], which is joint work with Mihaela Ifrim, Ben Pineau and Daniel Tataru.

Gravity	Capillarity	Depth	Existence
Yes	Yes	Infinite	Yes
Yes	No	Infinite	No
No	Yes	Infinite	No
Yes	Yes	Finite	Yes
Yes	No	Finite	Yes
No	Yes	Finite	Unknown

Table 3.1: Existence of 2D solitary waves in irrotational fluids

Our main result can be loosely formulated as follows:

Theorem 3.1.1. *No solitary waves exist in finite depth for the pure capillary irrotational water wave problem in 2D, even without the assumption that the free surface is a graph.*

A more precise formulation of the result is given later, in Theorem 3.4.1.

Historical perspectives

The mathematical study of travelling waves has been a fundamental – and longstanding – problem in fluid dynamics. Perhaps the first rigorous construction of 2D finite depth pure gravity solitary waves occurred in [118, 221]; further refinements can be found in [44, 246]. Solitary waves with large amplitudes were first constructed by Amick and Toland [16] in 1981 using global bifurcation techniques, leading to the existence of a limiting extreme wave with an angled crest [14]; see also [15, 45, 318]. By now, a vast literature exists on this subject, including both results for gravity and for gravity-capillary waves ([17, 66, 67, 86, 141, 142, 270, 283]). For water waves in deep water, solitary waves have been proved to exist provided that both gravity and surface tension are present, see [64, 65, 143, 179], following numerical work in [233, 234]. The forefront of current research on the mathematical theory of steady water waves is surveyed in [163].

The non-existence of 2D pure gravity solitary waves in infinite depth was originally proved in [172], under certain decay assumptions. The proof uses conformal mapping techniques, and the decay assumptions ultimately stem from difficulties in estimating commutators involving the Hilbert transform. The decay assumptions were completely removed in [174], as the authors were able to effectively deal with the aforementioned commutator issues – see [174, Lemma 3.1].

The proof of our result is loosely based on the ideas of [174]. The key difference is that the Tilbert transform (see Section 3.3 for the definition) does not enjoy the same commutator structure as the Hilbert transform. More precisely, we cannot simply replace Hilbert transforms with Tilbert transforms in [174, Lemma 3.1]. To circumvent this, we morally view the Tilbert transform as the Hilbert transform at high frequency, and a derivative at low frequency, and use these distinct regimes to close our argument.

For context, we mention that the problem we are considering in this chapter goes at least as far back as [37]. More specifically, in [37] it is noted that the systematic existence methods developed in [118, 194, 201] for the pure gravity problem in shallow water are unable to produce pure capillary solitary waves, but can be modified to produce gravity-capillary solitary waves. One may contrast the question of existence of solitary waves with that of the existence of periodic travelling waves. Indeed, for pure capillary irrotational waves in both finite and infinite depth, periodic travelling waves are known to exist. Most notably, one has the Crapper waves, which are quite explicit; see [88, 207] for the original results of Crapper and Kinnersley, and also the survey in [266]. Interestingly, the free surfaces of the Crapper waves need not be graphs, which makes the lack of graph assumption in Theorem 3.1.1 essential. The reader is referred to [5, 94, 238, 239, 326] for further literature on pure capillary waves, as well as gravity-capillary perturbations of these waves.

Finally, we mention a few recent directions that are somewhat outside the scope of this chapter. The first is the study of steady water waves with vorticity, for which we refer the interested reader to the surveys [140, 304]. As mentioned, our non-existence proof utilizes holomorphic coordinates, a technique which is not compatible with variable vorticity. However, such a restriction is quite natural, as heuristics dictate that one should expect solitary waves in problems with, say, constant non-zero vorticity. The other interesting direction – in situations where solitary waves are known to exist – is to determine which speeds are capable of sustaining solitary waves. Recently, it was shown in [211] that all finite depth, irrotational, pure gravity solitary waves must obey the inequality $c^2 > gh$. Here c is the speed, g the gravitational constant, and h the asymptotic depth. Heuristically, this result says that speeds that are precluded by the linearized problem are also precluded in the nonlinear problem. As a loose guideline, one expects solitary waves to travel at different speeds than the linear dispersive waves; the situations in Table 3.1 where solitary waves do not exist are exactly those in which the dispersion relation contains all speeds.

Our discussion above is fully confined to the 2D case, and that is for a good reason. All non-existence results discussed above in 2D are essentially open problems in 3D, so the 3D case is left for the future.

3.2 The equations in Eulerian coordinates

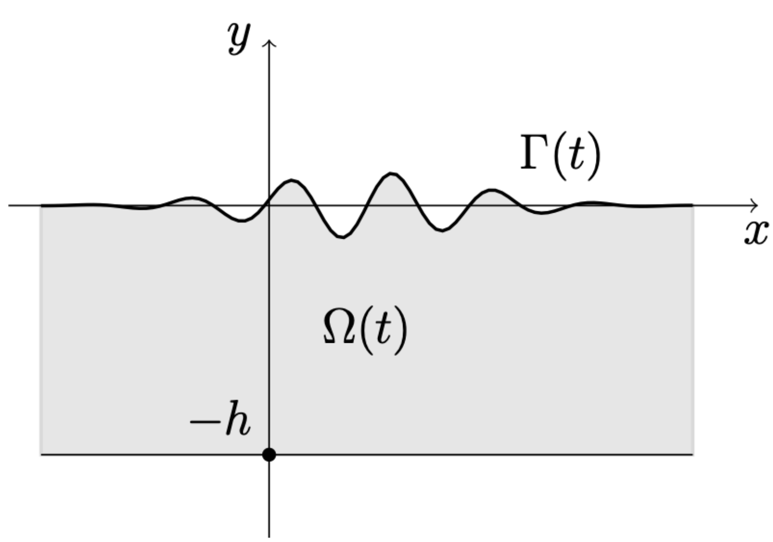
We consider the incompressible, finite depth water wave equations in two space dimensions. The motion of the water is governed by the incompressible Euler equations, with boundary conditions on the water surface and the flat, finite bottom. We emphasize that this section is purely for motivational purposes, and is not the formulation we will use to prove our non-existence result. In particular, for simplicity, this section assumes that $\Gamma(t)$ is a graph, but we will *not* assume this when working with the holomorphic formulation of our problem.

To describe the equations, denote the water domain at time t by $\Omega(t) \subseteq \mathbb{R}^2$; we assume that $\Omega(t)$ has a flat finite bottom $\{y = -h\}$, and let $\eta(x,t)$ denote the height of the free surface as a function of the horizontal coordinate:

$$\Omega(t) = \{(x,y) \in \mathbb{R}^2 : -h < y < \eta(x,t)\}. \tag{3.2.1}$$

The free surface of the water at time t will be denoted by $\Gamma(t)$. As we are interested in solitary waves, we think of $\Gamma(t)$ as being asymptotically flat at infinity to $y \approx 0$. Since the 2D finite depth capillary water wave equations do permit periodic travelling waves, this decay at infinity will factor heavily into our proof, even though we do not impose any specific rate of decay.

Figure 3.1: The fluid domain



We denote by u the fluid velocity and by p the pressure. The vector field u solves Euler's equations inside $\Omega(t)$,

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p - ge_2, \\ \text{div } u = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{3.2.2}$$

and the bottom boundary is impenetrable:

$$u \cdot e_2 = 0 \quad \text{when } y = -h. \tag{3.2.3}$$

On the upper boundary the atmospheric pressure is normalized to zero and we have the dynamic boundary condition

$$p = -\sigma \mathbf{H}(\eta) \quad \text{on } \Gamma(t), \tag{3.2.4}$$

and the kinematic boundary condition

$$\partial_t + u \cdot \nabla \quad \text{is tangent to } \bigcup \Gamma(t). \tag{3.2.5}$$

Here $g \geq 0$ represents the gravity,

$$\mathbf{H}(\eta) = \partial_x \left(\frac{\eta_x}{\sqrt{1+\eta_x^2}} \right) \tag{3.2.6}$$

is the mean curvature of the free boundary, and $\sigma > 0$ represents the surface tension coefficient.

We adhere to the classical assumption that the flow is irrotational, so we can write u in terms of a velocity potential ϕ as $u = \nabla \phi$. It is easy to see that ϕ is a harmonic function whose normal derivative is zero on the bottom. Thus, ϕ is determined by its trace $\psi = \phi|_{\Gamma(t)}$ on the free boundary $\Gamma(t)$. Under these assumptions, it is well-known that the fluid dynamics can be expressed in terms of a one-dimensional evolution of the pair of variables (η, ψ) via:

$$\begin{cases} \partial_t \eta - G(\eta)\psi = 0, \\ \partial_t \psi + g\eta - \sigma \mathbf{H}(\eta) + \dfrac{1}{2}|\nabla\psi|^2 - \dfrac{1}{2}\dfrac{(\nabla\eta \cdot \nabla\psi + G(\eta)\psi)^2}{1+|\nabla\eta|^2} = 0. \end{cases} \tag{3.2.7}$$

Here G denotes the Dirichlet to Neumann map associated to the fluid domain. This operator is one of the main analytical obstacles in this formulation of the problem, and in the next subsection we briefly discuss a change of coordinates that somewhat simplifies the analysis.

We now write down the solitary wave equations. We begin with the equations (3.2.1)–(3.2.6) as well as the irrotationality condition, and assume that the profile is uniformly translating in the horizontal direction with velocity c, i.e., $\phi(x,y,t) = \phi_0(x-ct,y)$, $\eta(x,y,t) = \eta_0(x-ct,y)$, and $p(x,y,t) = p_0(x-ct,y)$. This gives the steady water wave equations. To get to solitary waves (as opposed to, say, periodic waves), we impose some averaged decay on η_0 and u_0, so that in the far-field the water levels out and is essentially still. Contrary to many works which use a frame of reference travelling with the localized disturbance, we choose a frame so that the fluid is at rest near infinity. This allows us to set to zero the integration constant in the Bernoulli equation; the price to pay is that there are terms with c in the equations below.

We are thus interested in states (η, ϕ) satisfying the following equations:

$$\Delta\phi = 0 \quad \text{in } \Omega = \{(x,y) \in \mathbb{R}^2 : -h < y < \eta(x)\}, \tag{3.2.8}$$

$$-c\phi_x + \frac{1}{2}|\nabla\phi|^2 + g\eta - \sigma\partial_x\left(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right) = 0 \quad \text{on } \Gamma = \{(x,y) \in \mathbb{R}^2 : y = \eta(x)\}, \tag{3.2.9}$$

$$\phi_y = 0 \quad \text{when } y = -h, \tag{3.2.10}$$

$$-c\eta_x + \phi_x\eta_x = \phi_y \quad \text{on } \Gamma. \tag{3.2.11}$$

We prove that in the case $g = 0$ and $\sigma > 0$, the above equations admit no non-trivial solutions, with appropriate (averaged) decay at infinity. Such a claim, of course, presupposes certain regularity requirements on the solutions, but this will not play a major role due to ellipticity. Indeed, the above system can be shown to be locally elliptic whenever (η, ϕ) is above critical regularity, which corresponds to $\eta \in H^{\frac{3}{2}+}_{loc}$.

3.3 The equations in holomorphic coordinates

As mentioned, one of the main difficulties of (3.2.7) is the presence of the Dirichlet to Neumann operator $G(\eta)$, which depends on the free boundary. For this reason, we will reformulate the equations in holomorphic coordinates, which, in some sense, diagonalizes $G(\eta)$. We will only highlight briefly the procedure of changing coordinates; full details can be found in [153]. Moreover, although (3.2.7) assumes that $\Gamma(t)$ is a graph, the formulation below does not require this, which is another advantage of this approach. As we will see, making the solitary wave ansatz in holomorphic coordinates leads to remarkable simplifications, ultimately allowing us to derive (3.4.5), which we show admits no non-trivial solutions in appropriate function spaces. (3.4.5) is very similar to the equation analyzed in the original paper of Crapper, [88, Equation (15)], though in that paper they are viewed in infinite depth and in different function spaces.

The conditions we require on $\Gamma(t)$ are the same (or weaker, see the discussion below) as those listed in Section 2.3 of [153]; namely, that $\Gamma(t)$ can be parametrized to have sufficient Sobolev regularity, has no degeneracies or self-intersections, and never touches the bottom boundary. These assumptions are used in [153, Theorem 3] to justify the existence of the conformal map we refer to below.

In the holomorphic setting, the coordinates are denoted by $\alpha + i\beta \in S := \mathbb{R} \times (-h, 0)$, and the fluid domain is parameterized by the conformal map

$$z : S \to \Omega(t),$$

which takes the bottom $\mathbb{R} - ih$ into the bottom, and the top $\mathbb{R}$ into the top $\Gamma(t)$. The restriction of this map to the real line is denoted by Z, i.e., $Z(\alpha) := z(\alpha - i0)$, and can be viewed as a parametrization of the free boundary $\Gamma(t)$. We will work with the variables $W(\alpha) = Z(\alpha) - \alpha$, and the trace $Q(\alpha)$ of the holomorphic velocity potential on the free surface. W and Q are traditionally called holomorphic functions, which in this terminology means that they can be realized as the trace on the upper boundary $\beta = 0$ of holomorphic functions in the strip S which are purely real on the lower boundary $\beta = -h$. The space of holomorphic functions is a real algebra, but is not a complex algebra.

In terms of regularity, we note that the existence of the conformal map is guaranteed by the Riemann Mapping Theorem for any simply connected fluid domain. In order to have an equivalence between Sobolev norms, it suffices to assume that the free surface Γ has critical Besov regularity $B^{\frac{3}{2}}_{2,1}$. This, in particular, guarantees that Γ is a graph outside of a compact set. The conformal map, then, has the matching property $\Im(W) \in B^{\frac{3}{2}}_{2,1}$, and in particular $\Im(W)$ and W_α are bounded. For more details we refer the reader to both [153, Section 2] and the stronger results in [8], as well as the more general local results of [248].

The two-dimensional finite depth gravity-capillary water wave equations in holomorphic coordinates can be written as follows:

$$\begin{cases} W_t + F(1+W_\alpha) = 0, \\ Q_t + FQ_\alpha - g\mathcal{T}_h[W] + \mathbf{P}_h\left[\frac{|Q_\alpha|^2}{J}\right] + \sigma\mathbf{P}_h\left[i\left(\frac{W_{\alpha\alpha}}{J^{1/2}(1+W_\alpha)} - \frac{\overline{W_{\alpha\alpha}}}{J^{1/2}(1+\overline{W_\alpha})}\right)\right] = 0, \end{cases} \tag{3.3.1}$$

where

$$J = |1+W_\alpha|^2 \tag{3.3.2}$$

and

$$F = \mathbf{P}_h\left[\frac{Q_\alpha - \overline{Q_\alpha}}{J}\right]. \tag{3.3.3}$$

As before, g and σ are non-negative parameters, at least one of which is non-zero. $\mathcal{T}_h$ denotes the Tilbert transform, which is the Fourier multiplier with symbol $-i\tanh(h\xi)$, and arises in order to characterize what it means to be a holomorphic function. Precisely, holomorphic functions are described by the relation

$$\Im(u) = -\mathcal{T}_h\Re(u). \tag{3.3.4}$$

It is important to note that the Tilbert transform takes real-valued functions to real-valued functions, and satisfies the following product rule:

$$u\mathcal{T}_h[v] + \mathcal{T}_h[u]v = \mathcal{T}_h[uv - \mathcal{T}_h[u]\mathcal{T}_h[v]]. \tag{3.3.5}$$

Finally, $\mathbf{P}_h$ is the projection onto the space of holomorphic functions. In terms of $\mathcal{T}_h$ it can be written as

$$\mathbf{P}_h u = \frac{1}{2}\left[(1-i\mathcal{T}_h)\Re(u) + i(1+i\mathcal{T}_h^{-1})\Im(u)\right]. \tag{3.3.6}$$

In the case of no surface tension, equations (3.3.1) were derived in [153]. We begin with a brief outline of how the surface tension term arises, as we are particularly interested in the case when $g = 0$ and $\sigma > 0$.

Following [153], we arrive at the Bernoulli equation

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + gy + p = 0. \tag{3.3.7}$$

We then evaluate this equation on the top boundary and apply the dynamic boundary condition to replace p by $-\sigma\mathbf{H}$. We then pass to the strip S – so the equations are now defined on $\{\beta = 0\}$ – rewrite the equations in terms of the holomorphic variables, clear common factors of 2, and project. Running this procedure explicitly for the term with σ, we begin by parameterizing $\Gamma(t)$ by, say, $s \mapsto (\gamma_1(s), \gamma_2(s))$ and write $-\sigma\mathbf{H}$ in the standard parametric way. We then use the relations

$$\gamma_1(s) = \Re(Z(\alpha)), \quad \gamma_2(s) = \Im(Z(\alpha))$$

and formal calculations to write the capillary expression in terms of the holomorphic variables as:

$$\sigma i \left(\frac{W_{\alpha\alpha}}{J^{1/2}(1+W_\alpha)} - \frac{\overline{W_{\alpha\alpha}}}{J^{1/2}(1+\overline{W_\alpha})} \right),$$

which after projecting gives us the capillary term in (3.3.1).

Remark 3.3.1. Before proceeding, we would like to point out some inherent ambiguities of the above equations, which have to be properly interpreted. The first stems from the horizontal translation symmetry of the strip, which causes some arbitrariness in the choice of conformal mapping; precisely, $\Re(W)$ is only determined up to constants. A related issue is in the definition of the inverse Tilbert transform, as the Tilbert transform does not see constants. These ambiguities are built into the function spaces of [153], and play a much less significant role in our analysis than in the dynamic problem. Of course, a related, but easily resolved, ambiguity is that Q (and ϕ) are only defined up to addition of a real constant.

Remark 3.3.2. There are a few additional properties of z that we will note, all of which have been essentially verified in the proof of [153, Theorem 3]. The first is that the parameterization essentially moves "from left to right" or, more specifically, the parameterization on top satisfies $\frac{d\alpha}{ds} > 0$. This was implicitly used above in the derivation of the capillary term. Next, since z is holomorphic and a diffeomorphism, $|z_\alpha| > 0$ on S, which combined with the asymptotics at infinity implies that there is a $\delta > 0$ such that $|1+W_\alpha| = |Z_\alpha| \geq \delta$ on top. Note that we only require positivity conditions on $|1+W_\alpha|$; the boundary being a graph would assume positivity of $1+\Re(W_\alpha)$.

The solitary wave equations

In search for solitary wave solutions we fix a speed c and make the ansatz $(Q(\alpha,t), W(\alpha,t)) = (Q(\alpha-ct), W(\alpha-ct))$. The first equation in (3.3.1) then becomes

$$-cW_\alpha + F(1+W_\alpha) = 0 \tag{3.3.8}$$

while the second equation becomes

$$-cQ_\alpha + FQ_\alpha - g\mathcal{T}_h[W] + \mathbf{P}_h\left[\frac{|Q_\alpha|^2}{J}\right] + \sigma\mathbf{P}_h\left[i\left(\frac{W_{\alpha\alpha}}{J^{1/2}(1+W_\alpha)} - \frac{\overline{W_{\alpha\alpha}}}{J^{1/2}(1+\overline{W_\alpha})}\right)\right] = 0. \tag{3.3.9}$$

We rewrite the first equation as

$$F = \mathbf{P}_h\left[\frac{Q_\alpha - \overline{Q_\alpha}}{J}\right] = \frac{cW_\alpha}{1+W_\alpha}. \tag{3.3.10}$$

This gives that

$$\Im\left[\mathbf{P}_h\left[\frac{Q_\alpha - \overline{Q_\alpha}}{J}\right]\right] = c\Im\left(\frac{W_\alpha}{1+W_\alpha}\right) = \frac{c}{J}\Im\left(W_\alpha(1+\overline{W_\alpha})\right) = \frac{c}{J}\frac{W_\alpha - \overline{W_\alpha}}{2i}. \tag{3.3.11}$$

Recalling (3.3.6) and that the Tilbert transform maps real-valued functions to real-valued functions, we have

$$\Im(\mathbf{P}_h u) = \frac{1}{2}\left[\Im(u) - \mathcal{T}_h\Re(u)\right]. \tag{3.3.12}$$

Therefore,

$$\Im\left[\mathbf{P}_h\left[\frac{Q_\alpha - \overline{Q_\alpha}}{J}\right]\right] = \frac{1}{2}\Im\left(\frac{Q_\alpha - \overline{Q_\alpha}}{J}\right) = \frac{Q_\alpha - \overline{Q_\alpha}}{2iJ}. \tag{3.3.13}$$

The equation we end up with is, then,

$$\frac{Q_\alpha - \overline{Q_\alpha}}{2J} = \frac{c}{2}\frac{(W_\alpha - \overline{W_\alpha})}{J}, \tag{3.3.14}$$

which simplifies to

$$\Im(Q_\alpha) = c\Im(W_\alpha), \tag{3.3.15}$$

so that

$$Q_\alpha = cW_\alpha \tag{3.3.16}$$

because Q and W are holomorphic. Note that, formally, this argument only tells us that $Q_\alpha = cW_\alpha$ up to addition of a real constant. However, the decay properties of (W_α, Q_α) at infinity require the constant to vanish.

We now begin to simplify the second water wave equation. Beginning with (3.3.9), substituting (3.3.16) and the definition of F gives:

$$-c^2W_\alpha + \frac{c^2W_\alpha^2}{1+W_\alpha} - g\mathcal{T}_h[W] + c^2\mathbf{P}_h\left[\frac{|W_\alpha|^2}{J}\right] + \sigma\mathbf{P}_h\left[i\left(\frac{W_{\alpha\alpha}}{J^{1/2}(1+W_\alpha)} - \frac{\overline{W_{\alpha\alpha}}}{J^{1/2}(1+\overline{W_\alpha})}\right)\right] = 0. \tag{3.3.17}$$

Before continuing, we note a few things. First, we have

$$\mathbf{P}_h\left[\frac{|W_\alpha|^2}{J}\right] = \frac{1}{2}\left[(1-i\mathcal{T}_h)\frac{|W_\alpha|^2}{J}\right]. \tag{3.3.18}$$

This implies that

$$\Re\left(\mathbf{P}_h\left[\frac{|W_\alpha|^2}{J}\right]\right) = \frac{1}{2}\frac{|W_\alpha|^2}{J}. \tag{3.3.19}$$

Therefore, taking real part of (3.3.17) and then using the fact that holomorphic functions satisfy $\mathcal{T}_h\left[\Re(u)\right] = -\Im(u)$ we obtain:

$$-c^2\Re(W_\alpha)+c^2\Re\left(\frac{W_\alpha^2}{1+W_\alpha}\right)+g\Im(W)+\frac{c^2}{2}\frac{|W_\alpha|^2}{J}+\frac{\sigma}{2}i\left(\frac{W_{\alpha\alpha}}{J^{1/2}(1+W_\alpha)}-\frac{\overline{W_{\alpha\alpha}}}{J^{1/2}(1+\overline{W_\alpha})}\right)=0, \tag{3.3.20}$$

which can be re-written as

$$-c^2\Re(W_\alpha) + c^2\Re\left(\frac{W_\alpha^2}{1+W_\alpha}\right) + g\Im(W) + \frac{c^2}{2}\frac{|W_\alpha|^2}{J} + \frac{i\sigma}{1+W_\alpha}\partial_\alpha\left(\frac{1+W_\alpha}{|1+W_\alpha|}\right) = 0. \tag{3.3.21}$$

After straightforward manipulation of the terms with c^2 we arrive at

$$-\frac{c^2}{2}\frac{(W_\alpha+\overline{W_\alpha}+W_\alpha\overline{W_\alpha})}{|1+W_\alpha|^2} + g\Im(W) + \frac{i\sigma}{1+W_\alpha}\partial_\alpha\left(\frac{1+W_\alpha}{|1+W_\alpha|}\right) = 0. \tag{3.3.22}$$

As it turns out, these are exactly the same equations as the infinite-depth case considered in [174]. However, the function spaces are different, which plays a key role. In particular, as mentioned in the introduction, there are no infinite depth pure gravity solitary waves, but there are finite depth pure gravity solitary waves.

As a consistency check, we leave it as an exercise to show that (3.2.8)–(3.2.11) imply (3.3.22).

Notation for function spaces

The function spaces we use are standard, and similar to [175]. However, to set notation, we recall a few facts. Consider a standard dyadic Littlewood-Paley decomposition

$$1 = \sum_{k\in\mathbb{Z}} P_k,$$

where the projectors P_k select functions with frequencies $\approx 2^k$. We will place our (hypothetical) solutions in the critical Besov space $B^{\frac{1}{2}}_{2,1}$ defined via

$$\|u\|_{B^{\frac{1}{2}}_{2,1}} := \sum_{k\geq 1} 2^{\frac{k}{2}} \|P_k u\|_{L^2} + \|P_{\leq 0} u\|_{L^2}.$$

Our proof also makes use of the space $B^{\frac{3}{2}}_{2,1}$, which has the same norm as $B^{\frac{1}{2}}_{2,1}$ but with $2^{\frac{k}{2}}$ replaced by $2^{\frac{3k}{2}}$. Finally, we note the embedding of $B^{\frac{1}{2}}_{2,1}$ into L^∞, and the following Moser estimate:

Lemma 3.3.3. *Let $u \in B^{\frac{1}{2}}_{2,1}$, and suppose G is a smooth function with $G(0) = 0$. Then we have the Moser estimate*

$$\|G(u)\|_{B^{\frac{1}{2}}_{2,1}} \lesssim C(\|u\|_{L^\infty})\|u\|_{B^{\frac{1}{2}}_{2,1}}. \tag{3.3.23}$$

Proof. This is a standard result. For example, it follows from [175, Lemma 2.2] together with the analagous Moser estimate on the level of L^2. $\square$

3.4 No solitary waves when only surface tension is present

We are now able to state our main theorem. The result is stated in the low regularity function space $B^{\frac{1}{2}}_{2,1}$ defined above. However, part of the proof involves upgrading potential solutions to sufficient regularity to justify basic computations. Comparing with the infinite depth results in [174], our function space requires more regularity for W_α at low frequency, but this is to be expected, as the same happens in the dynamic problem [153]. From a technical standpoint, the issue is that $\mathcal{T}_h^{-1}$ does not have good mapping properties (it is not even bounded on L^2) compared to the Hilbert transform, which satisfies $H^{-1} = -H$. For justification of the other assumption – and conclusion – of Theorem 3.4.1, recall Remark 3.3.1 and Remark 3.3.2.

Theorem 3.4.1. *Suppose $W_\alpha \in B^{\frac{1}{2}}_{2,1}$ is holomorphic, solves* (3.3.22) *with $g = 0$ and $\sigma > 0$, $|1 + W_\alpha| > \delta > 0$ on the top, and its extension does not vanish on $\overline{S}$. Then $W_\alpha = 0$.*

Proof. We work with the equation

$$i\sigma\partial_\alpha\left(\frac{1+W_\alpha}{|1+W_\alpha|}\right) = c^2\left[W_\alpha + \frac{\overline{W_\alpha}}{1+\overline{W_\alpha}}\right], \tag{3.4.1}$$

which holds on the top and is just a rescaling of (3.3.22) with $g = 0$.

For what follows we slightly abuse notation by not distinguishing, notationally, between $1 + W_\alpha$ and its extension to the strip. First note that since $1 + W_\alpha$ is non-vanishing on the simply connected domain S, it admits a holomorphic logarithm. However, one has to be a little careful, to ensure that it is real on the bottom boundary. To see this, note that since, on the bottom, $1 + W_\alpha$ is real, non-vanishing and has limit 1 at infinity, it is positive on the bottom.

Define

$$T := \log(1 + W_\alpha) := U + iV. \tag{3.4.2}$$

The unknowns $U + iV$ are closely related to the unknowns $\tau + i\theta$ in [88]. It is easy to see that T can be chosen to be holomorphic; in particular, it can be chosen to be real on the bottom. Plugging into (3.4.1) we see that

$$-\sigma V_\alpha e^{iV} = c^2 \left[W_\alpha + \frac{\overline{W_\alpha}}{1 + \overline{W_\alpha}} \right] = c^2 \left(e^{U+iV} - e^{-U+iV} \right). \tag{3.4.3}$$

This implies that

$$-\sigma V_\alpha = 2c^2 \sinh(U). \tag{3.4.4}$$

Now, we upgrade regularity. By (3.4.2), $|1 + W_\alpha| > \delta$, and Lemma 3.3.3, it follows that $U, V \in B^{\frac{1}{2}}_{2,1}$. Again by Moser, we obtain $\sinh(U) \in B^{\frac{1}{2}}_{2,1}$ which in turn implies that $V_\alpha \in B^{\frac{1}{2}}_{2,1} \subseteq L^2$. From this we get $P_{>0} U_\alpha = -P_{>0} \mathcal{T}_h^{-1} V_\alpha \in B^{\frac{1}{2}}_{2,1}$. But since $U \in L^2$, it follows that $U_\alpha \in B^{\frac{1}{2}}_{2,1}$. This will be enough regularity to justify the calculations below, though H^∞ regularity for U and V_α could be obtained by reiteration.

Rescaling again and using that $-V_\alpha = \mathcal{T}_h U_\alpha$, it suffices to show that the equation

$$\mathcal{T}_h U_\alpha = 2c^2 \sinh U \tag{3.4.5}$$

admits no non-zero $B^{\frac{3}{2}}_{2,1}$ solutions. For this, we let χ be a smooth function with $\chi = 0$ on $(-\infty, -1]$ and $\chi = 1$ on $[1, \infty)$ with $\chi' \sim 1$ on $(-\frac{1}{2}, \frac{1}{2})$. Define $\chi_r(\alpha) = \chi(\frac{\alpha}{r})$.

Next, we multiply (3.4.5) by $-\chi_r U_\alpha$, and obtain

$$-\chi_r U_\alpha \mathcal{T}_h U_\alpha = -2c^2 \chi_r U_\alpha \sinh U = -2c^2 \chi_r \partial_\alpha \left(\cosh(U) - 1 \right). \tag{3.4.6}$$

An integration by parts yields the following identity:

$$-\int_{\mathbb{R}} \chi_r U_\alpha \mathcal{T}_h U_\alpha d\alpha = \frac{2c^2}{r} \int_{\mathbb{R}} \chi'(\frac{\alpha}{r})(\cosh(U) - 1) d\alpha. \tag{3.4.7}$$

Now, we treat the term on the left hand side of (3.4.7). From the product rule for the Tilbert transform we have

$$\chi_r \mathcal{T}_h U_\alpha = \mathcal{T}_h(\chi_r U_\alpha) - \mathcal{T}_h(\mathcal{T}_h \chi_r \mathcal{T}_h U_\alpha) - U_\alpha \mathcal{T}_h \chi_r. \tag{3.4.8}$$

Hence, using that the Tilbert transform is skew-adjoint and maps real-valued functions to real-valued functions,

$$\begin{aligned}
-\int_{\mathbb{R}} \chi_r U_\alpha \mathcal{T}_h U_\alpha d\alpha &= \int_{\mathbb{R}} U_\alpha \mathcal{T}_h(\mathcal{T}_h \chi_r \mathcal{T}_h U_\alpha) d\alpha + \int_{\mathbb{R}} |U_\alpha|^2 \mathcal{T}_h \chi_r d\alpha - \int_{\mathbb{R}} U_\alpha \mathcal{T}_h(\chi_r U_\alpha) d\alpha \\
&= \int_{\mathbb{R}} U_\alpha \mathcal{T}_h(\mathcal{T}_h \chi_r \mathcal{T}_h U_\alpha) d\alpha + \int_{\mathbb{R}} |U_\alpha|^2 \mathcal{T}_h \chi_r d\alpha + \int_{\mathbb{R}} \chi_r U_\alpha \mathcal{T}_h U_\alpha d\alpha \\
&= -\int_{\mathbb{R}} |\mathcal{T}_h U_\alpha|^2 \mathcal{T}_h \chi_r d\alpha + \int_{\mathbb{R}} |U_\alpha|^2 \mathcal{T}_h \chi_r d\alpha + \int_{\mathbb{R}} \chi_r U_\alpha \mathcal{T}_h U_\alpha d\alpha.
\end{aligned} \tag{3.4.9}$$

Hence, we obtain

$$-\int_{\mathbb{R}} \chi_r U_\alpha \mathcal{T}_h U_\alpha d\alpha = \frac{1}{2} \int_{\mathbb{R}} (|U_\alpha|^2 - |\mathcal{T}_h U_\alpha|^2) \mathcal{T}_h \chi_r d\alpha. \tag{3.4.10}$$

Combining this with (3.4.7), we get

$$\frac{2c^2}{r} \int_{\mathbb{R}} \chi'(\frac{\alpha}{r})(\cosh(U) - 1) d\alpha = \frac{1}{2} \int_{\mathbb{R}} (|U_\alpha|^2 - |\mathcal{T}_h U_\alpha|^2) \mathcal{T}_h \chi_r d\alpha. \tag{3.4.11}$$

The idea now is to use the fact that at low frequency, the Tilbert transform agrees with the multiplier $\xi \mapsto -hi\xi$ to third order. With this in mind, we rewrite the above equation as follows:

$$\begin{aligned}
\frac{2c^2}{r} \int_{\mathbb{R}} \chi'(\frac{\alpha}{r})(\cosh(U) - 1) d\alpha = &\frac{1}{2} \int_{\mathbb{R}} (|U_\alpha|^2 - |\mathcal{T}_h U_\alpha|^2)(\mathcal{T}_h + h\partial_\alpha) \chi_r d\alpha \\
&- \frac{h}{2r} \int_{\mathbb{R}} (|U_\alpha|^2 - |\mathcal{T}_h U_\alpha|^2) \chi'(\frac{\alpha}{r}) d\alpha.
\end{aligned} \tag{3.4.12}$$

Equivalently, we have

$$\begin{aligned}
2c^2 \int_{\mathbb{R}} \chi'(\frac{\alpha}{r})(\cosh(U) - 1) d\alpha &+ \frac{h}{2} \int_{\mathbb{R}} (|U_\alpha|^2 - |\mathcal{T}_h U_\alpha|^2) \chi'(\frac{\alpha}{r}) d\alpha \\
&= \frac{r}{2} \int_{\mathbb{R}} (|U_\alpha|^2 - |\mathcal{T}_h U_\alpha|^2)(\mathcal{T}_h + h\partial_\alpha) \chi_r d\alpha.
\end{aligned} \tag{3.4.13}$$

We are now in a position to estimate the right-hand side of (3.4.13). Indeed, by Cauchy Schwarz and Sobolev embedding, we have,

$$\begin{aligned}\frac{r}{2}\left|\int_{\mathbb{R}}(|U_\alpha|^2-|\mathcal{T}_h U_\alpha|^2)(\mathcal{T}_h+h\partial_\alpha)\chi_r d\alpha\right| &\le Cr(\|U_\alpha\|_4^2+\|\mathcal{T}_h U_\alpha\|_4^2)\|(\mathcal{T}_h+h\partial_\alpha)\chi_r\|_2 \\ &\le Cr\|U\|^2_{B^{\frac{3}{2}}_{2,1}}\|(\mathcal{T}_h+h\partial_\alpha)\chi_r\|_2.\end{aligned} \tag{3.4.14}$$

Using Plancherel's Theorem we then obtain the simple estimate,

$$\begin{aligned} r\|U\|^2_{B^{\frac{3}{2}}_{2,1}}\|(\mathcal{T}_h+h\partial_\alpha)\chi_r\|_2 &= Cr\|U\|^2_{B^{\frac{3}{2}}_{2,1}}\|(\tanh(h\xi)-h\xi)\widehat{\chi_r}\|_2 \\ &\le \frac{C}{r}\|U\|^2_{B^{\frac{3}{2}}_{2,1}}\left\|\frac{\tanh(h\xi)-h\xi}{\xi^2}\right\|_{L^\infty}\|\chi''(\frac{\alpha}{r})\|_2 \\ &\le \frac{C}{r^{1/2}}\|U\|^2_{B^{\frac{3}{2}}_{2,1}}\|\chi''\|_2.\end{aligned} \tag{3.4.15}$$

Hence, we obtain

$$2c^2\int_{\mathbb{R}}\chi'(\frac{\alpha}{r})(\cosh(U)-1)d\alpha+\frac{h}{2}\int_{\mathbb{R}}(|U_\alpha|^2-|\mathcal{T}_h U_\alpha|^2)\chi'(\frac{\alpha}{r})d\alpha=\mathcal{O}_{\|U\|_{B^{\frac{3}{2}}_{2,1}}}(r^{-1/2}). \tag{3.4.16}$$

Letting $r\to\infty$, dominated convergence gives

$$2c^2\int_{\mathbb{R}}(\cosh(U)-1)d\alpha=-\frac{h}{2}\int_{\mathbb{R}}(|U_\alpha|^2-|\mathcal{T}_h U_\alpha|^2)d\alpha=-\frac{h}{2}\int_{\mathbb{R}}|\xi|^2|\widehat{U}|^2\mathrm{sech}^2(h\xi)\le 0.$$

Therefore, since $\cosh(U)-1\ge 0$, we have

$$\cosh(U)=1,$$

so that $U\equiv 0$. Note that taking the limit is justified because $\cosh(U)-1$ is integrable. This is thanks to the fact that U is bounded, vanishes at infinity, and belongs to L^2.

We remark that if one assumes instead some stronger decay at infinity for U, then the above argument proving non-existence of solutions for (3.4.5) can be simplified somewhat by working directly with the choice $\chi(\alpha)=\alpha$. This of course leads to a weaker result; the details are left for the reader. □

Chapter 4

Stable phase retrieval in function spaces

4.1 Introduction

There are many situations in mathematics, science, and engineering where the goal is to recover some vector f from $|Tf|$, where T is a linear transformation into a function space. Note that if $|\lambda| = 1$ then it is impossible to distinguish f and λf in this way. The linear transformation T is said to do phase retrieval if this ambiguity is the only obstruction to recovering f. That is, given a vector space H and function space X, a linear operator $T : H \to X$ does *phase retrieval* if whenever $f, g \in H$ satisfy $|Tf| = |Tg|$ then $f = \lambda g$ for some scalar λ with $|\lambda| = 1$. Phase retrieval naturally arises in situations where one is only able to obtain the magnitude of linear measurements, and not the phase. Notable examples in physics and engineering which require phase retrieval include X-ray crystallography, electron microscopy, quantum state tomography, and cepstrum analysis in speech recognition. The study of phase retrieval in mathematical physics dates back to at least 1933 when in his seminal work *Die allgemeinen Prinzipien der Wellenmechanik* [272] W. Pauli asked whether a wave function is uniquely determined by the probability densities of position and momentum. In other words, Pauli asked whether $|f|$ and $|\widehat{f}|$ determine $f \in L_2(\mathbb{R})$ up to multiplication by a unimodular scalar. The mathematics of phase retrieval has since grown to be an important and well-studied topic in applied harmonic analysis.

As any application of phase retrieval would involve error, it is of fundamental importance that the recovery of f from $|Tf|$ not only be possible, but also be stable. We say that T

does *stable phase retrieval* if the recovery (up to a unimodular scalar) of f from $|Tf|$ is Lipschitz. If X is finite dimensional, then T does phase retrieval if and only if it does stable phase retrieval [41, 69]. However, if X is infinite dimensional and T is the analysis operator of a frame or a continuous frame, then T cannot do stable phase retrieval [7, 70]. Here, a collection of vectors $(\psi_t)_{t\in\Omega} \subseteq H$ is a *continuous frame* of a Hilbert space H over a measure space (Ω, Σ, μ) if the map $f \mapsto (\langle f, \psi_t\rangle)_{t\in\Omega}$ is an embedding of H into $L_2(\mu)$. One of the main goals of this chapter is to use the theory of subspaces of Banach lattices to present a unifying framework for stable phase retrieval which encompasses the previously studied cases and allows for stable phase retrieval in infinite dimensions.

Let $X = L_p(\mu)$, or, more generally, a Banach lattice. Let $E \subseteq X$ be a subspace. We say that E does *phase retrieval* as a subspace of X if whenever $|f| = |g|$ for some $f, g \in E$ we have that $f = \lambda g$ for some scalar λ with $|\lambda| = 1$. Given a constant $C > 0$, we say that E does *C-stable phase retrieval* as a subspace of X if

$$\inf_{|\lambda|=1} \|f - \lambda g\| \leq C\big\||f| - |g|\big\| \qquad \text{for all } f, g \in E. \tag{4.1.1}$$

We may define an equivalence relation $\sim$ on E by $f \sim g$ if $f = \lambda g$ for some scalar λ with $|\lambda| = 1$. Then, E does phase retrieval as a subspace of X if and only if the map $f \mapsto |f|$ from $E/\!\sim$ to X is injective. Furthermore, E does C-stable phase retrieval as a subspace of X if and only if the map $f \mapsto |f|$ from $E/\!\sim$ to X is injective and the inverse is C-Lipschitz. By introducing stable phase retrieval into the setting of Banach lattices, we are able to apply established methods from the subject to attack problems in phase retrieval, and conversely we hope that the ideas and questions in phase retrieval will inspire a new avenue of research in the theory of Banach lattices. Before starting the meat of the chapter, we present some additional motivation, give an outline of our major results, and state some of the important ideas and theorems from Banach lattices which we will be applying.

Motivation and applications

The inequality (4.1.1) arises in various circumstances. For instance, in crystallography and optics, one seeks to recover an unknown function $F \in L_2(\mathbb{R}^d)$ from the absolute value of its Fourier transform $\widehat{F}$. If one also seeks stability, this translates into an inequality of the form

$$\inf_{|\lambda|=1} \|F - \lambda G\|_{L_2} \leq C\||\widehat{F}| - |\widehat{G}|\|_{L_2}, \tag{4.1.2}$$

which one would want to be valid for F, G in a subspace $E \subseteq L_2(\mathbb{R}^d)$ which incorporates the additional constraints F, G are known to satisfy. Using Plancherel's theorem to write $\|F - \lambda G\|_{L_2} = \|\widehat{F} - \lambda \widehat{G}\|_{L_2}$, one sees that the inequality (4.1.2) reduces to (4.1.1), up to passing to Fourier space and making the change of notation $f = \widehat{F}$ and $g = \widehat{G}$. We refer the reader to the surveys [138, 180] and references therein for a further explanation of the importance of phase retrieval in optics, crystallography, and other areas. In particular, these articles explains why, in practice, physical experiments are often able to measure the magnitude of the Fourier transform, but are unable to measure the phase.

A second scenario where phase retrieval appears is quantum mechanics. In this case, one wants to identify situations where $|f|$ and $|\widehat{f}|$ determine $f \in L_2(\mathbb{R})$ uniquely. As already mentioned, Pauli asked whether this could true for all $f \in L_2(\mathbb{R})$. However, a counterexample to this conjecture was given in 1944: There exists $f, g \in L_2(\mathbb{R})$ such that $|f| = |g|$ and $|\widehat{f}| = |\widehat{g}|$ but f is not a multiple of g. This leads to the natural question of whether one can build "large" subspaces $G \subseteq L_2(\mathbb{R})$ for which $|f|$ and $|\widehat{f}|$ determine $f \in G \subseteq L_2(\mathbb{R})$ uniquely. By passing to the phase space $L_2(\mathbb{R}) \times L_2(\mathbb{R})$, we see that G has the above property if and only if $E := \{(f, \widehat{f}) : f \in G\}$ does phase retrieval as a subspace of $L_2(\mathbb{R}) \times L_2(\mathbb{R})$, i.e., knowing $h, k \in E$ and $|h| = |k|$ implies h is a unimodular multiple of k. This also naturally leads to the question of stability of Pauli phase retrieval, by requiring (4.1.1) hold on E. In this case, using Plancherel's theorem to return to G, (4.1.1) on E translates into the inequality

$$\inf_{|\lambda|=1} \|f - \lambda g\|_{L_2} \leq C \left(\||f| - |g|\|_{L_2} + \||\widehat{f}| - |\widehat{g}|\|_{L_2} \right) \text{ for } f, g \in G. \tag{4.1.3}$$

For a non-exhaustive collection of results on Pauli phase retrieval and its generalizations, see [18, 138, 182, 183] and references therein. To our knowledge, the question of stability in the Pauli Problem is essentially unexplored. However, the results presented here in conjunction with [82] give a relatively large class of subspaces of $L_2(\mathbb{R}^d)$ satisfying (4.1.3).

Finally, we mention that phase retrieval has grown to become an exciting and important topic of research in frame theory [41, 42, 43, 57, 76, 104, 138]. A frame for a separable Hilbert space H is a sequence of vectors $(\phi_j)_{j \in J}$ in H such that there exists uniform bounds $A, B > 0$ so that

$$A\|f\|^2 \leq \sum_{j \in J} |\langle f, \phi_j \rangle|^2 \leq B\|f\|^2 \qquad \text{for all } f \in H. \tag{4.1.4}$$

The *analysis operator* of a frame $(\phi_j)_{j\in J}$ of H is the map $\Theta : H \to \ell_2(J)$ given by $\Theta(f) = (\langle f, \phi_j\rangle)_{j\in J}$. Note that the uniform upper bound B in the frame inequality (4.1.4) guarantees that $\Theta : H \to \ell_2(J)$ is bounded, and the uniform lower bound A gives that Θ is an embedding of H into $\ell_2(J)$. Given a frame $(\phi_j)_{j\in J}$ of H, the *canonical dual frame* $(\widetilde{\phi}_j)_{j\in J}$ is defined by $\widetilde{\phi}_j = (\Theta^*\Theta)^{-1}\phi_j$ for all $j \in J$ and satisfies

$$f = \sum_{j\in J}\langle f, \widetilde{\phi}_j\rangle\phi_j = \sum_{j\in J}\langle f, \phi_j\rangle\widetilde{\phi}_j \qquad \text{for all } f \in H. \tag{4.1.5}$$

Frames have many applications and play a fundamental role in signal processing and applied harmonic analysis. One important reason for this is that the analysis operator Θ is an embedding of H into $\ell_2(J)$, which allows for the application of filters, thresholding, and other signal processing techniques. Another reason is that (4.1.5) gives a linear, stable, and unconditional reconstruction formula for a vector in terms of the frame coefficients.

A frame $(\phi_j)_{j\in J}$ is said to do *phase retrieval* if whenever $f, g \in H$ and $(|\langle f, \phi_j\rangle|)_{j\in J} = (|\langle g, \phi_j\rangle|)_{j\in J}$, there exists a unimodular scalar λ such that $f = \lambda g$. A frame is said to do *stable phase retrieval* if there exists a constant $C \geq 1$ such that for all $f, g \in H$,

$$\inf_{|\lambda|=1} \|f - \lambda g\|_H \leq C\||\Theta(f)| - |\Theta(g)|\|_{\ell_2(J)}. \tag{4.1.6}$$

Using the fact that the analysis operator $\Theta : H \to \ell_2(J)$ is an embedding, we see that a frame does stable phase retrieval if and only if the subspace $\Theta(H) \subseteq \ell_2(J)$ does stable phase retrieval in the sense of (4.1.1). In finite dimensions, phase retrieval for frames is automatically stable. However, in infinite dimensions, it is necessarily unstable. As we will see, this is due to the fact that the ambient Hilbert lattice $\ell_2(J)$ is atomic, whereas the construction of SPR subspaces from [71] is done in the non-atomic lattice $L_2(\mathbb{R})$. For further investigations on the instability of phase retrieval for frames - including generalizations to continuous frames and frames in Banach spaces - see [7, 70].

As mentioned previously, phase retrieval problems arise in applications when considering an operator $T : H \to X$, which embeds a Hilbert space H into a function space X. In particular, the inequality (4.1.2) arises by taking T to be the Fourier transform, and (4.1.6) arises by taking T to be the analysis operator of a frame. Another important choice for T is the Gabor transform (see [6, 139] for recent advances in Gabor phase retrieval). As should now be evident, the question of stability for each of these phase retrieval problems can be translated into a special case of (4.1.1), by taking $E := T(H)$.

An overview of the results

The examples from Section 4.1 show that the inequality (4.1.1) unifies various phase retrieval problems. However, as mentioned previously, phase retrieval for frames is unstable in infinite dimensions, and it was only recently that the first examples of infinite dimensional SPR subspaces of real $L_2(\mu)$ spaces were constructed [71]. The purpose of this chapter is twofold. First, we construct numerous examples of subspaces of $L_p(\mu)$ doing stable phase retrieval. For this, we use various isometric Banach space techniques, modifications of the "almost disjointness" methods in classical Banach lattice theory, random constructions, and analogues of some constructions from harmonic analysis. Secondly, we prove several structural results about SPR subspaces of $L_p(\mu)$, and even general Banach lattices. Notably, both the characterization of real SPR in terms of almost disjoint pairs (Theorem 4.3.4), as well as the equivalence of SPR and its Hölder analogue (Corollary 4.3.11) hold for all Banach lattices. Our results also extend those in the recent article [82] (reviewed in Section 4.7 below), which uses orthogonality and combinatorial arguments akin to Rudin's work [292] on $\Lambda(p)$-sets to produce examples of subspaces of (real or complex) $L_p(\mu)$ doing Hölder stable phase retrieval.

We now briefly overview the chapter. In Section 4.2, we recall some basic terminology and results from Banach lattice theory in order to make the chapter accessible to a wider audience. Most notably, in Section 4.2 we collect basic facts related to the Kadec-Pelczynski dichotomy. Such results give structural information about closed subspaces of Banach lattices that are *dispersed*, i.e., that do not contain normalized almost disjoint sequences. As we will show in Theorem 4.3.4, a subspace of a (real) Banach lattice does stable phase retrieval if and only if it does not contain normalized almost disjoint *pairs*. In Theorem 4.2.1, we collect various facts about dispersed subspaces; finding SPR analogues of these results will occupy much of the chapter. In particular, although SPR is much stronger than being dispersed, in Theorem 4.5.1 we will show that every closed infinite dimensional dispersed subspace of an order continuous Banach lattice contains a further closed infinite dimensional subspace doing SPR. The preliminary section finishes with Section 4.2, which recalls basic facts about complex Banach lattices.

Section 4.3 collects various results on stable phase retrieval that hold for general Banach lattices. In particular, in Section 4.3 we make the aforementioned connection between stable phase retrieval and almost disjoint pairs (see Theorem 4.3.4). In Section 4.3, we show that if the phase recovery map is Hölder continuous on the ball, then it is Lipschitz continuous on the whole space (Corollary 4.3.11). This follows from Theorem 4.3.9, which shows that failure of stable phase retrieval can be witnessed by "well-separated" vectors. The equivalence between stable phase retrieval and Hölder stable phase retrieval allows us to improve some results from [82], yielding the first examples of infinite dimensional closed subspaces of complex $L_2(\mu)$ doing stable phase retrieval.

In Section 4.4, we build infinite dimensional SPR subspaces using a variety of different techniques. In particular, we prove in Corollary 4.4.8 an analogue of statement (iii) of Theorem 4.2.1; namely, that for every dispersed subspace $E \subseteq L_p[0,1]$ $(1 \leq p \leq \infty)$, we can build a closed subspace $E' \subseteq L_p[0,1]$ isomorphic to E, and doing stable phase retrieval. Moreover, for $p < 2$ and $q \in (p, 2]$, we will show that any closed subspace of $L_p(\mu)$ isometric to $L_q(\mu)$ does SPR in $L_p(\mu)$, see Proposition 4.4.1. Regarding sequence spaces, in Section 4.4 we show that ℓ_∞ embeds into itself in an SPR way, while no infinite dimensional subspace of ℓ_p does SPR when $1 \leq p < \infty$. Section 4.4 constructs SPR subspaces of rearrangement invariant spaces using random variables. This, in particular, tells us that subspaces spanned by iid Gaussian and q-stable random variables will do SPR in a variety of spaces, including all L_p-spaces in which they can be found. Finally, Section 4.4 provides some basic stability properties of SPR subspaces.

Section 4.5 contains a study of the structure of SPR subspaces of $L_p(\mu)$, for a finite measure μ. We begin with the aforementioned Theorem 4.5.1, which is applicable for general order continuous Banach lattices, but for which much of the proof occurs in $L_1(\mu)$. Indeed, we will show that the generalization to order continuous Banach lattices follows from the result in $L_1(\mu)$ by arguing via the Kadec-Pelczynski dichotomy.

Note that from the classical results in Theorem 4.2.1 (a)-(d) it follows that if E is dispersed in $L_p(\mu)$ and $1 \leq q < p < \infty$, then E may be viewed as a closed subspace of $L_q(\mu)$, and it is dispersed in $L_q(\mu)$. In Theorem 4.5.3 we show that if $2 \leq p < \infty$, there are closed subspaces $E \subseteq L_p(\mu)$ which do SPR (and hence are dispersed in $L_q(\mu)$ for all $1 \leq q \leq p$), but fail to do SPR when viewed as a closed subspace of $L_q(\mu)$ for all $1 \leq q < p$. However, by Theorem 4.5.6, if $p < 2$, then any SPR subspace $E \subseteq L_p(\mu)$ also does SPR when viewed as a closed subspace of $L_q(\mu)$ for any $1 \leq q \leq p$. Whether there is an SPR analogue of statement (v) of Theorem 4.2.1 remains an open problem.

Section 4.6 is devoted to the study of infinite dimensional SPR subspaces of $C(K)$. The main result is Theorem 4.6.1 which states that for a compact Hausdorff space K, the space $C(K)$ of continuous functions over K admits a (closed) infinite dimensional SPR subspace if and only if the Cantor-Bendixson derivative K' of K is infinite.

The above sections are based on the joint work [115] with Dan Freeman, Timur Oikhberg and Ben Pineau. Section 4.7, which is joint with Michael Christ and Ben Pineau, gives various natural examples of real and complex subspaces doing SPR.

4.2 Preliminaries

As many of our results hold in the generality of Banach lattices, we briefly summarize some of the standard notations and conventions from this theory. For the most part, our conventions align with the references [12, 231]. Moreover, the statements of our results require minimal knowledge of Banach lattices to understand; it is simply the proofs that use the technology and terminology from this theory. Unless otherwise mentioned, all L_p-spaces, $C(K)$-spaces and Banach lattices are real. When a result is applicable for complex scalars, we will explicitly state this. The word "subspace" is to be interpreted in the vector space sense. If a result requires the subspace to be closed or (in)finite dimensional, we will state this.

Recall that a *vector lattice* is a vector space, equipped with a compatible lattice-ordering (see [12] for a precise definition). For a vector lattice X, the positive cone of X is denoted by $X_+ := \{f \in X : f \geq 0\}$. The infimum of $f, g \in X$ is denoted by $f \wedge g$, and the supremum is denoted by $f \vee g$. The modulus of f is defined as $|f| := f \vee (-f)$, and elements $f, g \in X$ are said to be *disjoint* if $|f| \wedge |g| = 0$. A *weak unit* is an element $e \in X_+$ for which $|f| \wedge e = 0$ implies $f = 0$. For a net (f_α) in X, the notation $f_\alpha \downarrow 0$ means that f_α is decreasing and has infimum 0. A subspace $E \subseteq X$ is a *sublattice* if it is closed under finite lattice operations; it is an *ideal* if $f \in E$ and $|g| \leq |f|$ implies $g \in E$.

A *Banach lattice* is a Banach space which is also a vector lattice, and for which one has the compatibility condition $\|f\| \leq \|g\|$ whenever $|f| \leq |g|$. Note that the SPR inequality (4.1.1) remains well-defined when $L_p(\mu)$ is replaced by an arbitrary Banach lattice. As we will see, several of our results on SPR are also valid in this level of generality. Common examples of Banach lattices include L_p-spaces, $C(K)$-spaces, Orlicz spaces, and various sequence spaces. In this case, the ordering is pointwise, i.e., $f \leq g$ means $f(t) \leq g(t)$ for all (or almost all in the case of measurable functions) t in the domain of f and g.

A Banach lattice X is *order continuous* if for each net (f_α) satisfying $f_\alpha \downarrow 0$ we have $f_\alpha \xrightarrow{\|\cdot\|_X} 0$. L_p-spaces are order continuous for $1 \leq p < \infty$, but $C(K)$-spaces are not (unless they are finite dimensional). To transfer results from $L_1(\mu)$ to more general Banach lattices, we will make use of the *AL-representation* procedure. For this, let X be an order continuous Banach lattice with a weak unit e. It is known that X can be represented as an order and norm dense ideal in $L_1(\mu)$ for some finite measure μ. That is, there is a vector lattice isomorphism $T : X \to L_1(\mu)$ such that Range$T$ is an order and norm dense ideal in $L_1(\mu)$. Note that T need not be a norm isomorphism, though T may be chosen to be continuous with $Te = \mathbb{1}$. Moreover, Range$T$ contains $L_\infty(\mu)$ as a norm and order dense ideal. It is common to identify X with Range$T$ and view X as an ideal of $L_1(\mu)$. Such an inclusion of X into $L_1(\mu)$ is called an *AL-representation* of X. We refer to [231, Theorem 1.b.14] or [121, Section 4] for details on AL-representations.

The Kadec-Pelczynski dichotomy

Here, we briefly recap the literature on subspaces which do not contain almost disjoint normalized sequences. Recall that a sequence (x_n) in a Banach lattice X is said to be a *normalized almost disjoint sequence* if $\|x_n\|_X = 1$ for all n, and there exists a disjoint sequence (d_n) in X such that $\|x_n - d_n\|_X \to 0$. Following [53, 133, 132], a closed subspace of a Banach lattice that fails to contain normalized almost disjoint sequences will be called *dispersed*. The classical *Kadec-Pelczynski dichotomy* (c.f. [231, Proposition 1.c.8]) states that for a subspace E of an order continuous Banach lattice X with weak unit, either

(i) E fails to be dispersed, i.e., E contains an almost disjoint normalized sequence, or,

(ii) E is isomorphic to a closed subspace of $L_1(\Omega, \Sigma, \mu)$.

As we will see in Theorem 4.3.4, for real scalars, a subspace does stable phase retrieval if and only if it does not contain normalized almost disjoint *pairs*. Hence, the Kadec-Pelczynski dichotomy will provide a tool to analyze such subspaces.

In $L_p(\mu)$ for $1 \leq p < \infty$ and a probability measure μ, the Kadec-Pelczynski dichotomy can be improved. Indeed, we summarize the literature in the following theorem.

Theorem 4.2.1. Let $1 \le p < \infty$ and μ be a probability measure. For a closed subspace E of $L_p(\mu)$, the following are equivalent:

(a) E is dispersed, i.e., E contains no almost disjoint normalized sequences;

(b) There exists $0 < q < p$ such that $\|\cdot\|_{L_p} \sim \|\cdot\|_{L_q}$ on E;

(c) For all $0 < q < p$, $\|\cdot\|_{L_p} \sim \|\cdot\|_{L_q}$ on E;

(d) E is strongly embedded in $L_p(\mu)$, i.e., convergence in measure coincides with norm convergence on E.

Moreover,

(i) For $p \neq 2$, a closed subspace of $L_p[0,1]$ is dispersed if and only if it contains no subspace isomorphic to ℓ_p.

(ii) For $p > 2$, a closed subspace of $L_p[0,1]$ is dispersed if and only if it is isomorphic to a Hilbert space.

(iii) For $p < 2$ and any $q \in (p, 2]$, there is a closed subspace of $L_p[0,1]$ which is both dispersed and isometric to $L_q[0,1]$.

(iv) For $p \neq 2$, $L_p[0,1]$ cannot be written as the direct sum of two dispersed subspaces.

(v) There exists an orthogonal decomposition $L_2[0,1] = E \oplus E^\perp$ with both E and $E^\perp$ dispersed in $L_2[0,1]$.

Proof. The equivalence of (b), (c) and (d) is [9, Proposition 6.4.5]. Other than the isometric portion of statement (iii), the rest of the statements are neatly summarized in [132, Propositions 3.4 and 3.5], with references to various textbooks for proofs. An isometric embedding of $L_q[0,1]$ into $L_p[0,1]$ for $q \in (p,2)$ is given in [231, Corollary 2.f.5]. An isometric embedding of ℓ_2 into $L_p[0,1]$ for $1 \le p < \infty$ is given in [9, Proposition 6.4.12]. $\square$

Remark 4.2.2. One of the goals of this chapter is to find SPR analogues of the results in Theorem 4.2.1. However, we should mention that the connection between Theorem 4.2.1 and SPR has already been implicitly made in [82]. Recall that a subset $\Lambda \subseteq \mathbb{Z}$ is called a *$\Lambda(p)$-set* if the closed subspace generated by the set of exponentials $\{e^{2\pi i n x} : n \in \Lambda\} \subseteq L_p(\mathbb{T})$ satisfies the equivalent conditions in Theorem 4.2.1. Such sets have been deeply studied [20, 61, 294], and have many interesting properties. For example, Rudin [292] showed that for all integers

$n > 1$, there are $\Lambda(2n)$-sets that are not $\Lambda(q)$-sets for every $q > 2n$. Moreover, Bourgain [58] extended Rudin's theorem to all $p > 2$. On the other hand, when $p < 2$, and Λ is $\Lambda(p)$, then it is automatically $\Lambda(p+\varepsilon)$ for some $\varepsilon > 0$ ([38, 152]). Since $|e^{2\pi inx}| \equiv 1$, complex exponentials cannot do stable phase retrieval. However, by replacing $e^{2\pi inx}$ by $\sin(2\pi nx)$ or other trigonometric polynomials with non-constant moduli, [82] is able to use combinatorial arguments in the spirit of Rudin to produce SPR subpaces of $L_p(\mu)$ when the dilation set Λ is sufficiently sparse.

Complex Banach lattices

Complex Banach lattices are defined as complexifications of real Banach lattices, and in the case of complex function spaces like $C(K)$ and $L_p(\mu)$, agree with the standard definition. More precisely, by a *complex Banach lattice* we mean the complexification $X_{\mathbb{C}} = X \oplus iX$ of a real Banach lattice, X, endowed with the norm $\|x+iy\|_{X_{\mathbb{C}}} = \||x+iy|\|_X$, where the modulus $|\cdot| : X_{\mathbb{C}} \to X_+$ is the mapping given by

$$|x+iy| = \sup_{\theta\in[0,2\pi]} \{x\cos\theta + y\sin\theta\}, \text{ for } x+iy \in X_{\mathbb{C}}. \tag{4.2.1}$$

We refer to [2, Section 3.2] and [298, Section 2.11] for a proof that the modulus function is well-defined, and behaves as expected.

With the above definition, one can define complex sublattices, complex ideals, etc. However, we will not need this. We do, however, note that if $T : X \to Y$ is a real linear operator between real Banach lattices, then we may define the *complexification* $T_{\mathbb{C}} : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ of T via $T_{\mathbb{C}}(x+iy) = Tx + iTy$. The map $T_{\mathbb{C}}$ is $\mathbb{C}$-linear, bounded, and if T is a lattice homomorphism then $T_{\mathbb{C}}$ preserves moduli, i.e., $T|z| = |T_{\mathbb{C}}z|$ for $z \in X_{\mathbb{C}}$. When we work with complex Banach lattices $X_{\mathbb{C}}$, we will use these facts to identify $X_{\mathbb{C}}$ as a space of measurable functions on some measure space, and then work pointwise. How to do this will be explained later in the chapter.

4.3 General theory

In this section, we present several results on (stable) phase retrieval that are valid in general Banach lattices. We begin with the definitions:

Definition 4.3.1. Let E be a subspace of a vector lattice X. We say that E does *phase retrieval* if for each $f, g \in E$ with $|f| = |g|$ there is a scalar λ such that $f = \lambda g$.

Definition 4.3.2. Let E be a subspace of a real or complex Banach lattice X. We say that E does C*-stable phase retrieval* if for each $f, g \in E$ we have

$$\inf_{|\lambda|=1} \|f - \lambda g\| \leq C \||f| - |g|\|. \tag{4.3.1}$$

If E does C-stable phase retrieval for some C, we simply say that E does *stable phase retrieval* (*SPR* for short).

Note that if a subspace E of a real or complex Banach lattice X does C-stable phase retrieval, then so does its closure.

Connections with almost disjoint pairs and sequences

When considering whether a subspace $E \subseteq X$ does phase retrieval, there is one obvious obstruction. If $f, g \in E$ are non-zero disjoint vectors, then $|f - g| = |f + g| = |f| + |g|$, but $f - g$ cannot be a multiple of $f + g$. Hence, if E is to do phase retrieval, then it cannot contain disjoint pairs. Similarly, if E is to do stable phase retrieval, then it cannot contain "almost" disjoint pairs. As we will now see, in the real case, these are the *only* obstructions to (stable) phase retrieval.

Definition 4.3.3. Let E be a subspace of a real or complex Banach lattice X. We say that E contains ε*-almost disjoint pairs* if there are $f, g \in S_E$ (here and below, $S_E = \{e \in E : \|e\| = 1\}$ stands for the unit sphere of E) such that $\||f| \wedge |g|\| < \varepsilon$. If E contains ε-almost disjoint pairs for all $\varepsilon > 0$, we say that E contains *almost disjoint pairs.*

Theorem 4.3.4. Let E be a subspace of a Banach lattice X, $C \geq 1$ and $\varepsilon > 0$. Then,

(i) If E does C-stable phase retrieval, then it contains no $\frac{1}{C}$-almost disjoint pairs;

(ii) If E contains no ε-almost disjoint pairs, then it does $\frac{2}{\varepsilon}$-stable phase retrieval.

In particular, E does stable phase retrieval if and only if it does not contain almost disjoint pairs.

Proof. (i)$\Rightarrow$(ii): Suppose that E does C-stable phase retrieval, but there are $f, g \in E$ such that $\|f\| = \|g\| = 1$ but $\||f| \wedge |g|\| < \frac{1}{C}$. Define $h_1 = f + g$ and $h_2 = f - g$. Then since the identity

$$||f+g| - |f-g|| = 2(|f| \wedge |g|)$$

holds in any vector lattice by [12, Theorem 1.7], we have

$$\||h_1| - |h_2|\| = 2\||f| \wedge |g|\| < \frac{2}{C}.$$

On the other hand, $h_1 + h_2 = 2f$ has norm 2, and $h_1 - h_2 = 2g$ also has norm 2. This contradicts that E does C-stable phase retrieval.

(ii)$\Rightarrow$(i): A classical Banach lattice fact (see, e.g., [29, Remark after Lemma 3.3]) is that every Banach lattice embeds lattice isometrically into some space of the form

$$\left(\bigoplus_{i \in I} L_1(\Omega_i, \Sigma_i, \mu_i)\right)_\infty.$$

Since both stable phase retrieval and existence of almost disjoint pairs are invariant under passing to and from closed sublattices, we may assume without loss of generality that X is of this form.

Suppose E does not do $\frac{2}{\varepsilon}$-stable phase retrieval. Find $f = (f_i), g = (g_i) \in E$ such that $\|f - g\|, \|f + g\| > \frac{2}{\varepsilon}\||f| - |g|\|$. For each $i \in I$ let

$$I_i = \{t \in \Omega_i : \text{sign}\,(f_i(t)) = \text{sign}\,(g_i(t))\,, \text{ or one of } f_i(t), g_i(t) \text{ is zero}\}.$$

Then

$$I_i^c := \Omega_i \setminus I_i = \{t \in \Omega_i : \text{sign}\,(f_i(t)) = -\text{sign}\,(g_i(t))\}.$$

We compute that

$$|f| - |g| = (|f_i| - |g_i|)_{i \in I} = (|f_{i|I_i}| - |g_{i|I_i}|)_{i \in I} + (|f_{i|I_i^c}| - |g_{i|I_i^c}|)_{i \in I}.$$

So, since the modulus is additive on disjoint vectors,

$$\big||f| - |g|\big| = \left(\big||f_{i|I_i}| - |g_{i|I_i}|\big|\right)_{i \in I} + \left(\big||f_{i|I_i^c}| - |g_{i|I_i^c}|\big|\right)_{i \in I}.$$

Now, by definition of I_i we have

$$\left(\big||f_{i|I_i}| - |g_{i|I_i}|\big|\right)_{i \in I} = \left(\big|f_{i|I_i} - g_{i|I_i}\big|\right)_{i \in I}$$

and

$$\big(\big||f_{i|I_i^c}| - |g_{i|I_i^c}|\big|\big)_{i\in I} = \big(\big|f_{i|I_i^c} + g_{i|I_i^c}\big|\big)_{i\in I}.$$

Notice next that $d_1 := (f_{i|I_i^c} - g_{i|I_i^c})_{i\in I}$ and $d_2 := (f_{i|I_i} + g_{i|I_i})_{i\in I}$ are disjoint. Moreover,

$$\begin{aligned}\|f - g - (f_{i|I_i^c} - g_{i|I_i^c})_{i\in I}\| = \|(f_{i|I_i} - g_{i|I_i})_{i\in I}\| &= \big\|\big(\big||f_{i|I_i}| - |g_{i|I_i}|\big|\big)_{i\in I}\big\| \\ &\le \||f| - |g|\| < \frac{\varepsilon}{2}\|f - g\|.\end{aligned}$$

Similarly,

$$\begin{aligned}\|f + g - (f_{i|I_i} + g_{i|I_i})_{i\in I}\| = \|(f_{i|I_i^c} + g_{i|I_i^c})_{i\in I}\| &= \big\|\big(\big||f_{i|I_i^c}| - |g_{i|I_i^c}|\big|\big)_{i\in I}\big\| \\ &\le \||f| - |g|\| < \frac{\varepsilon}{2}\|f + g\|.\end{aligned}$$

By assumption, we have that both $f+g$ and $f-g$ are non-zero. Hence, by [12, Lemma 1.4], and the fact that $|d_1| \wedge |d_2| = 0$ we have

$$\begin{aligned}\frac{|f-g|}{\|f-g\|} \wedge \frac{|f+g|}{\|f+g\|} &\le \frac{|f-g-d_1|}{\|f-g\|} \wedge \frac{|f+g|}{\|f+g\|} + \frac{|d_1|}{\|f-g\|} \wedge \frac{|f+g|}{\|f+g\|} \\ &\le \frac{|f-g-d_1|}{\|f-g\|} \wedge \frac{|f+g|}{\|f+g\|} + \frac{|d_1|}{\|f-g\|} \wedge \frac{|f+g-d_2|}{\|f+g\|}.\end{aligned}$$

It follows that

$$\Big\|\frac{|f-g|}{\|f-g\|} \wedge \frac{|f+g|}{\|f+g\|}\Big\| \le \frac{\|f-g-d_1\|}{\|f-g\|} + \frac{\|f+g-d_2\|}{\|f+g\|} < \varepsilon.$$

Thus, we have constructed normalized ε-almost disjoint vectors $\frac{f+g}{\|f+g\|}$ and $\frac{f-g}{\|f-g\|}$ in E. $\square$

Remark 4.3.5. Implication (i) of Theorem 4.3.4 holds when the Banach lattice X is replaced by any vector lattice equipped with an absolute norm. Here, a norm on a vector lattice X is *absolute* if $\||f|\| = \|f\|$ for all $f \in X$; see [56, 214, 284] for more information. The proof of Theorem 4.3.4 also shows that a subspace of a Banach lattice does phase retrieval if and only if it does not contain disjoint non-zero vectors. A compactness argument then yields that in finite dimensions, phase retrieval implies stable phase retrieval. Indeed, consider the map $S_E \times S_E \to \mathbb{R}$, $(f,g) \mapsto \||f| \wedge |g|\|$. Then this map is continuous, so its image is compact, which allows one to conclude that the existence of almost disjoint pairs implies the existence of a disjoint pair. In infinite dimensions, it is relatively easy to construct subspaces doing phase retrieval but failing stable phase retrieval.

Proposition 4.3.6. Every infinite dimensional Banach lattice has a closed subspace which does phase retrieval but not stable phase retrieval.

Proof. By [11, p. 46, Exercise 13], any infinite dimensional Banach lattice X contains a normalized disjoint positive sequence, which we shall index as consisting of vectors $(u_i)_{i\in\mathbb{N}}$ and $(v_s)_{s\in\mathcal{S}}$; here, $\mathcal{S}$ denotes the set of all two-element subsets of $\mathbb{N}$ (the order is not important). We fix an injection $\phi : \mathbb{N}^2 \to \mathbb{N}$ and consider the vectors

$$f_i = u_i + \sum_{j\neq i} 2^{-4\phi(i,j)} v_{\{i,j\}}.$$

The sum above converges, and we have

$$\|u_i - f_i\| \leq \varepsilon_i, \text{ where } \varepsilon_i = \sum_j 2^{-4\phi(i,j)}.$$

Then $\sum_i \varepsilon_i = \sum_{i,j} 2^{-4\phi(i,j)} \leq \sum_m 2^{-4m} = 1/15$, hence, by [9, Theorem 1.3.9], (f_i) is a Schauder basic sequence. Also, $1 \leq \|f_i\| \leq 16/15$ for each i, so this basis is semi-normalized. We shall show that $E = \overline{\mathrm{span}}[f_i : i \in \mathbb{N}]$ fails stable phase retrieval, but has phase retrieval.

To show the failure of SPR, let, for $i \neq j$, $\psi(i,j) = \max\{\phi(i,j), \phi(j,i)\}$. Clearly $\psi(i,j) = \psi(j,i)$, and $\lim_j \psi(i,j) = \infty$ for any i. Note that $f_i \wedge f_j = 2^{-4\psi(i,j)} v_{\{i,j\}}$, hence

$$\|f_i \wedge f_j\| = 2^{-4\psi(i,j)} \underset{i,j\to\infty}{\longrightarrow} 0.$$

Next we show that E does phase retrieval. Pick non-zero $f, g \in E$, with $|f| = |g|$; we have to show that $f = \pm g$. To this end, write $f = \sum_i a_i f_i$ and $g = \sum_i b_i f_i$. We can expand

$$f = \sum_i a_i u_i + \sum_{\{i,j\}\in\mathcal{S}} \big(a_i 2^{-4\phi(i,j)} + a_j 2^{-4\phi(j,i)}\big) v_{\{i,j\}},$$

and likewise for g. Comparing the coefficients with u_i, we conclude that, for every i, $|a_i| = |b_i|$. By switching signs in front of f and g, and by re-indexing, we can assume that $a_1 = b_1 > 0$. We have to show that the equality $a_i = b_i$ holds for every $i > 1$.

The preceding reasoning shows that $a_i = 0$ iff $b_i = 0$. Suppose both a_i and b_i are different from 0. Comparing the coefficients with $v_{\{i,j\}}$, we see that

$$\big|2^{-4\phi(1,i)} a_1 + 2^{-4\phi(i,1)} a_i\big| = \big|2^{-4\phi(1,i)} b_1 + 2^{-4\phi(i,1)} b_i\big|,$$

which is only possible if $\mathrm{sign}\, a_i = \mathrm{sign}\, b_i$. □

Example 4.3.7. Theorem 4.3.4 fails for complex spaces. Indeed, define E as the complex span of $\{(1,1,1),(i,1,-1)\} \subseteq \mathbb{C}^3$, where we equip $\mathbb{C}^3$ with the modulus $|(a,b,c)| := (|a|,|b|,|c|)$. Clearly, E contains vectors f,g with $|f|=|g|$ but such that $f-\lambda g$ is not zero for any $\lambda \in \mathbb{C}$. Hence, E fails phase retrieval. However, one can easily compute that E contains no disjoint vectors, which by compactness yields the non-existence of almost disjoint vectors. Moreover, as observed in [82], a complex subspace that contains two linearly independent real vectors cannot do complex phase retrieval. In particular, if $E \subseteq X$ is subspace of a Banach lattice X with $\dim E \geq 2$, then the canonical subspace $E_{\mathbb{C}} \subseteq X_{\mathbb{C}}$ fails to do phase retrieval.

Remark 4.3.8. Theorem 4.3.4 shows that for real scalars, the study of subspaces doing stable phase retrieval is equivalent to the study of subspaces lacking almost disjoint pairs. As mentioned in Section 4.2, there is a vast literature on closed subspaces lacking almost disjoint normalized *sequences*. Clearly, if E contains an almost disjoint normalized sequence, then it fails to do stable phase retrieval. However, the converse is not true. For example, the standard Rademacher sequence (r_n) in $L_p[0,1]$, $1 \leq p < \infty$, is dispersed by Khintchine's inequality, but $|r_n| \equiv 1$ for all n. Moreover, if one adds a single disjoint vector to a dispersed subspace, one produces a dispersed subspace failing phase retrieval. Nevertheless, as mentioned in Section 4.1, many of the results in Theorem 4.2.1 have SPR analogues.

Hölder stable phase retrieval and witnessing failure of SPR on orthogonal vectors

In [82] (see also Section 4.7 below, where these results are recalled), the following terminology was introduced in the setting of L_p-spaces: A subspace E of a real or complex Banach lattice X is said to do γ-*Hölder stable phase retrieval with constant* C if for all $f,g \in E$ we have

$$\inf_{|\lambda|=1} \|f-\lambda g\|_X \leq C \||f|-|g|\|_X^{\gamma} \left(\|f\|_X + \|g\|_X\right)^{1-\gamma}. \tag{4.3.2}$$

The utility of this definition arose from a construction in [82] of SPR subspaces of $L_4(\mu)$ which are dispersed in $L_6(\mu)$. Applying certain Hölder inequality arguments, [82] was then able to deduce that such subspaces do $\frac{1}{4}$-Hölder stable phase retrieval in $L_2(\mu)$. The idea in [82] is to begin with an orthonormal sequence (r_k), and instead of comparing $|f|$ to $|g|$, one compares $|f|^2$ to $|g|^2$. Assuming the integrability condition $r_k \in L_4(\mu)$ with uniformly bounded norm, and various orthogonality and mean-zero conditions on the products $r_k \overline{r}_j$, the orthogonal expansion $f = \sum_k a_k r_k$ leads to an orthogonal expansion

$$|f|^2 = \sum_{k\neq j} a_k \overline{a}_j r_k \overline{r}_j + \sum_k |a_k|^2 s_k + \|f\|_{L_2}^2 \mathbb{1},\ s_k = |r_k|^2 - \mathbb{1}.$$

The products $r_k\overline{r}_j$ encode how the subspace "sits" in $L_4(\mu)$, i.e., they encode the lattice structure. However, analyzing $|f|^2$ rather than $|f|$ allows one to work algebraically. As was shown in [82], if one imposes appropriate orthogonality conditions, the subspace E spanned by r_k will do stable phase retrieval in $L_4(\mu)$. [82] then gives examples of such r_k built from dilates of a single function P, with $|P|$ not identically constant. Verifying that such sequences (r_k) satisfy the required orthogonality conditions is then a combinatorial exercise, using sparseness of the dilates to get non-overlapping supports with respect to the basis expansion. This sparseness naturally leads to E lying in higher L_p-spaces, so that by interpolating, one concludes that E does Hölder stable phase retrieval in $L_2(\mu)$ with $\gamma = \frac{1}{4}$ if $p = 6$, and $\gamma \to \frac{1}{2}$ as $p \to \infty$.

The purpose of this section is to show that - at the cost of dilating the constant - Hölder stable phase retrieval is equivalent to stable phase retrieval. For real scalars, this can already be deduced from the almost disjoint pair characterization in Theorem 4.3.4. However, the proof below works equally well for complex scalars. The following theorem was proven in [10] for phase retrieval using a continuous frame for a Hilbert space. We extend it here to subspaces of Banach lattices.

Theorem 4.3.9. Let $(X, \|\cdot\|)$ be a Banach lattice, real or complex. Fix linearly independent $f, g \in X$, and suppose that $Y = \operatorname{span}\{f, g\}$ is equipped with a Hilbert space norm $\|\cdot\|_H$, which is K-equivalent to $\|\cdot\|$. Then there exist $f', g' \in Y$ so that

$$\min_{|\lambda|=1} \|f - \lambda g\| \leq K \min_{|\lambda|=1} \|f' - \lambda g'\|, \tag{4.3.3}$$

and

$$(\|f'\|^2 + \|g'\|^2)^{\frac{1}{2}} \leq K \min_{|\lambda|=1} \|f' - \lambda g'\|, \tag{4.3.4}$$

and

$$\||f| - |g|\| \geq \||f'| - |g'|\|. \tag{4.3.5}$$

Remark 4.3.10. Conditions (4.3.3) and (4.3.5) state that replacing (f, g) by (f', g') tightens the SPR inequality up to the universal factor K. The condition (4.3.4) states that f' and g' are "almost orthogonal" (in fact, the proof shows that they are orthogonal in the

Hilbert space $H = (Y, \|\cdot\|_H)$); it also permits us to witness the failure of SPR on f', g' with controlled norm.

In general, by John's Theorem, every 2-dimensional space is $\sqrt{2}$-equivalent to a Hilbert space, but in certain cases a better estimate can be obtained. For instance, if $X = L_2(\mu)$, then for the inherited norm on Y we have $K = 1$ and f' orthogonal to g'. If X is a Banach lattice, which is r-convex and s-concave ($1 < r \leq 2 < s < \infty$) with constants $M^{(r)}(X)$ and $M_{(s)}(X)$ respectively, then, by [319, Theorem 28.6], there exists $\|\cdot\|_H$ for which

$$K \leq M^{(r)}(X)M_{(s)}(X)2^{\alpha}, \ \text{ where } \alpha = \max\left\{\frac{1}{r} - \frac{1}{2}, \frac{1}{2} - \frac{1}{s}\right\}.$$

In particular, for $X = L_p$, there exists $\|\cdot\|_H$ for which $K \leq 2^{|1/p-1/2|}$. [319, Corollary 28.7] provides similar results for operator ideals (Schatten spaces).

In certain applications of Theorem 4.3.9 (such as Theorem 4.5.3), the norm $\|\cdot\|_H$ arises not from the Hilbert space with the minimal Banach-Mazur distance to Y, but from an equivalent Euclidean norm on some subspace E (with $Y \subseteq E \subseteq X$). In this setting K may exceed $\sqrt{2}$.

To prove Theorem 4.3.9, we need to represent elements of X as measurable functions. As mentioned in the proof of Theorem 4.3.4, every (real) Banach lattice X embeds lattice isometrically into a space of the form $\left(\bigoplus_{i\in I} L_1(\Omega_i, \Sigma_i, \mu_i)\right)_\infty$. Hence, throughout the proof we can assume that elements of X are functions on a measure space. In the complex case, a similar reduction is possible. Indeed, let X be a complex Banach lattice. By the discussion in Section 4.2, we can assume that $X = Z_{\mathbb{C}}$ is the complexification of some (real) Banach lattice Z. We can then let $T : Z \to \left(\bigoplus_{i\in I} L_1(\Omega_i, \Sigma_i, \mu_i)\right)_\infty$ be a lattice isometric embedding. The complexification $T_{\mathbb{C}}$ maps X into the complexification of $\left(\bigoplus_{i\in I} L_1(\Omega_i, \Sigma_i, \mu_i)\right)_\infty$. The codomain of this map is still $\left(\bigoplus_{i\in I} L_1(\Omega_i, \Sigma_i, \mu_i)\right)_\infty$, but now interpreted as a Banach lattice over the complex field (cf. [2, Exercises 3 and 5 on page 110]). Since T is one-to-one, the definition of $T_{\mathbb{C}}$ tells us that $T_{\mathbb{C}}$ is one-to-one. Moreover, as mentioned in Section 4.2, $T_{\mathbb{C}}$ preserves moduli. Finally, by [2, Lemma 3.18 or Corollary 3.23], $T_{\mathbb{C}}$ preserves norm. Thus, everything in the SPR inequality is preserved, so, analogously to the real case, we may assume throughout the proof that the complex Banach lattice X is a space of complex-valued functions.

Proof of Theorem 4.3.9. By scaling, we assume that, on Y, $\|\cdot\| \leq \|\cdot\|_H \leq K\|\cdot\|$. By replacing g by a unimodular scalar times g, we assume

$$\min_{|\lambda|=1} \|f - \lambda g\|_H = \|f - g\|_H.$$

This latter condition is equivalent to $\langle f, g\rangle \geq 0$. Indeed,

$$\|f - \lambda g\|_H^2 = \langle f, f\rangle + \langle g, g\rangle - 2\Re\left(\overline{\lambda}\langle f, g\rangle\right).$$

This is minimized when $\overline{\lambda}$ is the conjugate phase of $\langle f, g\rangle$. This is minimized when $\lambda = 1$ iff $\langle f, g\rangle \geq 0$.

Consider $f_r := f - r(f+g)$ and $g_r := g - r(f+g)$ for $r \in [0, 1/2]$. We let R be the first instance of $\langle f - r(f+g), g - r(f+g)\rangle = 0$. This is possible since when $r = 0$, the inner product is non-negative, and when $r = \frac{1}{2}$, it is negative. Note that

$$\|f_r - g_r\|_H = \|f - g\|_H.$$

Thus, since f_R and g_R are orthogonal,

$$\min_{|\lambda|=1} \|f_R - \lambda g_R\|_H = \min_{|\lambda|=1} \|f - \lambda g\|_H.$$

We will take $f' = f_R$ and $g' = g_R$. To see (4.3.3), we compute

$$\min_{|\lambda|=1} \|f - \lambda g\| \leq \min_{|\lambda|=1} \|f - \lambda g\|_H = \min_{|\lambda|=1} \|f' - \lambda g'\|_H \leq K \min_{|\lambda|=1} \|f' - \lambda g'\|. \tag{4.3.6}$$

Moreover, as f' and g' are orthogonal in H,

$$K \min_{|\lambda|=1} \|f' - \lambda g'\| \geq \min_{|\lambda|=1} \|f' - \lambda g'\|_H = (\|f'\|_H^2 + \|g'\|_H^2)^{\frac{1}{2}} \geq (\|f'\|^2 + \|g'\|^2)^{\frac{1}{2}}. \tag{4.3.7}$$

This gives (4.3.4).

We now verify (4.3.5). To see this, we prove

$$\||f_r| - |g_r|\| \leq \||f| - |g|\| \text{ for } r \in [0, \frac{1}{2}]. \tag{4.3.8}$$

We represent $X \subseteq L^0(\Omega)$ and let $t \in \Omega$. We will prove that

$$||f_r(t)| - |g_r(t)|| \leq ||f(t)| - |g(t)|| \text{ for } r \in [0, \frac{1}{2}]. \tag{4.3.9}$$

Note that (4.3.9) is simply a claim that an elementary inequality holds for complex numbers. Write $f(t) = a + ib$ and $g(t) = c + id$. Multiplying $f(t)$ and $g(t)$ by a unimodular scalar, we rotate so that $d = -b$. WLOG, $|a| \geq |c|$; then, multiplying by -1 if necessary, we also assume $a \geq 0$. We have

$$f_r(t) = a - r(a+c) + ib, \quad g_r(t) = c - r(a+c) - ib.$$

Now, we note that our assumptions give $||f_r(t)| - |g_r(t)|| = |f_r(t)| - |g_r(t)|$ for $0 \leq r \leq \frac{1}{2}$. Indeed, $\Im(f_r(t)) = -\Im(g_r(t))$ and $(\Re(f_r(t)))^2 \geq (\Re(g_r(t)))^2$ for $0 \leq r \leq \frac{1}{2}$ by elementary computations. Taking $r = 0$, $||f(t)| - |g(t)|| = |f(t)| - |g(t)|$. Hence, we must prove

$$|f_r(t)| - |g_r(t)| \leq |f(t)| - |g(t)| \text{ for } r \in [0, \frac{1}{2}].$$

This inequality is true for all $r \geq 0$. Indeed, recall first that $a \geq c$. By the Fundamental Theorem of Calculus, for any convex function ϕ and $w \geq 0$, we have $\phi(a-w) - \phi(c-w) \leq \phi(a) - \phi(c)$. In our case, the function $h(s) = \sqrt{s^2 + b^2}$ is convex and $r(a+c) \geq 0$; therefore,

$$|f_r(t)| - |g_r(t)| = h(a - r(a+c)) - h(c - r(a+c)) \leq h(a) - h(c) = |f(t)| - |g(t)|.$$

□

Corollary 4.3.11. Let E be a subspace of a real or complex Banach lattice X, and $\gamma \in (0,1]$. If E does γ-Hölder stable phase retrieval in X with constant $C > 0$ then E does stable phase retrieval in X with constant $\sqrt{2}(\sqrt{8}C)^{\frac{1}{\gamma}}$.

Proof. Let $f, g \in E$ with $\|f\| = 1$ and $\|g\| \leq 1$ such that

$$(\|f\|^2 + \|g\|^2)^{\frac{1}{2}} \leq \sqrt{2} \inf_{|\lambda|=1} \|f - \lambda g\|. \tag{4.3.10}$$

In particular,

$$2^{-1/2} \leq \inf_{|\lambda|=1} \|f - \lambda g\| \leq 2.$$

As E does C-stable γ-Hölder phase retrieval, we have that

$$2^{-1/2} \leq \inf_{|\lambda|=1} \|f - \lambda g\| \leq C\||f| - |g|\|^{\gamma}(\|f\| + \|g\|)^{1-\gamma} \leq 2^{1-\gamma}C\||f| - |g|\|^{\gamma}. \tag{4.3.11}$$

Thus, we have that $C^{1/\gamma}2^{3/(2\gamma)-1}\||f| - |g|\| \geq 1$ and $\inf_{|\lambda|=1} \|f - \lambda g\| \leq 2$. It follows that

$$\inf_{|\lambda|=1} \|f - \lambda g\| \leq (2^{3/2}C)^{1/\gamma}\||f| - |g|\|. \tag{4.3.12}$$

To prove (4.3.12) we have assumed that $\|f\| = 1$ and $\|g\| \leq 1$. However, by scaling we have that any $f, g \in E$ which satisfy (4.3.10) also satisfy (4.3.12).

We now consider any pair of linearly independent vectors $x, y \in E$. By Theorem 4.3.9 there exists $f, g \in E$ which satisfy (4.3.10) such that

$$\min_{|\lambda|=1} \|x - \lambda y\| \leq \sqrt{2} \min_{|\lambda|=1} \|f - \lambda g\| \text{ and } \||x| - |y|\| \geq \||f| - |g|\|.$$

Thus, we have that

$$\min_{|\lambda|=1} \|x - \lambda y\| \leq 2^{1/2}(2^{3/2}C)^{1/\gamma}\||x| - |y|\|.$$

This proves that E does $2^{1/2}(2^{3/2}C)^{1/\gamma}$-stable phase retrieval. $\square$

Remark 4.3.12. The constant $\sqrt{2}(\sqrt{8}C)^{\frac{1}{\gamma}}$ in Corollary 4.3.11 arises by using the worst case scenario $K = \sqrt{2}$ from Theorem 4.3.9. This constant can certainly be optimized; for example, if one also takes into account the distance from E to a Hilbert space.

To conclude this section we give a simple proof that in finite dimensions, phase retrieval is automatically stable.

Corollary 4.3.13. Let X be a real or complex Banach lattice, and E a finite dimensional subspace of X. If E does phase retrieval, then E does stable phase retrieval.

Proof. The real case has already been dealt with in Remark 4.3.5, but the argument we provide below works for both real and complex scalars. Indeed, by Theorem 4.3.9, if E fails to do stable phase retrieval then we can find, for each $N \in \mathbb{N}$, functions f_N, g_N with $\|f_N\| = 1$, $\|g_N\| \leq 1$,

$$(\|f_N\|^2 + \|g_N\|^2)^{\frac{1}{2}} \leq \sqrt{2} \min_{|\lambda|=1} \|f_N - \lambda g_N\|, \tag{4.3.13}$$

and

$$2 \geq \min_{|\lambda|=1} \|f_N - \lambda g_N\| > N\||f_N| - |g_N|\|. \tag{4.3.14}$$

By compactness, after passing to subsequences, we may assume that $f_N \xrightarrow{\|\cdot\|} f$ and $g_N \xrightarrow{\|\cdot\|} g$, for some $f, g \in E$. Since $\|f_N\| = 1$ for all N, it follows that $\|f\| = 1$. Moreover, from (4.3.14) and continuity of lattice operations, we see that $\||f| - |g|\| = 0$. Hence, $|f| = |g| \neq 0$. Fix a phase λ. By (4.3.13), we have

$$(\|f_N\|^2 + \|g_N\|^2)^{\frac{1}{2}} \leq \sqrt{2}\|f_N - \lambda g_N\|.$$

Passing to the limit, we see that

$$1 \leq (\|f\|^2 + \|g\|^2)^{\frac{1}{2}} \leq \sqrt{2}\|f - \lambda g\|.$$

Hence, $f \neq \lambda g$. It follows that E fails to do phase retrieval. $\square$

Remark 4.3.14. Note that the Banach lattice X in Corollary 4.3.13 is not assumed to be finite dimensional. This is of some note, as, unlike for closed spans, the closed sublattice generated by a finite set can be infinite dimensional.

4.4 Examples

Building SPR subspaces via isometric theory

As mentioned in Theorem 4.2.1, when $1 \leq p < 2$ and $q \in (p, 2]$, one can find isometric copies of $L_q[0,1]$ in $L_p[0,1]$. As we will now see, such subspaces must do SPR.

Proposition 4.4.1. Suppose $p, q \in [1, \infty)$, and either (1) $1 \leq p < q \leq 2$, or (2) $q = 2 < p < \infty$. There exists an $\varepsilon > 0$ such that if $E \subseteq L_p[0,1]$ is $(1+\varepsilon)$-isomorphic to $F \subseteq L_q[0,1]$, then E does SPR in $L_p[0,1]$.

Proof. We only handle case (1), as (2) is very similar. Suppose, for the sake of contradiction, that E fails SPR. Then by Theorem 4.3.4, E contains c-isomorphic copies of ℓ_p^2, for any $c > 1$. Consequently, for any such c we can find norm one $f, g \in E$ so that $\|f + g\|_{L_p}, \|f - g\|_{L_p} \geq c^{-1}2^{1/p}$. However, by the Clarkson inequality in L_q,

$$\|f + g\|_{L_q}^{q'} + \|f - g\|_{L_q}^{q'} \leq 2(\|f\|_{L_q}^q + \|g\|_{L_q}^q)^{q'-1} \leq (1+\varepsilon)^{q'} 2^{q'},$$

where $1/q + 1/q' = 1$. However, the left side is $\geq c^{-q'}2^{1+q'/p}$, and it is easy to see that $1 + q'/p > q'$. Hence, we get a contradiction if $\varepsilon > 0$ is sufficiently small. $\square$

Corollary 4.4.2. If either $1 \leq p < q \leq 2$, or $q = 2 < p < \infty$, then $L_p[0,1]$ contains an SPR subspace isometric to $L_q[0,1]$.

Proof. It is well known (see e.g. [190, Section 9]) that, under the above conditions, $L_p[0,1]$ contains an isometric copy of $L_q[0,1]$. By Proposition 4.4.1, that copy does SPR. $\square$

Existence of SPR embeddings into sequence spaces

Proposition 4.4.3. If a Banach space E embeds into $\ell_\infty(\alpha)$ for some infinite cardinal α (which happens, in particular, when E itself has density character α), then there is an isomorphic SPR embedding of E inside of $\ell_\infty(\alpha)$.

The fact that any Banach space E of density character α embeds isometrically into $\ell_\infty(\alpha)$ is standard. We recall the construction for the sake of completeness: Let $(x_i)_{i\in I}$ be a dense subset of E of cardinality α; for each i find $x_i^* \in S_{E^*}$ so that $x_i^*(x_i) = \|x_i\|$. Then $E \to \ell_\infty(\alpha) : x \mapsto (x_i^*(x))_{i\in I}$ is the desired embedding. Similarly, one can show that if E is a dual space, with a predual of density character α, then E embeds isometrically into $\ell_\infty(\alpha)$.

To establish Proposition 4.4.3, it therefore suffices to prove:

Lemma 4.4.4. For any infinite cardinal α, there exists an isometric SPR embedding of $\ell_\infty(\alpha)$ into itself.

To prove Lemma 4.4.4, we rely on the following.

Lemma 4.4.5. Suppose E is a (real or complex) Banach space, and $x, y \in E$ have norm 1. Then there exists a norm 1 functional $f \in E^*$ so that $|f(x)| \wedge |f(y)| \geq 1/5$.

Proof. Suppose first that $\mathrm{dist}\,(y, \mathbb{F}x) \leq 2/5$ (here $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$). Find $t \in \mathbb{F}$ so that $\|y - tx\| \leq 2/5$. By the triangle inequality, $|t| \geq 3/5$. Find $f \in E^*$ so that $\|f\| = 1 = f(x)$. Then $|f(y)| \geq |t||f(x)| - \|y - tx\| \geq 1/5$. The case of $\mathrm{dist}\,(x, \mathbb{F}y) \leq 2/5$ is handled similarly.

Now suppose $\mathrm{dist}\,(x, \mathbb{F}y), \mathrm{dist}\,(y, \mathbb{F}x) > 2/5$. By Hahn-Banach Theorem, there exist norm one $g, h \in E^*$ so that $g(x) \geq 2/5$, $g(y) = 0$, $h(y) \geq 2/5$, and $h(x) = 0$. Then $f := (g + h)/\|g + h\|$ has the desired properties. Indeed, $\|g + h\| \leq 2$, hence

$$|f(x)| \geq \frac{1}{2}\big(|g(x)| - |h(x)|\big) \geq \frac{1}{5},$$

and likewise, $|f(y)| \geq 1/5$. □

Proof of Lemma 4.4.4. For the sake of brevity, we shall use the notation $E = \ell_\infty(\alpha)$, and $E_* = \ell_1(\alpha)$. Pick a dense set $(f_i)_{i\in I}$ in S_{E_*}, with $|I| = \alpha$. Define an isometric embedding $J : E \to \ell_\infty(I) : x \mapsto (f_i(x))_{i\in I}$. We shall show that, for every $x, y \in S_E$ and $\varepsilon > 0$, there exists i so that $|f_i(x)| \wedge |f_i(y)| \geq 1/5 - \varepsilon$. Once this is done, we will conclude that $\||Jx| \wedge |Jy|\| \geq 1/5$ for any $x, y \in S_E$, which by Theorem 4.3.4 tells us that J is indeed an

SPR embedding.

By Lemma 4.4.5, there exists $f \in S_{E^*}$ so that $|f(x)| \wedge |f(y)| \geq 1/5$. By Goldstine's Theorem, there exists $f' \in S_{E_*}$ so that $|f'(x)| \wedge |f'(y)| \geq 1/5 - \varepsilon/2$. Find i so that $\|f' - f_i\| \leq \varepsilon/2$. Then

$$|f_i(x)| \wedge |f_i(y)| \geq |f'(x)| \wedge |f'(y)| - \|f' - f_i\| \geq \frac{1}{5} - \varepsilon,$$

which proves our claim. □

Remark 4.4.6. We can define the canonical embedding of E into $C(B_{E^*})$ (with $B_{E^*} = \{e^* \in E^* : \|e^*\| \leq 1\}$ equipped with its weak* topology) by sending $e \in E$ to the function $e^* \mapsto e^*(e)$. The above reasoning shows that this embedding is SPR. For separable E, more can be said - see Proposition 4.6.2 below.

Remark 4.4.7. If an atomic lattice is order continuous (which ℓ_∞ of course is not), then the "gliding hump" argument shows the non-existence of infinite dimensional dispersed subspaces. The lattice c is not order continuous, but it has no infinite dimensional dispersed subspaces. This is because c contains c_0 as a subspace of finite codimension, hence any infinite dimensional subspace of c has an infinite dimensional intersection with c_0.

Combining the results from this and the previous subsection, we see that, often, the collection of dispersed subspaces of a Banach lattice coincides with those that do SPR, up to isomorphism. Indeed, we have the following:

Corollary 4.4.8. For every dispersed subspace $E \subseteq L_p[0,1]$ $(1 \leq p \leq \infty)$, there exists a closed subspace $E' \subseteq L_p[0,1]$ isomorphic to E, and doing stable phase retrieval. The same result holds with $L_p[0,1]$ replaced by $C[0,1]$, $C(\Delta)$, c or any order continuous atomic Banach lattice.

Proof. By Theorem 4.2.1, for $1 \leq p < \infty$ and $p \neq 2$, a closed subspace of $L_p[0,1]$ is dispersed if and only if it contains no subspace isomorphic to ℓ_p. A result of Rosenthal [289] states that for $1 \leq p < 2$, a subspace of $L_p[0,1]$ that does not contain ℓ_p must be isomorphic to a subspace of L_r for some $r \in (p, 2]$. By Corollary 4.4.2, one can build an SPR copy of L_r in L_p.

In the case $2 \leq p < \infty$, Theorem 4.2.1 states that any dispersed subspace of $L_p[0,1]$ must be isomorphic to a Hilbert space. By Corollary 4.4.2, $L_p[0,1]$ contains an SPR copy of ℓ_2. To deal with the case $p = \infty$, note that $L_\infty[0,1]$ is isomorphic (as a Banach space) to ℓ_∞, and use Lemma 4.4.4 together with the fact that ℓ_∞ lattice isometrically embeds in $L_\infty[0,1]$.

For order continuous atomic lattices and c, there are no infinite dimensional dispersed subspaces by Remark 4.4.7. The claim for $C[0,1]$ and $C(\Delta)$ will be proven in Proposition 4.6.2 below, when we analyze SPR subspaces of $C(K)$-spaces. As we will see in the proof of Proposition 4.6.2, the fact that every separable Banach space embeds into $C[0,1]$ and $C(\Delta)$ in an SPR fashion ultimately follows from Remark 4.4.6. $\square$

Explicit constructions of SPR subspaces using random variables

In this subsection, we construct SPR subspaces of a rather general class of function spaces by considering the closed span of certain independent random variables. The use of sub-Gaussian random vectors has been widely successful in building random frames for finite dimensional Hilbert spaces which do stable phase retrieval whose stability bound is independent of the dimension [72, 73, 106, 213, 212]. However, different distributions for random variables will allow for the construction of subspaces which do stable phase retrieval and are not isomorphic to Hilbert spaces. We begin by presenting a technical criterion for SPR.

Proposition 4.4.9. Suppose X is a Banach lattice of measurable functions on a probability measure space (Ω, μ) which contains the indicator functions and has the property that for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ so that $\|\chi_S\| > \delta$ whenever $\mu(S) > \varepsilon$. Suppose, furthermore, that E is a subspace of X, which has the following property: There exist $\alpha > 1/2$ and $\beta > 0$ so that, for any norm one $f \in E$, we have

$$\mu\left(\left\{\omega \in \Omega : |f(\omega)| \geq \beta\right\}\right) \geq \alpha. \tag{4.4.1}$$

Then E is an SPR-subspace.

Proof. Suppose $f, g \in E$ have norm 1. By the Inclusion-Exclusion Principle,

$$\mu\left(\left\{\omega \in \Omega : |f(\omega)| \geq \beta, |g(\omega)| \geq \beta\right\}\right) \geq 2\alpha - 1.$$

Thus, $\||f| \wedge |g|\| \geq \beta\delta(2\alpha - 1)$. $\square$

The above proposition is applicable, for instance, when X is a rearrangement invariant (r.i. for short; see [231] for an in-depth treatment) space on $(0,1)$, equipped with the canonical Lebesgue measure λ. Examples include L_p spaces, and, more generally, Lorentz and Orlicz spaces (once again, described in great detail in [231]; for Lorentz spaces, see also [101]). Below we describe some SPR subspaces, spanned by independent identically distributed random variables.

Suppose f is a random variable, realized as a measurable function on $(0,1)$ (with the usual Lebesgue measure λ). Then independent copies of f – denoted by $f_1, f_2, \ldots$ – can be realized on $((0,1),\lambda)^{\aleph_0}$. By Caratheodory's Theorem (see e.g. [219, p. 121]), there exists a measure-preserving bijection between $((0,1),\lambda)^{\aleph_0}$ and $((0,1),\lambda)$. Therefore, we view $f_1, f_2, \ldots$ as functions on $(0,1)$.

Suppose now that, in the above setting, the following statements hold:

(i) f belongs to X, and has norm one in that space;

(ii) There exists r so that, if $f_1, \ldots, f_n$ are independent copies of f, and $\sum_i |a_i|^r = 1$, then $\sum_i a_i f_i$ is equidistributed with f;

(iii) There exists $\beta > 0$ so that $\mathbb{P}(|f| > \beta) > 1/2$.

In this situation, if $f_1, f_2, \ldots$ are independent copies of f (viewed as elements of X, per the preceding paragraph), then $\overline{\mathrm{span}}[f_i : i \in \mathbb{N}]$ is an SPR copy of ℓ_r in X.

We should mention two examples of random variables with the above properties: Gaussian ((ii) holds with $r = 2$) and q-stable ($q \in (1,2)$; (ii) holds with $r = q$). The details can be found in [9, Section 6.4]. For the Gaussian variables, the probability density function is $d_f(x) = ce^{-x^2/2}$, with c depending on the normalization. For the q-stable variables with characteristic function $t \mapsto ce^{-|t|^q}$ (with c ensuring normalization), the Fourier inversion formula gives the density function

$$d_f(x) = \frac{c}{\pi} \int_0^\infty \cos(tx) e^{-t^q} \, dt.$$

In both cases, d_f is continuous (in the latter case, due to Dominated Convergence Theorem), hence there exists $\beta > 0$ so that

$$\mathbb{P}(|f| > \beta) = 1 - \int_{-\beta}^{\beta} d_f > \frac{3}{4}.$$

It is known that Gaussian random variables belong to L_p for $p \in [1,\infty)$, while the r-stable random variables ($1 < r < 2$) lie in L_p if and only $p \in [1, r)$. Moreover, the results from

[231, p. 142-143] tell us that $L_s(0,1) \subset L_{p,q}(0,1)$ for $s > p$ (this is a continuous inclusion, not an isomorphic embedding). If $r > p$, then the r-stable variables belong to $L_{p,q}(0,1)$ (indeed, take $s \in (p,r)$; then the r-stable variables live in $L_s(0,1)$, which in turn sits inside of $L_{p,q}(0,1)$). Likewise, one shows that any Lorentz space $L_{p,q}(0,1)$ contains Gaussian random variables.

The above reasoning implies:

Proposition 4.4.10. Suppose $1 \leq p < \infty$ and $1 \leq q \leq \infty$ (when $p = 1$, assume in addition $q < \infty$). Then $L_{p,q}(0,1)$ contains a copy of ℓ_2 that does SPR. If, in addition, $1 \leq p < r < 2$, then $L_{p,q}(0,1)$ contains a copy of ℓ_r that does SPR.

Stability of SPR subspaces under ultraproducts and small perturbations

We show that SPR subspaces are stable under ultraproducts, and under small perturbations (in the sense of Hausdorff distance). These results hold for both real and complex spaces.

Proposition 4.4.11. Suppose $\mathfrak{U}$ is an ultrafilter on a set I, and, for each $i \in I$, E_i is a C-SPR subspace of a Banach lattice X_i. Then $\prod_{\mathfrak{U}} E_i$ is a C-SPR subspace of $\prod_{\mathfrak{U}} X_i$.

We refer the reader to [166] or [97, Chapter 8] for information on ultraproducts of Banach spaces and Banach lattices.

Proof. We have to show that, for any $x, y \in \prod_{\mathfrak{U}} E_i$, there exists a modulus one λ so that $\|x - \lambda y\| \leq C\||x| - |y|\|$. To this end, find families (x_i) and (y_i), representing x and y respectively. Then for each i there exists λ_i so that $|\lambda_i| = 1$ and $\|x_i - \lambda_i y_i\| \leq C\||x_i| - |y_i|\|$. As ultraproducts preserve lattice operations, $|x|$ and $|y|$ are represented by $(|x_i|)$ and $(|y_i|)$, respectively, hence $\||x| - |y|\| = \lim_{\mathfrak{U}} \||x_i| - |y_i|\|$. By the compactness of the unit torus, there exists $\lambda = \lim_{\mathfrak{U}} \lambda_i$, with $|\lambda| = 1$. Then $\|x - \lambda y\| = \lim_{\mathfrak{U}} \|x_i - \lambda_i y_i\|$, which leads to the desired inequality. $\square$

Remark 4.4.12. Proposition 4.4.11 can be used to give an alternative proof of Corollary 4.4.2. First find a family of finite dimensional subspaces $F_k \subseteq L_q(0,1)$, ordered by inclusion, so that $\cup_k F_k$ is dense in $L_q(0,1)$, and each F_k is isometric to $\ell_q^{n_k}$ for some n_k (one can, for instance, take subspaces spanned by certain step functions). A reasoning similar to that of [97, Theorem 8.8] permits us to find a free ultrafilter $\mathfrak{U}$ so that $\prod_{\mathfrak{U}} F_k$ contains an isometric copy of $L_q(0,1)$. *A fortiori*, $\prod_{\mathfrak{U}} \ell_q$ contains an isometric copy of $L_q(0,1)$ (call it E).

Proposition 4.4.10 proves that $L_p(0,1)$ contains a subspace, isometric to ℓ_q (spanned by Gaussian random variables for $q = 2$, q-stable random variables for $q < 2$) which does SPR. By Proposition 4.4.11, $\prod_{\mathfrak{U}} \ell_q$ embeds isometrically into $\prod_{\mathfrak{U}} L_p(0,1)$, in an SPR fashion. By [166], $\prod_{\mathfrak{U}} L_p(0,1)$ can be identified (as a Banach lattice) with $L_p(\Omega,\mu)$, for some measure space (Ω,μ).

Let X be the (separable) sublattice of $L_p(\Omega,\mu)$ generated by E. By [231, Corollary 1.b.4], X is an L_p space. [219, Corollary, p. 128] gives a complete list of all separable L_p spaces; all of them lattice embed into $L_p(0,1)$. Thus, we have established the existence of an SPR embedding of $E = L_q(0,1)$ into $L_p(0,1)$.

To examine stability of SPR under small perturbations, we introduce the notion of *one-sided Hausdorff distance* between subspaces of a given Banach space. If E, F are subspaces of X, define $d_{1H}(E,F)$ as the infimum of all $\delta > 0$ so that, for every $x \in F$ with $\|x\| \le 1$ there exists $x' \in E$ with $\|x - x'\| < \delta$ (this "distance" is not reflexive, hence "one-sided"). Note also that, for x as above, there exists $x'' \in E$ with $\|x''\| = \|x\|$ and $\|x - x''\| < 2\delta$; indeed, one can take $x'' = \frac{\|x\|}{\|x'\|}x'$.

By "symmetrizing" d_{1H}, we obtain the classical *Hausdorff distance*: if E and F are subspaces of X, let $d_H(E,F) = \max\{d_{1H}(E,F), d_{1H}(F,E)\}$. For interesting properties of d_H, see [62], and references therein.

Proposition 4.4.13. Suppose E is an SPR subspace of a Banach lattice X. Then there exists $\delta > 0$ so that any subspace F with $d_{1H}(E,F) < \delta$ is again SPR.

From this we immediately obtain:

Corollary 4.4.14. For any Banach lattice X, the set of its SPR subspaces is open in the topology determined by the Hausdorff distance.

Remark 4.4.15. See [132, Proposition 3.10] for a similar stability result for dispersed subspaces of a Banach lattice.

Proof of Proposition 4.4.13. Suppose E does C-SPR. We shall show that, if $d_{1H}(E,F) < 1/(2\sqrt{2}(C+1))$, then F does C'-SPR, with

$$\frac{1}{C'} = \frac{1}{C}\Big(\frac{1}{\sqrt{2}} - 2d_{1H}(E,F)\Big) - 2d_{1H}(E,F).$$

Suppose, for the sake of contradiction, that F fails to do C'-SPR. Find $f, g \in F$ so that $\min_{|\lambda|=1} \|f - \lambda g\| = 1$ and $\||f| - |g|\| = c < 1/C'$. By Theorem 4.3.9, we can find $f', g' \in F$ so that

$$\min_{|\lambda|=1} \|f' - \lambda g'\| \geq \frac{1}{\sqrt{2}}, \||f'| - |g'|\| \leq c, \text{ and } \|f'\| + \|g'\| \leq 2.$$

For any $\delta > d_{1H}(E,F)$, there exist $f'', g'' \in E$ so that $\|f'' - f'\| < \delta\|f'\|$ and $\|g'' - g'\| < \delta\|g'\|$. The triangle inequality implies:

$$\||f''| - |g''|\| \leq \||f'| - |g'|\| + \delta(\|f'\| + \|g'\|) \leq c + 2\delta;$$
$$\min_{|\lambda|=1} \|f'' - \lambda g''\| \geq \min_{|\lambda|=1} \|f' - \lambda g'\| - \delta(\|f'\| + \|g'\|) \geq \frac{1}{\sqrt{2}} - 2\delta.$$

As E does C-SPR, we conclude that

$$\frac{1}{\sqrt{2}} - 2\delta \leq C(c + 2\delta),$$

and consequently,

$$\frac{1}{\sqrt{2}} - 2d_{1H}(E,F) \leq C(c + 2d_{1H}(E,F)) < C\Big(\frac{1}{C'} + 2d_{1H}(E,F)\Big),$$

which contradicts our choice of C'. □

R. Balan proved that frames which do stable phase retrieval for finite dimensional Hilbert spaces are stable under small perturbations [41]. The following extends this to infinite dimensional subspaces of Banach lattices.

Corollary 4.4.16. Suppose (e_i) is a semi-normalized basic sequence in a Banach lattice X, so that $\overline{\text{span}}[e_i : i \in \mathbb{N}]$ does SPR in X. Then there exists $\varepsilon > 0$ so that if $(f_i) \subseteq X$ and $\sum_i \|e_i - f_i\| < \varepsilon$ then $\overline{\text{span}}[f_i : i \in \mathbb{N}]$ does SPR in X.

Remark 4.4.17. In real L_2, Corollary 4.4.16 can be strengthened. Suppose (e_i) is a sequence of normalized independent mean-zero random variables, spanning an SPR-subspace of L_2. Then there exists an $\varepsilon > 0$ with the following property: if (f_i) is a collection of normalized independent mean-zero random variables so that (e_i, f_j) are independent whenever $i \neq j$, and $\sup_i \|e_i - f_i\| \leq \varepsilon$, then $\mathrm{span}[f_i : i \in \mathbb{N}] \subseteq L_2$ does SPR as well. For the proof, recall that there exists $\gamma > 0$ so that the inequality $\||u| \wedge |v|\| \geq \gamma$ holds for any norm one $u, v \in \mathrm{span}[e_i : i \in \mathbb{N}]$. Let $\varepsilon = \gamma/4$. We will show that, for any norm one $x, y \in F = \mathrm{span}[f_i : i \in \mathbb{N}]$, we have $\||x| \wedge |y|\| \geq \gamma/2$.

Write $x = \sum_i \alpha_i f_i$ and $y = \sum_i \beta_i f_i$, and define $x' = \sum_i \alpha_i e_i$, $y' = \sum_i \beta_i e_i$. Then

$$\|x - x'\|^2 = \Big\| \sum_i \alpha_i (f_i - e_i) \Big\|^2 = \sum_i |\alpha_i|^2 \|f_i - e_i\|^2 \leq \varepsilon^2 \sum_i \alpha_i^2 = \varepsilon^2.$$

Similarly, $\|y - y'\| \leq \varepsilon$. Therefore, $\||x| - |x'|\|, \||y| - |y'|\| \leq \varepsilon$, hence $\||x| \wedge |y|\| \geq \||x'| \wedge |y'|\| - 2\varepsilon$. But $\||x'| \wedge |y'|\| \geq \gamma$, hence $\||x| \wedge |y|\| \geq \gamma/2$.

4.5 SPR in Lebesgue spaces

In this section, we investigate the relations between dispersed and SPR subspaces of L_p, as well as the relation between doing SPR in L_p versus doing SPR in L_q.

Theorem 4.5.1. Every infinite dimensional dispersed subspace of an order continuous Banach lattice X contains a further closed infinite dimensional subspace that does SPR.

Proof. We first prove the claim for $L_1(\Omega, \mu)$, with μ a finite measure. Let E be a closed infinite dimensional subspace of $L_1(\Omega, \mu)$ containing no normalized almost disjoint sequence. By Theorem 4.2.1, E also does not contain ℓ_1. By [215], every closed infinite dimensional subspace of $L_1(\Omega, \mu)$ almost isometrically contains ℓ_r for some $1 \leq r \leq 2$. Since E does not contain ℓ_1, it follows that there exists $r > 1$ such that for all $\varepsilon > 0$, ℓ_r is $(1+\varepsilon)$-isomorphic to a subspace of E. Let $\alpha > 0$ be such that ℓ_1^2 is not $(1+\alpha)$-isomorphic to a subspace of ℓ_r. Such an α exists by the Clarkson argument in Proposition 4.4.1. We now claim that for $0 < \varepsilon < \alpha$, every subspace of L_1 that is $(1+\varepsilon)$-isomorphic to ℓ_r must do stable phase retrieval. Indeed, if E failed SPR, it would contain for all $\gamma > 0$ a $(1+\gamma)$-copy of ℓ_1^2. Thus, for all $\gamma > 0$, we have that ℓ_1^2 is $(1+\gamma)(1+\varepsilon)$-isomorphic to a subspace of ℓ_r. However, this gives a contradiction if $\gamma > 0$ is small enough such that $(1+\gamma)(1+\varepsilon) < 1 + \alpha$.

Now let E be a closed infinite dimensional dispersed subspace of an order continuous Banach lattice X. Replacing E be a separable subspace of E, we may assume that E is separable. Using that every closed sublattice of an order continuous Banach lattice is order continuous, replacing X by the closed sublattice generated by E in X, we may assume that X is separable. It follows in particular that X has a weak unit. By the AL-representation theory, there exists a finite measure space (Ω, μ) such that X can be represented as an ideal of $L_1(\Omega, \mu)$ satisfying

(i) X is dense in $L_1(\Omega, \mu)$ and $L_\infty(\Omega, \mu)$ is dense in X;

(ii) $\|f\|_1 \leq \|f\|_X$ and $\|f\|_X \leq 2\|f\|_\infty$ for all $f \in X$.

Since E contains no almost disjoint normalized sequence, the Kadec-Pelczynski dichotomy [231, Proposition 1.c.8] guarantees that $\|\cdot\|_X \sim \|\cdot\|_{L_1}$ on E. In particular, we may view E as a closed infinite dimensional subspace of $L_1(\mu)$. We claim that E contains no almost disjoint sequence when viewed as a subspace of L_1. Indeed, suppose there exists a sequence (x_n) in E with $\|x_n\|_{L_1} = 1$ for all n, and a disjoint sequence (d_n) in L_1 with $\|x_n - d_n\|_{L_1} \to 0$. Then in particular, x_n converges to 0 in measure. By [96, Theorem 4.6], $x_n \xrightarrow{un} 0$ in X. That is, for all $u \in X$, we have that $\||x_n| \wedge |u|\|_X \to 0$. Thus, by [96, Theorem 3.2] there exists a subsequence (x_{n_k}) and a disjoint sequence (d_k) in X such that $\|x_{n_k} - d_k\|_X \to 0$. Since $\|x_n\|_{L_1} = 1$ and $\|\cdot\|_X \sim \|\cdot\|_{L_1}$ on E, this contradicts that E contains no normalized almost disjoint sequence.

By the beginning part of the proof, we may select an infinite dimensional closed subspace E' of E that does SPR in L_1. In other words, there exists $\varepsilon > 0$ such that for all $f, g \in E'$ with $\|f\|_{L_1} = \|g\|_{L_1} = 1$ we have

$$\||f| \wedge |g|\|_{L_1} \geq \varepsilon.$$

Since $\|\cdot\|_X \sim \|\cdot\|_{L_1}$ on E, the same is true on E', so we may view E' as a closed infinite dimensional subspace of X. We claim that it contains no normalized almost disjoint pairs. Indeed, if $f, g \in E'$ with $\|f\|_X = \|g\|_X = 1$, then $\|f\|_{L_1} \sim \|g\|_{L_1} \sim 1$. Now, using that E' does SPR in L_1 and property (ii) of the embedding, we have

$$\||f| \wedge |g|\|_X \geq \||f| \wedge |g|\|_{L_1} \gtrsim \varepsilon.$$

Thus, E' contains no normalized almost disjoint pairs when viewed as a subspace of X. It follows that E' does SPR in X. □

Question 4.5.2. With Corollary 4.4.8 and Theorem 4.5.1 in mind, we ask the following: If a Banach lattice X contains an infinite dimensional dispersed subspace E, does it contains an infinite dimensional SPR subspace? If so, can we construct an infinite dimensional SPR subspace E' with $E' \subseteq E \subseteq X$?

Our next results are motivated by the equivalence between statements (a)-(d) in Theorem 4.2.1 and the discussion in Remark 4.2.2. Note that it follows from Theorem 4.2.1 (a)-(d) that if E is dispersed in $L_p(\mu)$ and $1 \leq q < p$, then E may be viewed as a closed subspace of $L_q(\mu)$, and it is dispersed in $L_q(\mu)$. It is then natural to ask the following question: Let μ be a finite measure and $1 \leq q < p$. Let E be a subspace of $L_p(\mu) \subseteq L_q(\mu)$. What is the relation between E doing SPR in $L_p(\mu)$ versus E doing SPR in $L_q(\mu)$? It is easy to see that if E does SPR in $L_q(\mu)$, then E does SPR in $L_p(\mu)$ if and only if $\|\cdot\|_{L_p} \sim \|\cdot\|_{L_q}$ on E. We will now show that E doing SPR in $L_p(\mu)$ does not imply E does SPR in $L_q(\mu)$, even though the property of being dispersed passes from $L_p(\mu)$ to $L_q(\mu)$.

Theorem 4.5.3. For all $2 \leq p < \infty$ there exists a closed subspace $E \subseteq L_p[0,1]$ such that E does stable phase retrieval in $L_p[0,1]$ but E fails to do stable phase retrieval in $L_q[0,1]$ for all $1 \leq q < p$.

Proof. Let $2 \leq p < \infty$. It will be convenient to build the subspace $E \subseteq L_p[0,2]$ instead of $L_p[0,1]$. Let $(r_j)_{j=1}^{\infty}$ be the Rademacher sequence of independent, mean-zero, ± 1 random variables on $[0,1]$. For all $j \in \mathbb{N}$, we let $g_j = r_j + 2^{j/p}\mathbb{1}_{[1+2^{-j},1+2^{-j+1})}$. Let $E = \overline{\text{span}}_{j\in\mathbb{N}} g_j$.

We first prove for all $1 \leq q < p$ that E fails to do stable phase retrieval in $L_q[0,2]$. We have for all $j \neq i$ that $\|g_j - g_i\|_{L_q}^q = \|g_j + g_i\|_{L_q}^q \geq 2^{q-1}$. On the other hand, $|r_j| = |r_{j+1}|$ and $\lim \|2^{j/p}\mathbb{1}_{[1+2^{-j},1+2^{-j+1})}\|_{L_q}^q = 0$. Thus, $\lim \||g_j| - |g_{j+1}|\|_{L_q}^q = 0$. This shows that E fails to do stable phase retrieval in $L_q[0,2]$.

We now prove that E does stable phase retrieval in $L_p[0,2]$. Note that by Khintchine's Inequality there exists $B \geq 0$ so that $(\sum |a_j|^2)^{1/2} \leq \|\sum a_j r_j\|_{L_p} \leq B(\sum |a_j|^2)^{1/2}$ for all scalars $(a_j) \in \ell_2$. Thus, we have for all $f = \sum a_j r_j$ and $x = f + \sum a_j 2^{j/p}\mathbb{1}_{[1+2^{-j},1+2^{-j+1})} \in E$ that

$$\begin{aligned}
\|f\|_{L_2([0,1])}^p \leq \|x\|_{L_p([0,2])}^p &= \|\sum a_j r_j\|_{L_p([0,1])}^p + \sum |a_j|^p \\
&\leq B^p(\sum |a_j|^2)^{p/2} + (\sum |a_j|^2)^{p/2} \\
&= (B^p + 1)\|f\|_{L_2([0,1])}^p.
\end{aligned}$$

This computation shows that the map $g_j \mapsto r_j$ extends linearly to a map $E \subseteq L_p[0,2] \to L_2[0,1]$, $x \mapsto f$, establishing an isomorphism between E and a Hilbert space. By Theorem 4.3.9 and Remark 4.3.10 it suffices to prove that there exists a constant $\delta > 0$ so that if $x, y \in E$ and $f, g \in L_2[0,1]$ with $f = \mathbb{1}_{[0,1]}x$ and $g = \mathbb{1}_{[0,1]}y$ such that $\|f\|_{L_2} = 1$, $\|g\|_{L_2} \leq 1$, and $\langle f, g\rangle = 0$ then $\||x| - |y|\|_{L_p} \geq \delta$.

We now claim that it suffices to prove that there exists $\varepsilon > 0$ such that

$$\text{if } \||x\mathbb{1}_{(1,2)}| - |y\mathbb{1}_{(1,2)}|\|_{L_p} < \varepsilon \text{ then } \||f|^2 - |g|^2\|_{L_2}^2 \geq \delta. \tag{4.5.1}$$

Indeed, as all the L_q norms are equivalent on the span of the Rademacher sequence, there exists a uniform constant $K > 0$ so that the following holds:

$$\begin{aligned}
\||f|^2 - |g|^2\|_{L_2}^2 &= \int (|f|^2 - |g|^2)^2 \\
&= \int (|f| - |g|)(|f| + |g|)(|f|^2 - |g|^2) \\
&\leq \||f| - |g|\|_{L_2} \|(|f| + |g|)(|f|^2 - |g|^2)\|_{L_2} \\
&\leq K\||f| - |g|\|_{L_2} \leq K\||f| - |g|\|_{L_p}.
\end{aligned}$$

Here, the constant K comes from bounding

$$\|(|f| + |g|)(|f|^2 - |g|^2)\|_{L_2} \leq K. \tag{4.5.2}$$

To get this upper estimate, note that, by Hölder's Inequality,

$$\begin{aligned}
&\|(|f| + |g|)(|f|^2 - |g|^2)\|_{L_2} = \|(|f| + |g|)(|f| + |g|)(|f| - |g|)\|_{L_2} \\
&\leq \||f| + |g|\|_{L_6}^2 \||f| - |g|\|_{L_6},
\end{aligned}$$

hence, by Triangle Inequality,

$$\|(|f| + |g|)(|f|^2 - |g|^2)\|_{L_2} \leq \big(\|f\|_{L_6} + \|g\|_{L_6}\big)^3. \tag{4.5.3}$$

Further, both f and g belong to the span of independent Rademachers, on which all the L_p norms are equivalent (for finite p). Since we know that $\|f\|_{L_2} = 1$ and $\|g\|_{L_2} \leq 1$, this gives a bound for the right-hand side of (4.5.3), which, in turn, implies (4.5.2).

To finish the proof of the claim, note that if $\||x\mathbb{1}_{(1,2)}| - |y\mathbb{1}_{(1,2)}|\|_{L_p} \geq \varepsilon$ then $\||x| - |y|\|_{L_p} \geq \varepsilon$ and if $\||x\mathbb{1}_{(1,2)}| - |y\mathbb{1}_{(1,2)}|\|_{L_p} < \varepsilon$ then $\||x| - |y|\|_{L_p} \geq \delta K^{-1}$.

We now establish (4.5.1) with $\varepsilon = 1/8$ and $\delta = 1$. Let $x = \sum a_j(r_j + 2^{j/p}\mathbb{1}_{[1+2^{-j}, 1+2^{-j+1})})$ and $y = \sum b_j(r_j + 2^{j/p}\mathbb{1}_{[1+2^{-j}, 1+2^{-j+1})})$. We let $f = \sum a_j r_j$ and $g = \sum b_j r_j$ and assume that $\|f\|_{L_2}^2 = \sum |a_j|^2 = 1$, $\|g\|_{L_2}^2 = \sum |b_j|^2 \leq 1$ and $\langle f, g\rangle = \sum a_j b_j = 0$. We may assume that

$$\Big(\sum ||a_j| - |b_j||^p\Big)^{1/p} = \||x\mathbb{1}_{(1,2)}| - |y\mathbb{1}_{(1,2)}|\|_{L_p} < \varepsilon = 1/8. \tag{4.5.4}$$

All that remains is to prove that $\||f|^2-|g|^2\|_{L_2}^2 \geq \delta$. We have from (4.5.4) that $||a_j|-|b_j|| \leq 1/8$ for all $j \in \mathbb{N}$. Hence, $||a_j|^2-|b_j|^2| \leq 1/4$ for all $j \in \mathbb{N}$ as $|a_j|+|b_j| \leq 2$. As $r_j^2 = \mathbb{1}_{[0,1]}$ for all $j \in \mathbb{N}$, we have that

$$f^2-g^2 = (f-g)(f+g) = 2\sum_{j>i}(a_ja_i - b_jb_i)r_jr_i + \sum(a_j^2-b_j^2)\mathbb{1}. \tag{4.5.5}$$

Note that (4.5.5) gives an expansion for f^2-g^2 in terms of the ortho-normal collection of vectors $\{\mathbb{1}_{[0,1]}\} \cup \{r_jr_i\}_{j>i}$. Thus we have that

$$\begin{aligned}
2^{-1}\||f|^2-|g|^2\|_{L_2}^2 &\geq 2\sum_{j>i}|a_ja_i-b_jb_i|^2 \\
&= \sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}}|a_ja_i-b_jb_i|^2 - \sum_{j\in\mathbb{N}}|a_j^2-b_j^2|^2 \\
&= \sum_{j\in\mathbb{N}}\Big((\sum_{i\in\mathbb{N}}|a_ja_i|^2+|b_jb_i|^2) - (2a_jb_j\sum_{i\in\mathbb{N}}a_ib_i)\Big) - \sum_{j\in\mathbb{N}}|a_j^2-b_j^2|^2 \\
&= \sum_{j\in\mathbb{N}}(\sum_{i\in\mathbb{N}}|a_ja_i|^2+|b_jb_j|^2) - \sum_{j\in\mathbb{N}}|a_j^2-b_j^2|^2 \qquad \text{as } \sum a_ib_i = 0 \\
&= \big(\|f\|_{L_2}^4+\|g\|_{L_2}^4\big) - \sum_{j\in\mathbb{N}}|a_j^2-b_j^2|^2 \\
&\geq \big(\|f\|_{L_2}^4+\|g\|_{L_2}^4\big) - \frac{1}{4}\sum_{j\in\mathbb{N}}|a_j^2-b_j^2| \qquad \text{as } |a_j^2-b_j^2| \leq 1/4 \\
&\geq \big(\|f\|_{L_2}^4+\|g\|_{L_2}^4\big) - \frac{1}{4}(\|f\|_{L_2}^2+\|g\|_{L_2}^2) \\
&= \frac{3}{4} + \|g\|_{L_2}^2\left(\|g\|_{L_2}^2 - \frac{1}{4}\right) \qquad \text{as } \|f\|_{L_2} = 1 \\
&\geq \frac{3}{4} - \frac{1}{8} \qquad \text{as } \|g\|_{L_2} \leq 1.
\end{aligned}$$

Hence, $\||f|^2-|g|^2\|_{L_2}^2 \geq 3/2 - \frac{1}{4} > 1 = \delta$. $\square$

Example 4.5.4. In the special case $p=2$, Theorem 4.5.3 could have been proven using a result in [71]. Indeed, as above, let (r_j) denote the Rademacher sequence, realized on the interval $[0,1]$. Define $g_j = r_j + 2^{\frac{j}{2}}\mathbb{1}_{[1+2^{-j},1+2^{-j+1})}$. We can think of the sequence (g_j) as being defined on a finite measure space. Note that $\|2^{\frac{j}{2}}\mathbb{1}_{[1+2^{-j},1+2^{-j+1})}\|_{L_2} = 1$. Hence, for the same reason as in [71], $\overline{\text{span}}\{g_j\}$ does SPR in L_2. However, recall that the Rademacher sequence does not do phase retrieval; we've also scaled the additional indicator functions to be perturbative in L_1. Hence, for $i \neq j$ we have $\||g_i|-|g_j|\|_{L_1} = \frac{1}{2^i}+\frac{1}{2^j}$, whereas the other side of the SPR inequality is of order 1. This provides an example of a subspace $E \subseteq L_2(\mu) \subseteq L_1(\mu)$ that does SPR in $L_2(\mu)$ but not in $L_1(\mu)$.

The next result contrasts with Theorem 4.5.3 by showing, in particular, that if E does SPR in both L_p and L_q then we can both "interpolate" the SPR, and "extrapolate it downward".

Theorem 4.5.5. Suppose μ is a probability measure and $1 \le q < p < \infty$. Let E be a closed subspace of L_p (real or complex). Assume that $\|\cdot\|_{L_p} \sim \|\cdot\|_{L_q}$ on E, and E does stable phase retrieval in L_q. Then for all $1 \le r \le p$, $\|\cdot\|_{L_r} \sim \|\cdot\|_{L_p}$ on E, and E does stable phase retrieval in L_r.

Proof. From the discussion on the Kadec-Pelczynski dichotomy (see Section 4.2), we know that, as $E \subseteq L_q(\mu)$ does SPR, then $\|\cdot\|_{L_p} \sim \|\cdot\|_{L_r}$ on E, whenever $r < p$.

Assume first that $q < r \le p$. Let $C > 0$ so that the L_q and L_p norms are C-equivalent on E, and let $K > 0$ so that E does K-stable phase retrieval in L_q. As $q < r \le p$ we have for all $f, g \in E$ that

$$\inf_{|\lambda|=1} \|f - \lambda g\|_{L_r} \le C \inf_{|\lambda|=1} \|f - \lambda g\|_{L_q} \le CK \||f| - |g|\|_{L_q} \le CK \||f| - |g|\|_{L_r}. \tag{4.5.6}$$

Thus, E does stable phase retrieval in L_r.

We now turn to the case $1 \le r < q$. By the previous argument, E does stable phase retrieval in L_p. Hence, the L_p norm is equivalent to the L_1 norm on E, and hence the L_p norm is equivalent to the L_r norm on E. Let $C > 0$ so that the L_p and L_r norms are C-equivalent on E, and let $K > 0$ so that E does K-stable phase retrieval in L_p. Let θ be the value so that $q^{-1} = \theta r^{-1} + (1-\theta)p^{-1}$. By Hölder's inequality, for any $f, g \in E$,

$$\||f|-|g|\|_{L_q} \le \||f|-|g|\|^{\theta}_{L_r} \left(\|f\|_{L_p} + \|g\|_{L_p}\right)^{1-\theta} \le C\||f|-|g|\|^{\theta}_{L_r} \left(\|f\|_{L_r} + \|g\|_{L_r}\right)^{1-\theta}. \tag{4.5.7}$$

Therefore, for any $f, g \in E$, we have

$$\inf_{|\lambda|=1} \|f - \lambda g\|_{L_r} \le \inf_{|\lambda|=1} \|f - \lambda g\|_{L_q} \le K \||f| - |g|\|_{L_q} \le CK \||f| - |g|\|^{\theta}_{L_r} \left(\|f\|_{L_r} + \|g\|_{L_r}\right)^{1-\theta}.$$

Thus, E does θ-Hölder stable phase retrieval in L_r. By Corollary 4.3.11, it follows that E does stable phase retrieval in L_r. $\square$

In Theorem 4.5.3, we showed that when $2 \leq p < \infty$ an SPR-subspace $E \subseteq L_p[0,1]$ need not do SPR in $L_q[0,1]$ for any $1 \leq q < p$. Our next result shows that the case $1 \leq p < 2$ is completely different.

Theorem 4.5.6. Let (Ω, μ) be a probability space and let E be a closed infinite dimensional subspace of $L_p(\Omega, \mu)$. Consider the following statements:

(i) E does stable phase retrieval in $L_p(\Omega, \mu)$.

(ii) E does stable phase retrieval in $L_1(\Omega, \mu)$ and $\|\cdot\|_{L_p} \sim \|\cdot\|_{L_1}$ on E.

(iii) There exists $\alpha > 0$ such that for all $x, y \in E$,

$$\mu(\{t \in \Omega : |x(t)| \geq \alpha\|x\|_{L_p} \text{ and } |y(t)| \geq \alpha\|y\|_{L_p}\}) > \alpha. \tag{4.5.8}$$

Then for all $1 \leq p < \infty$, (iii)$\Leftrightarrow$(ii)$\Rightarrow$(i). Moreover, if $1 \leq p < 2$, all three statements are equivalent.

Proof. *(iii)* $\Rightarrow$ *(ii)*: Note that condition (iii) implies that E contains no normalized $\alpha^{1+\frac{1}{p}}$-disjoint pairs, when viewed in the L_p norm. Hence, E does SPR in L_p, which implies that $\|\cdot\|_{L_p} \sim \|\cdot\|_{L_1}$ on E. Using this in condition (iii), we conclude that E contains no normalized almost disjoint pairs, when viewed in the L_1 norm, hence does SPR in L_1.

(ii) $\Rightarrow$ *(i)*: Let $C > 0$ so that $\|x\|_{L_p} \leq C\|x\|_{L_1}$ for all $x \in E$. Let $K > 0$ so that E does K-stable phase retrieval in L_1. Thus, for all $x, y \in E$ we have that

$$\min_{|\lambda|=1} \|x - \lambda y\|_{L_p} \leq C \min_{|\lambda|=1} \|x - \lambda y\|_{L_1} \leq CK\||x| - |y|\|_{L_1} \leq CK\||x| - |y|\|_{L_p}.$$

Thus, E does CK-stable phase retrieval in $L_p(\Omega)$.

(i) $\Rightarrow$ *(iii)*: Let $1 \leq p < 2$ and assume that (i) is true but (iii) is false. We first note that condition (i) implies that $\|\cdot\|_{L_1} \sim \|\cdot\|_{L_p}$ on E. We may choose a sequence of pairs $(x_n, y_n)_{n=1}^\infty$ in E and $\alpha > 0$ such that $\|x_n\|_{L_p} = \|y_n\|_{L_p} = 1$, with

$$\mu(\{t \in \Omega : |x_n| \wedge |y_n| \geq n^{-1}\}) \to 0, \text{ but } \||x_n| \wedge |y_n|\|_{L_p} \geq 2\alpha. \tag{4.5.9}$$

As $(|x_n| \wedge |y_n|)_{n=1}^\infty$ converges in measure to 0 and is uniformly bounded below in L_p norm, after passing to a subsequence we may find a sequence of disjoint subsets $(\Omega_n)_{n=1}^\infty \subseteq \Omega$ such that

$$\|(|x_n| \wedge |y_n|)\mathbb{1}_{\Omega_n^c}\|_{L_p} = \||x_n| \wedge |y_n| - (|x_n| \wedge |y_n|)\mathbb{1}_{\Omega_n}\|_{L_p} \to 0. \tag{4.5.10}$$

Let $\varepsilon_n \searrow 0$ with $\varepsilon_1 < \alpha/2$. After passing to a subsequence, we may assume that $\|x_n \mathbb{1}_{\Omega_n}\|_{L_p} \geq \alpha$ for all $n \in \mathbb{N}$. As $(\Omega_n)_{n=1}^\infty$ is a sequence of disjoint subsets of the probability space (Ω, μ), we have that $\mu(\Omega_n) \to 0$. Thus, after passing to a further subsequence we may assume that $\|x_j \mathbb{1}_{\Omega_n}\|_{L_p} < \varepsilon_n$ for all $j < n$. Again, after passing to a further subsequence we may assume that there exists values $(\beta_n)_{n=1}^\infty$ such that $\lim_{j\to\infty} \|x_j \mathbb{1}_{\Omega_n}\|_{L_p} = \beta_n$ for all $n \in \mathbb{N}$. Furthermore, we may assume that $\|x_j \mathbb{1}_{\Omega_n}\|_{L_p} < \beta_n + \varepsilon_n/2$ for all $j > n$. As $(\Omega_j)_{j=1}^\infty$ is a sequence of disjoint sets, we have for all $N \in \mathbb{N}$ that

$$\lim_{j\to\infty} \|x_j\|_{L_p}^p \geq \lim_{j\to\infty} \sum_{n=1}^N \|x_j \mathbb{1}_{\Omega_n}\|_{L_p}^p = \sum_{n=1}^N \beta_n^p.$$

In particular, we have that $\beta_n \to 0$. Hence, after passing to a further subsequence of $(x_n)_{n=1}^\infty$ we may assume that $\beta_n < \varepsilon_n/2$ for all $n \in \mathbb{N}$. Thus, $\|x_j \mathbb{1}_{\Omega_n}\|_{L_p} < \varepsilon_n$ for all $j > n$. In summary, we have that for all $n \in \mathbb{N}$, $\|x_n \mathbb{1}_{\Omega_n}\|_{L_p} \geq \alpha$ and for all $j \neq n$, we have $\|x_j \mathbb{1}_{\Omega_n}\|_{L_p} < \varepsilon_n$.

As $\varepsilon_1 < \alpha/2$, we have in particular that $\|x_j - x_n\|_{L_p} \geq \alpha/2$ for all $j \neq n$. We have that $(x_n)_{n=1}^\infty$ is a semi-normalized sequence in a closed subspace of L_p which does not contain ℓ_p. Thus, by [289, Theorem 8], $(x_n)_{n=1}^\infty$ is equivalent to a semi-normalized sequence in $L_{p'}(\nu)$ for some $p < p' \leq 2$ and probability measure ν. We may assume after passing to a subsequence that $(x_n)_{n=1}^\infty$ is weakly convergent in $L_{p'}(\nu)$. Thus, the sequence $(x_{2n} - x_{2n-1})_{n=1}^\infty$ converges weakly to 0 in $L_{p'}(\nu)$. As $L_{p'}(\nu)$ has an unconditional basis, after passing to a further subsequence, we may assume that $(x_{2n} - x_{2n-1})_{n=1}^\infty$ is C-unconditional for some constant C.

As $L_{p'}(\nu)$ has type p' and $(x_{2n} - x_{2n-1})_{n=1}^\infty$ is unconditional, we have that $(x_{2n} - x_{2n-1})_{n=1}^\infty$ is dominated by the unit vector basis of $\ell_{p'}$. We will prove that there exists a constant K so that for all $N \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that the finite sequence $(x_{2n} - x_{2n-1})_{n=k+1}^{k+N}$ K-dominates the unit vector basis of ℓ_p^N. As $p < p'$, this would contradict that $(x_{2n} - x_{2n-1})_{n=1}^\infty$ is dominated by the unit vector basis of $\ell_{p'}$. Alternatively, one could use that L_p has type p, the uniform containment of ℓ_p^N, and [289, Theorem 13] to get that E contains a subspace isomorphic to ℓ_p, which, in view of Theorem 4.2.1, contradicts that E does stable phase retrieval in L_p.

Let $N \in \mathbb{N}$ and $\varepsilon > 0$. Let $k \in \mathbb{N}$ be large enough so that $2\varepsilon_k N < 2^{-1}\alpha$. Let $(a_j)_{j=k+1}^{k+N}$ be a sequence of scalars. We have that

$$
\begin{aligned}
\| \sum_{j=k+1}^{k+N} a_j(x_{2j}-x_{2j-1})\|_{L_p}^p &\geq \sum_{n=k+1}^{k+N} \| \sum_{j=k+1}^{k+N} a_j(x_{2j}-x_{2j-1})\|_{L_p(\Omega_{2n})}^p \\
&\geq \sum_{n=k+1}^{k+N} \left(2^{1-p}\|a_n(x_{2n}-x_{2n-1})\|_{L_p(\Omega_{2n})}^p - \|\sum_{j\neq n} a_j(x_{2j}-x_{2j-1})\|_{L_p(\Omega_{2n})}^p\right) \\
&\geq \sum_{n=k+1}^{k+N} \left(2^{1-p}\alpha^p|a_n|^p - 2^p\varepsilon_k^p\left(\sum_{j\neq n}|a_j|\right)^p\right) \\
&\geq \sum_{n=k+1}^{k+N} \left(2^{1-p}\alpha^p|a_n|^p - 2^p\varepsilon_k^p N^{p-1}\sum_{j\neq n}|a_j|^p\right) \\
&\geq \left(2^{1-p}\alpha^p - 2^p\varepsilon_k^p N^p\right)\sum_{n=k+1}^{k+N}|a_n|^p \\
&\geq 2^{-p}\alpha^p \sum_{n=k+1}^{k+N}|a_n|^p.
\end{aligned}
$$

Now that we have established that all three statements in Theorem 4.5.6 are equivalent for $1 \leq p < 2$, we can show the implication (ii)$\Rightarrow$(iii) for $1 \leq p < \infty$. Indeed, we assume (ii) holds. Since E does SPR in L_1, by (ii)$\Rightarrow$(iii) for $p=1$, we deduce that there exists $\alpha > 0$ such that for all $x, y \in E$,

$$\mu(\{t \in \Omega : |x(t)| \geq \alpha\|x\|_{L_1} \text{ and } |y(t)| \geq \alpha\|y\|_{L_1}\}) > \alpha. \tag{4.5.11}$$

Now we use the second assumption of (ii) to replace the L_1 norm with the L_p norm in (4.5.11). □

4.6 C(K)-spaces with SPR subspaces

Throughout this section, subspaces are assumed to be closed and infinite dimensional, unless otherwise mentioned. Recall that a non-empty compact Hausdorff space is called *perfect* if it has no isolated points, and *scattered* (or *dispersed*) if it contains no perfect subsets. For a compact Hausdorff space K, we define its *Cantor-Bendixson derivative* K' to be the set of all non-isolated points of K. Clearly K' is closed, and $K = K'$ iff K is perfect; otherwise, K' is a proper subset of K. Also, if K contains a perfect set S, then S lies inside of K' as well.

Theorem 4.6.1. Suppose K is a compact Hausdorff space. Then $C(K)$ has an SPR subspace if and only if K' is infinite.

The proof depends on an auxiliary result, strengthening Remark 4.4.6.

Proposition 4.6.2. Every separable Banach space embeds isometrically into $C(\Delta)$, and into $C[0,1]$, as a 10-SPR subspace (here Δ is the Cantor set).

Proof. Fix a separable Banach space E. Let K be the unit ball of E^*, with its weak* topology. By Lemma 4.4.5 and Remark 4.4.6, the natural isometric embedding $j : E \to C(K)$ (taking e into the function $K \to \mathbb{R} : e^* \mapsto e^*(e)$) is such that $\||jx| \wedge |jy|\| \geq 1/5$ whenever $\|x\| = 1 = \|y\|$. As K is compact and metrizable, there exists a continuous surjection $\Delta \to K$ [200, Theorem 4.18]; this generates a lattice isometric embedding of $C(K)$ into $C(\Delta)$, hence one can find an isometric copy of $E \subseteq C(\Delta)$ so that $\||x| \wedge |y|\| \geq 1/5$ whenever x, y are norm one elements of E.

View Δ as a subset of $[0,1]$. Then there exists a positive unital isometric extension operator $T : C(\Delta) \to C[0,1]$ – that is, for $f \in C(\Delta)$, $Tf|_\Delta = f$; $T1 = 1$; $\|T\| = 1$; and $Tf \geq 0$ whenever $f \geq 0$. The "standard" construction of T involves piecewise-affine extensions of functions from Δ to $[0,1]$; for a more general approach, see the proof of [9, Theorem 4.4.4]. One observes that $\||Tx| \wedge |Ty|\| \geq \||x| \wedge |y|\|$, hence, if $E \subseteq C(\Delta)$ has the property described in the preceding paragraph, then $\||Tx| \wedge |Ty|\| \geq 1/5$ whenever $x, y \in E$ have norm 1.

By Theorem 4.3.4, the copies of E in $C(\Delta)$ and $C[0,1]$ described above do 10-SPR. $\square$

The next result is standard topological fare (cf. [252, Theorem 29.2]).

Lemma 4.6.3. Suppose K is a compact Hausdorff space, and $t \in U \subseteq K$, where U is an open set. Then there exists an open set V so that $t \in V \subseteq \overline{V} \subseteq U$.

Proof of Theorem 4.6.1. Suppose first that K' is finite (in this case, K must be scattered). To show that any subspace $E \subseteq C(K)$ fails SPR, consider $C_0(K, K') = \{f \in C(K) : f|_{K'} = 0\}$. Then $\dim C(K)/C_0(K, K') = |K'| < \infty$, hence $E \cap C_0(K, K')$ is infinite dimensional as well. It suffices therefore to show that every infinite dimensional subspace of $C_0(K, K')$ fails SPR.

Note that, in the case of finite K', $C_0(K, K')$ can be identified with $c_0(K \backslash K')$ as a Banach lattice. Indeed, any $f \in c_0(K \backslash K')$ is continuous on $K \backslash K'$, since this set consists of isolated points only. Extend f to a function $\widetilde{f} : K \to \mathbb{R}$ with $\widetilde{f}|_{K'} = 0$, $\widetilde{f}|_{K\backslash K'} = f$. Note that for any $c > 0$, the set $\{t \in K\backslash K' : |f(t)| \geq c\} = \{t \in K : |\widetilde{f}(t)| \geq c\}$ is finite, hence closed; consequently, $\{t \in K : |\widetilde{f}(t)| < c\}$ is an open neighborhood of any element of K'. From this it follows that $\widetilde{f}$ is continuous.

On the other hand, pick $h \in C_0(K, K')$. We claim that $h|_{K\backslash K'} \in c_0(K\backslash K')$ – that is, $\{t \in K\backslash K' : |h(t)| > c\}$ is finite for any $c > 0$. Suppose, for the sake of contradiction, that this set is infinite for some c. By the compactness of K, this set must have an accumulation point, which must lie in K'. This, however, contradicts the continuity of h.

A “gliding hump” argument shows that no subspace of $c_0(K\backslash K')$ does SPR. From this we conclude that no subspace of $C(K)$ does SPR if K' is finite.

Now suppose K contains a perfect set. By [219, Theorem 2, p. 29], there exists a continuous surjection $\phi : K \to [0, 1]$. This map generates a lattice isometric embedding $T : C[0, 1] \to C(K) : f \mapsto f \circ \phi$. However, $C[0, 1]$ contains SPR subspaces, by Proposition 4.6.2.

It remains to prove that $C(K)$ contains an SPR copy of c_0 when K is scattered, and K' is infinite. Note first that $K'\backslash K''$ must be infinite. Indeed, otherwise any point of $K'' = K'\backslash(K'\backslash K'')$ will be an accumulation point of the same set, and K'' will be perfect, which is impossible.

Observe also that any $t \in K'\backslash K''$ is an accumulation point of $K\backslash K'$. Indeed, suppose otherwise, for the sake of contradiction. Then t has an open neighborhood W, disjoint from $K\backslash K'$. If U is another open neighborhood of t, then so is $U \cap W$. As t is an accumulation point of K, $U \cap W$ must meet K, hence also K'. This implies $t \in K''$, providing us with the desired contradiction.

Find distinct points $t_1, t_2, \ldots \in K'\backslash K''$. For each i find an open set $A_i \ni t_i$ so that $t_j \notin A_i$ for $j \neq i$. Lemma 4.6.3 permits us to find an open set U_i so that $t_i \in U_i \subseteq \overline{U_i} \subseteq A_i$. Replacing U_2 by $U_2\backslash\overline{U_1}$, U_3 by $U_3\backslash\overline{U_1 \cup U_2}$, and so on, we can assume that the sets U_i are disjoint. Lemma 4.6.3 guarantees the existence of open sets V_i so that, for every i, $t_i \in V_i \subseteq \overline{V_i} \subseteq U_i$.

As noted above, each t_i is an accumulation point of $K\backslash K'$. Therefore, we can find distinct points $(s_{ji})_{j=1}^\infty \subseteq (K\backslash K') \cap V_i$. For each n, let S_n be the closure of $\{s_{j,2n} : j \in \mathbb{N}\}$ (note $S_n \subseteq \overline{V_{2n}} \subseteq U_{2n}$). Note that there exists $x^{(n)} \in C(K)$ such that:

(i) $0 \leq x^{(n)} \leq 1$ everywhere.

(ii) $x^{(n)}|_{S_n} = 1/2$.

(iii) $x^{(n)}(s_{1,2n-1}) = 1$.

(iv) $x^{(n)}(s_{n,2i}) = 1/2$ for $1 \leq i \leq n-1$.

(v) $x^{(n)} = 0$ on $(K\backslash U_{2n})\backslash\{s_{1,2n-1}, s_{n,2}, s_{n,4}, \ldots, s_{n,2n-2}\}$.

To construct such an $x^{(n)}$, recall that $s_{1,2n-1}, s_{n,2}, s_{n,4}, \ldots, s_{n,2n-2}$ are isolated points of K, hence the function g, defined by $g(s_{1,2n-1}) = 1$, $g(s_{n,2i}) = 1/2$ for $1 \leq i \leq n-1$, and $g = 0$ everywhere else, is continuous. Further, by Urysohn's Lemma, there exists $h \in C(K)$ so that $0 \leq h \leq 1/2 = h|_{S_n}$, vanishing outside of U_{2n}. Then $x^{(n)} = g + h$ has the desired properties.

We claim that $(x^{(n)})$ is equivalent to the standard c_0-basis. Indeed, suppose $(\alpha_n) \in c_{00}$, with $\vee_n|\alpha_n| = 1$. We need to show $\|\sum_n \alpha_n x^{(n)}\| = 1$. The lower estimate on the norm is clear, since $x = \sum_n \alpha_n x^{(n)}$ attains the value of α_n at $s_{1,2n-1}$.

For an upper estimate, note that x vanishes outside of $\cup_m U_m$, and on U_m if m is large enough. If m is odd ($m = 2n-1$), then the only point of U_m where x does not vanish is $s_{1,2n-1}$, which we have already discussed. If m is even ($m = 2n$), then $|x| \leq 1/2$ except for the points $s_{i,2n}$ ($i > n$); at these points, x equals $(\alpha_n + \alpha_i)/2$, which has absolute value not exceeding 1.

It remains to show that $E = \mathrm{span}[x^{(n)} : n \in \mathbb{N}]$ does SPR. In light of Theorem 4.3.4, if suffices to prove that $\||x| \wedge |y|\| \geq 1/3$ for any norm one $x, y \in E$. Write $x = \sum_n \alpha_n x^{(n)}$ and $y = \sum_n \beta_n x^{(n)}$. Find n and m so that $|\alpha_n| = 1 = |\beta_m|$. If $n = m$, then both $|x|$ and $|y|$ equal 1 at $s_{1,2n-1}$, so $\||x| \wedge |y|\| = 1$.

Otherwise, assume, by relabeling, that $n < m$. If $|\alpha_m| \geq 1/3$, then

$$\||x| \wedge |y|\| \geq |x(s_{1,2m-1})| \wedge |y(s_{1,2m-1})| = |\alpha_m| \wedge |\beta_m| \geq \frac{1}{3}.$$

The case of $|\beta_n| \geq 1/3$ is treated similarly. If $|\alpha_m|, |\beta_n| < 1/3$, then $|x(s_{m,2n})| = |\alpha_n + \alpha_m|/2 > 1/3$, and similarly, $|y(s_{m,2n})| > 1/3$, which again gives us $\||x| \wedge |y|\| \geq 1/3$. □

Question 4.6.4. The proof of Theorem 4.6.1 shows that K' is infinite iff $C(K)$ contains an SPR copy of c_0. If K is "large" enough (in terms of the smallest ordinal α for which $K^{(\alpha)}$ is finite), what SPR subspaces (other than c_0) does $C(K)$ have? Note that c_0 is isomorphic to $c = C[0, \omega]$ (ω is the first infinite ordinal). If $K^{(\alpha)}$ is infinite, does $C(K)$ contain an SPR copy of $C[0, \omega^\alpha]$? This question is of interest even for separable $C(K)$, i.e., metrizable K.

In the spirit of Proposition 4.4.1, it is natural to ask which (isometric) subspaces of $C(K)$ are necessarily SPR. Below we give a "very local" condition on a Banach space E (finite or infinite dimensional) which guarantees that any isometric embedding of E into $C(K)$ has SPR.

Recall (see [181]) that a Banach space E is called *uniformly non-square* if there exists $\varepsilon > 0$ so that, for any norm one $f, g \in E$ we have $\min\{\|f + g\|, \|f - g\|\} < 2 - \varepsilon$. Note that E fails to be uniformly non-square iff for every $\varepsilon > 0$ there exist norm one $f, g \in E$ so that $\|f + g\|, \|f - g\| > 2 - \varepsilon$. In the real case, this means that E contains ℓ_1^2 (equivalently, ℓ_∞^2) with arbitrarily small distortion. This is incompatible with uniform convexity or uniform smoothness.

Proposition 4.6.5. Any uniformly non-square subspace of $C(K)$ does SPR.

Proof. Suppose E is a non-SPR subspace of $C(K)$; we shall show that it fails to be uniformly non-square. To this end, fix $\varepsilon \in (0, 1/2)$; by Theorem 4.3.4, there exist norm one $f, g \in E$ with $\||f| \wedge |g|\| < \varepsilon$. Pointwise evaluation shows that

$$|f| \vee |g| + |f| \wedge |g| \geq |f + g| \geq |f| \vee |g| - |f| \wedge |g|.$$

As the ambient lattice is an M-space, we have $\||f| \vee |g|\| = 1$, hence

$$1 - \varepsilon < \||f| \vee |g|\| - \||f| \wedge |g|\| \leq \|f + g\| \leq \||f| \vee |g|\| + \||f| \wedge |g|\| < 1 + \varepsilon.$$

Replacing g by $-g$, we conclude that $1 - \varepsilon < \|f - g\| < 1 + \varepsilon$.

Let $u = (f + g)/\|f + g\|$ and $v = (f - g)/\|f - g\|$. Then

$$\big\|u - (f + g)\big\| = \big|1 - \|f + g\|\big| < \varepsilon,$$

and similarly, $\big\|v - (f - g)\big\| < \varepsilon$. Then

$$\|u + v\| \geq \big\|(f + g) + (f - g)\big\| - \big\|u - (f + g)\big\| - \big\|v - (f - g)\big\| > 2 - 2\varepsilon,$$

and likewise, $\|u - v\| > 2 - 2\varepsilon$. As ε is arbitrary, E fails to be uniformly non-square. $\square$

For infinite dimensional subspaces, Proposition 4.6.5 is only meaningful when K is not scattered. Indeed, if K is scattered, then $C(K)$ is c_0-saturated [109, Theorem 14.26], hence any infinite dimensional subspace of $C(K)$ contains an almost isometric copy of c_0 [230, Proposition 2.e.3]. In particular, such subspaces contain almost isometric copies of ℓ_1^2, hence they cannot be uniformly non-square.

In light of Proposition 4.6.5, we ask:

Question 4.6.6. Which Banach spaces E isometrically embed into $C(K)$ in a non-SPR way?

Note that containing an isometric copy of ℓ_∞^2 (and consequently, failing to be uniformly non-square) does not automatically guarantee the existence of a non-SPR embedding into $C(K)$ (in this sense, the converse to Proposition 4.6.5 fails). In the following example we look at isometric embeddings only; one can modify this example to allow for sufficiently small distortions.

Proposition 4.6.7. There exists a 3-dimensional space E, containing ℓ_∞^2 isometrically (and consequently, failing to be uniformly non-square), so that, if K is a Hausdorff compact, and $J : E \to C(K)$ is an isometric embedding, then $\||Jx| \wedge |Jy|\| \geq 1/3$ for any norm one $x, y \in E$.

The following lemma is needed for the proof, and may be of interest in its own right.

Lemma 4.6.8. Suppose K is a Hausdorff compact, E is a Banach space, and $J : E \to C(K)$ is an isometric embedding. Denote by $\mathcal{F}$ the set of all extreme points of the unit ball of E^*. Then, for any $x, y \in E$, $\||Jx| \wedge |Jy|\| \geq \sup_{e^* \in \mathcal{F}} |e^*(x)| \wedge |e^*(y)|$.

Proof. Standard duality considerations tell us that $J^* : M(K) \to E^*$ ($M(K)$ stands for the space of Radon measures on K) is a strict quotient – that is, for any $e^* \in E^*$ there exists $\mu \in M(K)$ so that $\|\mu\| = \|e^*\|$ and $J^*\mu = e^*$. Further, we claim that, for any $e^* \in \mathcal{F}$, there exists $t \in K$ so that $J^*\delta_t \in \{e^*, -e^*\}$. Indeed, the set $S = \{\mu \in M(K) : \|\mu\| \leq 1, J^*\mu = e^*\}$ is weak*-compact, hence it is the weak*-closure of the convex hull of its extreme points. We claim that any such extreme point is also an extreme point of $\{\mu \in M(K) : \|\mu\| \leq 1\}$. Indeed, suppose $\mu = (\mu_1 + \mu_2)/2$, with $\|\mu_1\|, \|\mu_2\| \leq 1$. Then $e^* = (J^*\mu_1 + J^*\mu_2)/2$, which guarantees that $e^* = J^*\mu_1 = J^*\mu_2$, so $\mu_1, \mu_2 \in S$, and therefore, they coincide with μ.

To finish the proof, recall that the extreme points of $\{\mu \in M(K) : \|\mu\| \leq 1\}$ are point evaluations and their opposites. $\square$

Proof of Proposition 4.6.7. To obtain E, equip $\mathbb{R}^3$ with the norm

$$\|(x, y, z)\| = \max\left\{|x|, |y|, \frac{1}{2}\big(|x| + |y| + |z|\big)\right\}. \tag{4.6.1}$$

Clearly $\{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{R}\}$ gives us an isometric copy of ℓ_∞^2 in E. Note that the unit ball of E^* is a polyhedron with vertices $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, and $(\pm 1/2, \pm 1/2, \pm 1/2)$; we denote this set of vertices by $\mathcal{F}$. In light of Lemma 4.6.8, we have to show that, for any norm one $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in E, there exists $e^* \in \mathcal{F}$ so that $|e^*(x)| \wedge |e^*(y)| \geq 1/3$.

In searching for e^*, we deal with several cases separately. Note first that, if $|x_1| \wedge |y_1| \geq 1/3$, then $e^* = (1, 0, 0)$ has the desired properties. The case of $|x_2| \wedge |y_2| \geq 1/3$ is treated similarly. Henceforth we assume $|x_1| \wedge |y_1|, |x_2| \wedge |y_2| < 1/3$. In light of (4.6.1), we need to consider three cases:

(i) $|x_1| = 1 = |y_2|$ or $|x_2| = 1 = |y_1|$.

(ii) Either $|x_1| \vee |x_2| = 1$ and $|y_1| + |y_2| + |y_3| = 2$, or $|y_1| \vee |y_2| = 1$ and $|x_1| + |x_2| + |x_3| = 2$.

(iii) $|x_1| + |x_2| + |x_3| = 2 = |y_1| + |y_2| + |y_3|$.

In all the three cases, we look for $e^* = (\varepsilon_1, \varepsilon_2, \varepsilon_3)/2$, with $\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1$ selected appropriately.

Case (i). We shall assume $x_1 = 1 = y_2$, as other permutations of indices and choices of sign are handled similarly. Select $\varepsilon_1 = 1$, and take ε_3 so that $\varepsilon_3 x_3 \geq 0$. Pick $\varepsilon_2 = 1$ if $\varepsilon_1 y_1 + \varepsilon_3 y_3 \geq 0$ and $\varepsilon_2 = -1$ otherwise. Then $|x_2| < 1/3$, hence

$$e^*(x) = \frac{1}{2}\big(\varepsilon_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3\big) \geq \frac{1 - |x_2|}{2} > \frac{1 - 1/3}{2} = \frac{1}{3}.$$

Further,

$$|e^*(y)| = \frac{|\varepsilon_1 y_1 + \varepsilon_2 + \varepsilon_3 y_3|}{2} \geq \frac{1}{2}.$$

Case (ii). We deal with $x_1 = 1$ (and consequently, $|y_1| < 1/3$) and $|y_1| + |y_2| + |y_3| = 2$, as other possible settings can be treated similarly. Let $\varepsilon_1 = 1$. If $|x_2| < 1/3$, select ε_3 so that $\varepsilon_3 x_3 \geq 0$. Pick ε_2 so that $\varepsilon_2 y_2$ and $\varepsilon_3 y_3$ have the same sign. Then

$$|e^*(x)| \geq \frac{1 + |x_3| - |x_2|}{2} \geq \frac{1 - |x_2|}{2} \geq \frac{1 - 1/3}{2} = \frac{1}{3},$$

and

$$|e^*(y)| \geq \frac{|y_2| + |y_3| - |y_1|}{2} = \frac{2 - 2|y_1|}{2} \geq \frac{2 - 2 \cdot 1/3}{2} = \frac{2}{3}.$$

Suppose, conversely, that $|x_2| \geq 1/3$, hence $|y_2| < 1/3$. Let $\varepsilon_2 = \operatorname{sign} x_2$. Select ε_3 so that $\varepsilon_1 y_1$ and $\varepsilon_3 y_3$ are of the same sign. Then $|x_3| \leq 2 - (1 + |x_2|) = 1 - |x_2|$, hence

$$|e^*(x)| \geq \frac{1 + |x_2| - |x_3|}{2} \geq \frac{2|x_2|}{2} \geq \frac{1}{3}.$$

On the other hand, $2 - |y_2| = |y_1| + |y_3|$ and

$$|e^*(y)| \geq \frac{|y_1| + |y_3| - |y_2|}{2} = \frac{2 - 2|y_2|}{2} \geq \frac{2 - 2 \cdot 1/3}{2} \geq \frac{2}{3}.$$

Case (iii). If $|x_1|, |x_2| < 1/3$, let $\varepsilon_3 = \operatorname{sign} x_3$, and select $\varepsilon_1, \varepsilon_2$ so that both $\varepsilon_1 y_1$ and $\varepsilon_2 y_2$ have the same sign as $\varepsilon_3 y_3$. Then

$$|e^*(x)| \geq \frac{|x_3| - |x_1| - |x_2|}{2} = \frac{2 - 2(|x_1| + |x_2|)}{2} \geq \frac{2 - 4 \cdot 1/3}{2} = \frac{1}{3},$$

and

$$|e^*(y)| = \frac{|y_1| + |y_2| + |y_3|}{2} = 1.$$

The case of $|y_1|, |y_2| < 1/3$ is handled similarly.

Now suppose neither of the above holds. Up to a permutation of indices, we assume that $|x_1| \geq 1/3$ (hence $|y_1| < 1/3$), and $|y_2| \geq 1/3$ (hence $|x_2| < 1/3$). Then let $\varepsilon_1 = \operatorname{sign} x_1$ and $\varepsilon_3 = \operatorname{sign} x_3$. Pick ε_2 so that $\operatorname{sign} \varepsilon_2 y_2 = \operatorname{sign} \varepsilon_3 y_3$, then

$$|e^*(x)| \geq \frac{|x_1| + |x_3| - |x_2|}{2} = \frac{2 - 2|x_2|}{2} \geq \frac{2 - 2 \cdot 1/3}{2} = \frac{2}{3},$$

and likewise,

$$|e^*(y)| \geq \frac{|y_2| + |y_3| - |y_1|}{2} \geq \frac{2}{3}.$$

□

4.7 Examples of subspaces and subsets doing Hölder stable phase retrieval

In this final section, we construct various examples of subspaces and subsets of $L_p(\mu)$ doing stable phase retrieval. This section is based on the work [82], which is joint with Michael

Christ and Ben Pineau.

We begin by setting notation. Let (X, μ) be a measure space and let V be a closed subspace of the (real or complex) Hilbert space $L^2 = L^2(\mu)$. In previous sections, we have mainly studied subspaces of real-valued L^2 for which there exists $C < \infty$ satisfying

$$\min\big(\|f-g\|_{L^2}\,,\, \|f+g\|_{L^2} \big) \leqslant C \big\| \, |f| - |g| \, \big\|_{L^2} \ \forall\, f, g \in V, \tag{4.7.1}$$

and have constructed various infinite-dimensional examples. The present section develops simple examples of infinite-dimensional subspaces in which versions of stable phase retrieval hold. These examples include certain variants of Rademacher series and lacunary Fourier series. More importantly, our constructions work over the complex field, and give the first examples of infinite dimensional complex SPR subspaces.

For complex-valued functions, the natural quantity on the left-hand side of the inequality (4.7.1) becomes $\min_{|z|=1} \|f - zg\|_{L^2}$, with the minimum taken over all complex numbers z of modulus 1. As in previous sections, we say that a subspace V of a complex L_2-space satisfies stable phase retrieval if there exists $C < \infty$ such that

$$\min_{|z|=1} \|f - zg\|_{L^2} \leqslant C \big\| \, |f| - |g| \, \big\|_{L^2} \ \forall\, f, g \in V. \tag{4.7.2}$$

We generalize the stable phase retrieval inequality in the following way.

Definition 4.7.1. Let $p \in [1, \infty]$ and let V be a subset of the complex Banach space $L^p(\mu)$ for some measure μ. We say that V satisfies L^p-Hölder-stable phase retrieval if there exist parameters $\gamma \in (0, 1]$ and $C < \infty$ such that

$$\min_{|z|=1} \|f - zg\|_{L^p} \leqslant C \big\| \, |f| - |g| \, \big\|_{L^p}^{\gamma} \cdot (\|f\|_{L^p} + \|g\|_{L^p})^{1-\gamma} \quad \forall\, f, g \in V. \tag{4.7.3}$$

We say that V satisfies L^p-stable phase retrieval if (4.7.3) holds with $\gamma = 1$.

Stable phase retrieval in the sense (4.7.2) is thus L^p-stable phase retrieval for $p = 2$. The notion of Hölder-stable phase retrieval for subsets has appeared in work of Cahill, Casazza, and Daubechies [70]. We are primarily interested in subspaces V, but in Example 4.7.12 below, V is not a subspace. We will abbreviate, writing L^p-Hölder-SPR and L^p-SPR, and occasionally writing L^p-Lipschitz-SPR as a synonym for L^p-SPR. For real Hilbert spaces $L^2(\mu, \mathbb{R})$, this definition is modified by replacing $\{z \in \mathbb{C} : |z| = 1\}$ by $\{\pm 1\}$. We will write "real L^p-SPR". Only the exponents $p = 2, 4$ arise in the examples below.

By defining the equivalence relation $\sim$ on a subspace V by $f \sim g$ if and only if $f = zg$ for some unimodular scalar z, we see that $\min_{|z|=1} \|f - zg\|_{L^p}$ is exactly the distance between f and g in the quotient space $V/\sim$. In particular, V satisfies L^p-SPR with constant C if and only if the recovery map of $f \in V/\sim$ from $|f|$ is well-defined and C-Lipschitz.

Some of our proofs only directly establish L^2-Hölder-SPR with certain specific exponents $\gamma < 1$, rather than the formally stronger property of L^2-Lipschitz-SPR. However, recall that from Corollary 4.3.11 (originally proved in [115, Corollary 3.12]) we know that for both the real and the complex cases that for any exponent $p \in [1, \infty]$, for subspaces V, L^p-Hölder-SPR implies L^p-Lipschitz-SPR. We will exploit this general result to upgrade conclusions from L^2-Hölder-SPR to L^2-Lipschitz-SPR.

Let μ be a probability measure. Consider an orthonormal subset $\{r_j : j \in \mathbb{N}\}$ of the complex Hilbert space $L^2 = L^2(\mu) = L^2(\mu, \mathbb{C})$. Let $V \subset L^2$ be the closure of the span of $\{r_j\}$ over $\mathbb{C}$. Let $\mathbb{1}$ be the function $\mathbb{1}(x) \equiv 1$. Define associated functions

$$s_j = |r_j|^2 - \mathbb{1}. \tag{4.7.4}$$

In the case of $L^2(\mu, \mathbb{C})$, we consider closed subspaces spanned by orthogonal sets $\{r_j : j \in \mathbb{N}\}$ satisfying the following three hypotheses:

$$\left\{\mathbb{1},\, s_i,\, r_j\overline{r_k} : i, j, k \in \mathbb{N} \text{ and } j \neq k\right\} \text{ is an orthogonal set.} \tag{4.7.5}$$

$$\sup_j \|r_j\|_{L^4} < \infty. \tag{4.7.6}$$

$$\text{There exists } \delta > 0 \text{ such that } \inf_i \|r_i\|_4^4 \geqslant 1 + \delta \text{ and } \inf_{j \neq k} \|r_j\overline{r_k}\|_2^2 \geqslant \delta. \tag{4.7.7}$$

Since $\|s_i\|_2^2 = \|r_i\|_4^4 - 2\|r_i\|_2^2 + 1 = \|r_i\|_4^4 - 1$ by the hypothesis that $\|r_i\|_2 = 1$, the first part of hypothesis (4.7.7) can be equivalently restated as $\|s_i\|_2^2 \geqslant \delta > 0$.

A consequence of these hypotheses is that $V \subset L^4$ and there exists $C < \infty$ such that

$$\|f\|_{L^4} \leqslant C\|f\|_{L^2} \;\; \forall f \in V. \tag{4.7.8}$$

Indeed, if $f = \sum_k a_k r_k$ with $(a_k : k \in \mathbb{N}) \in \ell^2$ then $|f|^2$ is represented as the pairwise orthogonal sum

$$|f|^2 = \sum_{i \neq j} a_i\overline{a_j}r_i\overline{r_j} + \sum_k |a_k|^2 s_k + \|f\|_2^2 \cdot \mathbb{1}. \tag{4.7.9}$$

The L^4 norm bound follows using orthogonality and the Cauchy-Schwarz inequality, since $\|r_i\overline{r_j}\|_2 \leqslant \|r_i\|_4 \|r_j\|_4$ and $\|s_k\|_2 \leqslant 1 + \|r_k^2\|_2 \leqslant 1 + \|r_k\|_4^2$ are uniformly bounded by (4.7.6). The inequality (4.7.8), and a similar L^6 norm inequality that holds under stronger hypotheses, are pillars of our reasoning.

Let $\{r_j\} \subset L^2(\mu, \mathbb{C})$ be an orthonormal set of complex-valued functions satisfying hypotheses (4.7.5),(4.7.6),(4.7.7), and let V be as above. We begin by observing that $|f|$ determines f uniquely, up to multiplication by a unimodular complex scalar, for each $f \in V$. Indeed, $|f|$ certainly determines f if $|f| = 0$ almost everywhere. Consider next any $0 \neq f \in V$. Expand $f = \sum_k a_k r_k$, with $a \in \ell^2$. Then $|f|^2 \in L^2$, and has expansion (4.7.9). The terms of this sum are mutually orthogonal, and the series converges in L^2 norm. Therefore $|f|^2$ determines each of the coefficients in this expansion; it determines each $|a_n|^2$ and each product $a_i\overline{a_j}$. Choose some n_0 satisfying $a_{n_0} \neq 0$. Writing $a_n = |a_n|e^{i\arg(a_n)}$, $\arg(a_n) - \arg(a_{n_0})$ is determined modulo $2\pi\mathbb{Z}$ by $|a_n|^2$, $|a_{n_0}|^2$, and $a_n\overline{a_{n_0}}$. Therefore $|f|^2$ and $\arg(a_{n_0})$ together determine all coefficients a_n, and hence determine f, up to multiplication by $z = e^{i\arg(a_{n_0})}$.

Note that this reconstruction of f from $|f|$ is not stable in the sense desired, since it requires division by $|a_{n_0}|$, for which no *a priori* positive lower bound is available. Note also that it exploits only the coefficients of s_k and of $r_n\overline{r}_{n_0}$. The next result asserts that under these same hypotheses, the reconstruction of f from $|f|$ can be done stably.

Proposition 4.7.2. Let μ be a probability measure. Let $\{r_j\} \subset L^2(\mu, \mathbb{C})$ be an orthonormal set of complex-valued functions satisfying hypotheses (4.7.5),(4.7.6),(4.7.7). Then V satisfies L^4-SPR.

Under a supplementary hypothesis, Proposition 4.7.2 has an almost immediate implication for L^2-stable phase retrieval.

Corollary 4.7.3. Let $\{r_n\}$ satisfy the hypotheses of Proposition 4.7.2. Assume also that there exist $q > 4$ and $C < \infty$ such that $V \subset L^q(\mu)$ and

$$\|f\|_{L^q} \leqslant C \|f\|_{L^2} \;\; \forall\, f \in V. \tag{4.7.10}$$

Then V satisfies L^2-stable phase retrieval.

Proposition 4.7.2 and Corollary 4.7.3 will be proved below.

As is well known, for any even integer $q \geqslant 6$, the inequality (4.7.10) holds whenever the functions r_j are independent random variables, have uniformly bounded L^q norms, and satisfy $r_j \perp \mathbb{1}$. Indeed, consider the case $q = 6$. If $\|r_n\|_6 \leqslant A < \infty$ for all n then

$$\begin{aligned}\Big\|\sum_n a_n r_n\Big\|_6^6 &= \sum_{i_1,i_2,i_3}\sum_{j_1,j_2,j_3}\prod_{k=1}^{3} a_{i_k}\prod_{l=1}^{3}\overline{a_{j_l}}\,\Big\langle r_{i_1}r_{i_2}r_{i_3},\, r_{j_1}r_{j_2}r_{j_3}\Big\rangle\\ &\leqslant \sum_n |a_n|^6 A^6 + \binom{6}{2} A^6 \sum_m\sum_n |a_m|^4|a_n|^2 + \binom{6}{3} A^6 \sum_m\sum_n |a_m|^3|a_n|^3\end{aligned}$$

since $\Big\langle r_{i_1}r_{i_2}r_{i_3},\, r_{j_1}r_{j_2}r_{j_3}\Big\rangle = 0$ unless each of the six indices that appear in the inner product, appears at least twice. The same reasoning applies for arbitrary even integers $q \geqslant 8$.

We next present a class of examples based on Proposition 4.7.2 and Corollary 4.7.3. The construction involves sums of independent random variables, and may be contrasted with a more elaborate construction in [71], which combines independent summands with summands having pairwise disjoint supports. A more direct comparison can be made with Example 4.7.9 below, which is a real analogue of Example 4.7.4.

Example 4.7.4. Let μ be a probability measure. Let r_n be independent identically distributed complex-valued random variables in $L^6(\mu)$ satisfying $\|r_n\|_{L^2} = 1$. Assume that

$$r_n \perp \mathbb{1} \text{ and } r_n^2 \perp \mathbb{1} \tag{4.7.11}$$

$$\mu(\{x : |r_n(x)| \neq 1\}) > 0. \tag{4.7.12}$$

Then $\{r_n\}$ satisfies the hypotheses of Proposition 4.7.2, and satisfies those of Corollary 4.7.3 with $q = 6$. Therefore the closure of its span in $L^2(\mu)$ satisfies both L^4-SPR and L^2-SPR.

Example 4.7.4 and Example 4.7.9 do not apply to Rademacher series, for which $r_n = \pm 1$ each with probability $\frac{1}{2}$, violating hypothesis (4.7.12). Nor do Rademacher series satisfy phase retrieval, since $|r_m| \equiv |r_n|$ for all m, n.

In the formulation of Example 4.7.4, the hypothesis $r_n^2 \perp \mathbb{1}$, together with independence, ensure that $r_i\overline{r_j} \perp r_j\overline{r_i}$ whenever $i \neq j$, since

$$\langle r_i\overline{r_j},\, r_j\overline{r_i}\rangle = \int r_i^2\,\overline{r_j}^2\,d\mu = \int r_i^2\,d\mu \cdot \overline{\int r_j^2\,d\mu} = \langle r_i^2, \mathbb{1}\rangle \cdot \overline{\langle r_j^2, \mathbb{1}\rangle} = 0.$$

The hypothesis that $|r_n|$ is not equal almost everywhere to 1 ensures that $\|s_n\|_2 \neq 0$. The other hypotheses of Proposition 4.7.2, and the embedding of V into L^6, are consequences of independence, identical distribution, and the assumption that $r_n \perp \mathbb{1}$. Details of the verifications are left to the reader. Note that the assumption that $r_n \in L^6$ is an easy way to deduce SPR in L^2. The proof directly establishes SPR in L^4 when $r_n \in L^4$. It is an open problem to prove variants of Example 4.7.4 and Example 4.7.9 which deduce SPR in L^2 without any higher order integrability assumptions, i.e., only assuming $r_n \in L^2$.

Before indicating other classes of examples with stable phase retrieval, we prove Corollary 4.7.3 and Proposition 4.7.2.

Proof of Corollary 4.7.3. By Hölder's inequality,

$$\||f|-|g|\|_4 \leqslant \||f|-|g|\|_2^{\theta}\left(\|f\|_q+\|g\|_q\right)^{1-\theta} \leqslant C^{1-\theta}\||f|-|g|\|_2^{\theta}\left(\|f\|_2+\|g\|_2\right)^{1-\theta}$$

where $\theta \in (0,1)$ is defined by the relation $\frac{1}{4}=\frac{\theta}{2}+\frac{1-\theta}{q}$. Therefore for any $f, g \in V$, by Hölder's inequality and Proposition 4.7.2,

$$\min_{|z|=1}\|f-zg\|_2 \leqslant \min_{|z|=1}\|f-zg\|_4 \leqslant C'\||f|-|g|\|_4 \leqslant C''\||f|-|g|\|_2^{\theta}\left(\|f\|_2+\|g\|_2\right)^{1-\theta}.$$

Thus L^2-Hölder SPR holds. L^2-Lipschitz SPR follows from Corollary 4.3.11. □

The proof of Proposition 4.7.2 relies on the following elementary inequality.

Lemma 4.7.5. Let $\{r_j\}$ satisfy hypotheses (4.7.5), (4.7.6), and (4.7.7). For any $f, g \in V$,

$$\left\||f|^2-|g|^2\right\|_2^2 \geqslant \delta\Big[\|f\|_2^2\|g\|_2^2-|\langle f, g\rangle|^2\Big]+\left(\|f\|_2^2-\|g\|_2^2\right)^2. \tag{4.7.13}$$

We prove Proposition 4.7.2 assuming Lemma 4.7.5, and then prove Lemma 4.7.5 below.

Proof of Proposition 4.7.2. By multiplying by scalars and interchanging the roles of f, g if necessary, we may assume with no loss of generality that $\|f\|_2 \leqslant \|g\|_2=1$. By Cauchy-Schwarz,

$$\begin{aligned}\left\||f|^2-|g|^2\right\|_2 \leqslant \big\||f|+|g|\big\|_4 \cdot \big\||f|-|g|\big\|_4 &\leqslant C\left(\|f\|_2+\|g\|_2\right)\big\||f|-|g|\big\|_4 \\ &\leqslant 2C\big\||f|-|g|\big\|_4.\end{aligned} \tag{4.7.14}$$

Write $f=re^{i\theta}g+h$ with $r \geqslant 0$, $\theta \in \mathbb{R}$, and $h \perp g$. Then $|\langle f, g\rangle|^2=r^2$ and

$$\|f\|_2^2\|g\|_2^2-|\langle f, g\rangle|^2=\left(r^2+\|h\|_2^2\right)-r^2=\|h\|_2^2. \tag{4.7.15}$$

Let $\delta \in (0,1]$ be a parameter for which the conclusion (4.7.13) of Lemma 4.7.5 holds. Inserting (4.7.15) into (4.7.13) gives

$$\delta\|h\|_2^2+\left(1-r^2-\|h\|_2^2\right)^2 \leqslant \left\||f|^2-|g|^2\right\|_2^2 \leqslant 4C^2\||f|-|g|\|_4^2.$$

Therefore since $0 < \delta \leqslant 1$,

$$\delta \|h\|_2^2 + \tfrac{1}{4}\delta(1 - r^2 - \|h\|_2^2)^2 \leqslant 4C^2 \||f| - |g|\|_4^2 .$$

The left-hand side is

$$(\delta - \tfrac{1}{2}\delta(1-r^2)) \|h\|_2^2 + \tfrac{1}{4}\delta(1-r^2)^2 + \tfrac{1}{4}\delta \|h\|_2^4 \geqslant \tfrac{1}{2}\delta \|h\|_2^2 + \tfrac{1}{4}\delta(1-r^2)^2$$

and therefore since $(1 - r) \leqslant (1 - r^2)$,

$$\|h\|_2^2 + (1-r)^2 \leqslant 16C^2\delta^{-1} \||f| - |g|\|_4^2 .$$

Defining $z = e^{i\theta}$, $\|f - zg\|_2^2 = \|h\|_2^2 + (1-r)^2$ and therefore

$$\|f - zg\|_2^2 \leqslant 16C^2\delta^{-1} \||f| - |g|\|_4^2 .$$

Since $f - zg \in V$, its L^4 norm is majorized by a constant multiple of its L^2 norm. Thus $\|f - zg\|_4 \leqslant C' \||f| - |g|\|_4$ for another finite constant C' which depends on δ. □

Proof of Lemma 4.7.5. Under the hypothesis that $\|r_j\|_2 = 1$, $\|s_j\|_2^2 = \|r_j\|_4^4 - 1$. Therefore the hypothesis $\inf_j \|r_j\|_4^4 \geqslant 1 + \delta$ is equivalent to $\inf_j \|s_j\|_2^2 \geqslant \delta$.

Express $f, g \in V$ as $f = \sum_k a_k r_k$ and $g = \sum_k b_k r_k$. By (4.7.9),

$$|f|^2 - |g|^2 = \sum_{i \neq j} (a_i\overline{a_j} - b_i\overline{b_j}) r_i\overline{r_j} + (\|f\|_2^2 - \|g\|_2^2)\mathbb{1} + \sum_k (|a_k|^2 - |b_k|^2) s_k \tag{4.7.16}$$

where $\mathbb{1}$ is the constant function 1. The functions $\mathbb{1}$, s_k, and $r_i\overline{r_j}$ with $i \neq j$ are pairwise orthogonal by hypothesis (4.7.5). Therefore

$$\begin{aligned} \left\||f|^2 - |g|^2\right\|_2^2 &= \sum_k \left||a_k|^2 - |b_k|^2\right|^2 \|s_k\|_2^2 + (\|f\|_2^2 - \|g\|_2^2)^2 + \sum_{i \neq j} |a_i\overline{a_j} - b_i\overline{b_j}|^2 \|r_i\overline{r_j}\|_2^2 \\ &\geqslant \delta \sum_k \left||a_k|^2 - |b_k|^2\right|^2 + (\|f\|_2^2 - \|g\|_2^2)^2 + \delta \sum_{i \neq j} |a_i\overline{a_j} - b_i\overline{b_j}|^2 \end{aligned} \tag{4.7.17}$$

by hypothesis (4.7.7). Algebraic manipulation of the last term on the right-hand side gives

$$\begin{aligned} \sum_{i \neq j} |a_i\overline{a_j} - b_i\overline{b_j}|^2 &= (\sum_k |a_k|^2)^2 + (\sum_k |b_k|^2)^2 - 2|\sum_k a_k\overline{b_k}|^2 - \sum_k (|a_k|^2 - |b_k|^2)^2. \\ &= \|f\|_2^4 + \|g\|_2^4 - 2|\langle f, g\rangle|^2 - \sum_k (|a_k|^2 - |b_k|^2)^2 \\ &= 2\Big[\|f\|_2^2 \|g\|_2^2 - |\langle f, g\rangle|^2 \Big] + (\|f\|_2^2 - \|g\|_2^2)^2 - \sum_k (|a_k|^2 - |b_k|^2)^2. \end{aligned}$$

Substituting this expression into the preceding lower bound, two terms cancel, leaving

$$\begin{aligned}\big\||f|^2 - |g|^2\big\|_2^2 &\geqslant 2\delta\Big[\,\|f\|_2^2\,\|g\|_2^2 - |\langle f, g\rangle|^2\Big] + (1+\delta)(\|f\|_2^2 - \|g\|_2^2)^2 \\ &\geqslant 2\delta\Big[\,\|f\|_2^2\,\|g\|_2^2 - |\langle f, g\rangle|^2\Big] + (\|f\|_2^2 - \|g\|_2^2)^2.\end{aligned}$$

□

A well-known theme is the analogy between lacunary Fourier series and sums of independent random variables. Our next two examples express this theme.

Example 4.7.6. Let $N \geqslant 2$ and let $P \in L^2([0,1], \mathbb{C})$ be a trigonometric polynomial

$$P(x) = \sum_{k=1}^{N} \alpha_k e^{2\pi i k x}$$

with coefficients $\alpha_k \in \mathbb{C}$. Suppose that $|P|$ is not constant. Let $A \in \mathbb{N}$ satisfy $A > 2N$. Let $V \subset L^2([0,1], \mathbb{C})$ be the closure of the span of $\{P(A^n x) : n \in \mathbb{N}\}$. Then V satisfies both L^4-SPR and L^2-SPR.

Example 4.7.6 is an instance of Corollary 4.7.3, with arbitrarily large $q < \infty$. Verification of the hypotheses of the corollary is left to the reader. The L^q norm inequality (4.7.10) holds since $\sum_{n=1}^{\infty} a_n \sum_{k=1}^{N} \alpha_k e^{2\pi i A^n k x}$ is a sum of N lacunary Fourier series, and since any lacunary series with ℓ^2 coefficients defines a function in L^q for all $q < \infty$. The next example is a real analogue of Example 4.7.6.

Example 4.7.7. The closure of the subspace of $L^2([0,1], \mathbb{R})$ spanned by $\{\sin(2\pi 4^n x) : n \in \mathbb{N}\}$ satisfies L^4-SPR and L^2-SPR.

Example 4.7.11, below, is a more efficient version of Example 4.7.7. If complex rather than real linear combinations are allowed, then phase retrieval cannot hold in Example 4.7.7, nor in any example with two real-valued basis functions r, r'. Indeed, $f = r + ir'$ and $g = \overline{f} = r - ir'$ satisfy $|f| \equiv |g|$, but f is not a constant multiple of g.

Proposition 4.7.2 and Corollary 4.7.3 do not apply to Example 4.7.7, since with $r_n(x) = 2^{1/2}\sin(2\pi 4^n x)$ one has $r_i \overline{r_j} = r_j \overline{r_i}$ for all i, j. However, a small modification of the reasoning underlying those two results gives Proposition 4.7.8, whose hypotheses are satisfied in Example 4.7.7.

For Hilbert spaces $L^2(\mu, \mathbb{R})$ of real-valued functions with orthonormal bases of real-valued functions r_n we modify the orthogonality hypothesis (4.7.5) as follows:

$$\left\{\mathbb{1},\, s_i,\, r_j r_k : i, j, k \in \mathbb{N} \text{ and } j < k\right\} \text{ is an orthogonal set.} \tag{4.7.18}$$

Proposition 4.7.8. Let μ be a probability measure. Let $\{r_j\} \subset L^2(\mu)$ be an orthonormal set of real-valued functions satisfying hypotheses (4.7.6),(4.7.7),(4.7.18). Then the closure $V \subset L^2(\mu, \mathbb{R})$ of the span of $\{r_j : j \in \mathbb{N}\}$ over $\mathbb{R}$ satisfies real L^4-SPR.

If there exist $q > 4$ and $C < \infty$ such that the L^q norm inequality (4.7.10) holds for all functions in V then V satisfies real L^2-SPR.

The only changes from the proof of Proposition 4.7.2 are that in (4.7.16), the first term becomes $2\sum_{i<j}(a_i a_j - b_i b_j) r_i r_j$, and consequently that on the right-hand side of (4.7.17), the last term is changed to

$$4\sum_{i<j}(a_i a_j - b_i b_j)^2 \left\|r_i r_j\right\|_2^2 = 2\sum_{i\neq j}\left|a_i a_j - b_i b_j\right|^2 \left\|r_i \overline{r_j}\right\|_2^2.$$

The corresponding quantity in the proof of Proposition 4.7.2 is $\sum_{i\neq j}|a_i a_j - b_i b_j|^2 \left\|r_i\overline{r_j}\right\|_2^2$. The new factor of 2 thus arising is favorable for our purpose. □

If 4^n is replaced by 3^n or 2^n in Example 4.7.7 then Proposition 4.7.8 no longer applies. Indeed, if 3^n is used the desired orthogonality between s_n and $r_{n+1}r_n$ fails to hold; $e^{2\pi i\cdot 2\cdot 3^n x}$ occurs with nonzero coefficient in the Fourier series for s_n, while $e^{2\pi i\cdot 3^{n+1}x}\cdot e^{-2\pi i\cdot 3^n x} = e^{2\pi i\cdot 2\cdot 3^n x}$ also occurs with nonzero coefficient in the Fourier series for $r_{n+1}r_n$. A similar issue arises for 2^n.

Another application of Proposition 4.7.8 is a real analogue of Example 4.7.4.

Example 4.7.9. Let μ be a probability measure. Let $q > 4$ be an even integer. Let r_n be independent identically distributed real-valued random variables in $L^q(\mu)$ satisfying $\|r_n\|_{L^2} = 1$. Assume that

$$\begin{cases} r_n \perp \mathbb{1} \\ \mu(\{x : |r_n(x)| \neq 1\}) > 0. \end{cases} \tag{4.7.19}$$

Then $\{r_n\}$ satisfies the hypotheses of Proposition 4.7.8, and consequently the closure of its span in $L^2(\mu, \mathbb{R})$ satisfies real L^4-SPR and real L^2-SPR.

Remark 4.7.10. The hypothesis $q > 4$ is only needed to get SPR in L^2; without this hypothesis, one deduces SPR in L^4.

We proceed by lightly modifying a construction of Rudin [292] to create examples of trigonometric series related to the theory of $\Lambda(p)$ sets that satisfy stable phase retrieval, yet are rather far from being lacunary in nature. To simplify matters, we set this example in the ambient Hilbert space $L^2([0,1]\times[0,1],\mathbb{C})$, with respect to two-dimensional Lebesgue measure, rather than in $L^2([0,1],\mathbb{C})$. Define r_ν to be

$$r_\nu(x,y) = 2^{1/2}\sin(2\pi\nu y)\,e^{2\pi i n_\nu x}, \tag{4.7.20}$$

where $(n_\nu : \nu\in\mathbb{N})$ is a subsequence of $\mathbb{N}$ to be specified.

To quantify the asymptotic density of a subsequence (n_ν) of $\mathbb{N}$, define $\alpha(N)$ to be the number of indices ν satisfying $n_\nu \leqslant N$.

Example 4.7.11. There exists a strictly increasing sequence $(n_\nu : \nu\in\mathbb{N})$, satisfying the asymptotic density lower bound $\limsup_{N\to\infty} N^{-1/2}\alpha(N) > 0$ such that the closed subspace V of $L^2([0,1]\times[0,1])$ spanned by the functions r_ν defined in (4.7.20) satisfies L^4-SPR.

There exists such a sequence satisfying $\limsup_{N\to\infty} N^{-1/3}\alpha(N) > 0$ such that V also satisfies L^2-SPR.

Thus these sequences (n_ν) are far denser than lacunary sequences.

Proof. In §4.7 of [292], Rudin constructs a sequence n_ν satisfying $\limsup N^{-1/2}\alpha(N) > 0$ such that $n_i+n_j = n_k+n_l$ if and only if (i,j) is a permutation of (k,l), and deduces from this property the inequality $\|f\|_4 \leqslant C\,\|f\|_2$ for all L^2 functions of the form $f(x) = \sum_\nu c_\nu e^{2\pi i n_\nu x}$. Let (n_ν) be any such sequence, and define $\{r_\nu\}$ by (4.7.20). Hypothesis (4.7.6), the uniform upper bound for $\|r_\nu\|_4$, certainly holds. The nonconstant factors $\sin(2\pi\nu y)$ ensure a uniform lower bound $\|r_\nu\|_4^4 \geqslant 1+\delta$, so (4.7.7) holds.

To verify hypothesis (4.7.5), consider any inner product $\langle r_j\overline{r_k},\, r_l\overline{r_m}\rangle$ with $j\neq k$ and $l\neq m$. Calculation of this inner product yields a factor of $\int_0^1 e^{2\pi i(n_j-n_k-n_l+n_m)x}\,dx$, which vanishes unless $n_j-n_k-n_l+n_m = 0$. Equivalently, $n_j+n_m = n_l+n_k$. Therefore by Rudin's construction, (l,k) is a permutation of (j,m). If $j\neq k$, this implies that $(j,k)=(l,m)$. The associated functions $s_k(x,y) = 2\sin^2(2\pi ky)-1 = -\cos(4\pi ky)$ are independent of x, hence satisfy $s_k\perp r_i\overline{r_j}$ whenever $i\neq j$. Finally, if $k\neq l$ the $s_k\perp s_l$ since $\cos(4\pi ky)\perp\cos(4\pi ly)$ in $L^2([0,1])$.

Rudin [292] likewise constructs a sequence satisfying $\limsup N^{-1/3}\alpha(N) > 0$, satisfying the same conditions in the preceding paragraph, and satisfying $\|\sum_\nu b_\nu e^{2\pi i n_\nu x}\|_6 \leqslant C\,\|b\|_{\ell^2}$ for all coefficient sequences $b \in \ell^2$. Consequently for any function $f(x,y)$ of the form $\sum_\nu a_\nu \sin(2\pi\nu y)e^{2\pi i n_\nu x}$ with $a \in \ell^2$,

$$\begin{aligned}\int_{[0,1]^2} \big|\sum_\nu a_\nu \sin(2\pi\nu y)e^{2\pi i n_\nu x}\big|^6\,dx\,dy &\leqslant C\int_{[0,1]} (\sum_\nu |a_\nu \sin(2\pi\nu y)|^2)^{6/2}\,dy\\ &\leqslant C\int_{[0,1]} (\sum_\nu |a_\nu|^2)^3\,dy = C\,\|a\|_{\ell^2}^6 \leq 8C\,\|f\|_{L^2}^6 .\end{aligned}$$

In each of these two situations, V has the indicated properties. □

Remark. In this example, the subspace V is in a sense larger, relative to other ambient subspaces naturally associated to it, than is the case for corresponding examples involving lacunary series. To formulate this assertion more precisely, for each degree $D \in \mathbb{N}$ let $V_{N,D}$ be the subspace of L^2 spanned by polynomials of degrees $\leqslant D$ in $\{r_\nu : 1 \leqslant \nu \leqslant N\}$. Let N tend to infinity, while D remains fixed. The dimensions $\dim(V_{N,D})$ satisfy $\liminf_{N\to\infty} N^{-3}\dim(V_{N,D}) < \infty$ for any D in Example 4.7.11, while for the lacunary series example $r_\nu = 2^{1/2}\sin(2\pi 4^\nu x)$, $\dim(V_{N,D})$ has order of magnitude N^D. Thus the span of $\{r_\nu : 1 \leqslant \nu \leqslant N\}$, for these N, is a comparatively large subspace of the associated spaces $V_{N,D}$ in Example 4.7.11.

We conclude by giving an example of a subset that satisfies Hölder-stable phase retrieval and is invariant under multiplication by unimodular scalars, but is not a subspace. The aforementioned Corollary 4.3.11 applies only to subspaces, so we are unable to upgrade the conclusion from Hölder-SPR to Lipschitz-SPR.

Example 4.7.12. Let $\Lambda \subset \mathbb{Z}$, and let E be the set of all $f \in L^2([0,1],\mathbb{C})$ such that $\widehat{f}$ is supported on Λ. Suppose that Λ has the property that if $n_j \in \Lambda$ and $n_1 - n_2 = n_3 - n_4$ then either $n_1 = n_2$ or $n_1 = n_3$. Fix $c = (c_n)_{n\in\Lambda} \in \ell^2(\Lambda)_+$ and define

$$E_c = \{f \in E : |\widehat{f}| = c\} = \Big\{\sum_{n\in\Lambda} \gamma_n c_n e^{2\pi i n x} : \gamma_n \in \mathbb{C} \text{ and } \ |\gamma_n| = 1\ \forall n \in \Lambda\Big\}.$$

Then E_c, equipped with the L^4 norm, satisfies (4.7.3) with $\gamma = 1$. Moreover, if for some $q > 4$ all $f \in E$ satisfy the L^q bound $\|f\|_q \leq C'_\Lambda \|f\|_2$ then E_c also satisfies (4.7.3) with $p = 2$ and $\gamma = \frac{q-4}{2q-4}$.

Proof. We begin by noting that $E \subset L^4([0,1],\mathbb{C})$ and $\|f\|_4 \leq C_\Lambda \|f\|_2$ for all $f \in E$. To prove our claim that E_c satisfies (4.7.3) with $p=4$ and $\gamma = 1$, notice that we may assume without loss of generality that $\|c\|_{\ell^2} = 1$. In this case, $\|f\|_2 = \|g\|_2 = 1$, and

$$\||f|^2 - |g|^2\|_2 \leq \||f|-|g|\|_4 \||f|+|g|\|_4 \leq 2C_\Lambda \||f|-|g|\|_4.$$

We claim that the following identity holds for $f, g \in E$:

$$\left|\|f\|_2^2 - \|g\|_2^2\right|^2 + \left[\|f\|_2^4 + \|g\|_2^4 - 2|\langle f,g\rangle|^2\right] = \||f|^2-|g|^2\|_2^2 + \sum_{m\in\Lambda}\left(|\widehat{f}(m)|^2 - |\widehat{g}(m)|^2\right)^2. \quad (4.7.21)$$

The identity (4.7.21) implies that E_c satisfies L^4-Lipschitz-stable phase retrieval. Indeed, the second term on the right-hand side of (4.7.21) vanishes. Since $f, g \in E_c$, they have equal L^2 norms, implying that the first term on the left-hand side vanishes. Write $f = re^{i\theta}g + h$, with $0 \leq r \leq 1$, $\theta \in \mathbb{R}$, and $h \perp g$. Then, $\|f\|_2^2\|g\|_2^2 - |\langle f,g\rangle|^2 = \|h\|_2^2 = 1-r^2$. To finish the proof, note that $\|f - e^{i\theta}g\|_2^2 = \|h\|_2^2 + (1-r)^2 \leq 2\|h\|_2^2$, use the inequality $\|f\|_4 \leq C_\Lambda\|f\|_2$, and combine the above inequalities.

The derivation of (4.7.21) is similar to the proof of Lemma 4.7.5, but easier. The details are left to the reader. That the supplementary L^q bound implies that E_c satisfies (4.7.3) with $p=2$ and $\gamma = \frac{q-4}{2q-4}$ follows from an invocation of Hölder's inequality similar to the one in the proof of Corollary 4.7.3. □

Remark. The subspace E in Example 4.7.12 will not satisfy phase retrieval unless Λ has cardinality at most one, as if $m, n \in \Lambda$ and $f = e^{2\pi i n x}$ and $g = e^{2\pi i m x}$ then $|f| \equiv |g|$. Observe that on the Fourier side, f, g are disjoint unit vectors in ℓ^2 when $m \neq n$. Subsets of the form E_c have an opposite behavior on the Fourier side and appear in the study of random Fourier series.

We now list some general open questions, which may be of interest.

Question 4.7.13. Given a trigonometric polynomial P and a dilation set Λ, when does $\{P(\lambda\cdot) : \lambda \in \Lambda\}$ generate an SPR subspace of L^p? If $P(x) = \sin(2\pi x)$, does any lacunary dilation set give a real SPR subspace of L^2?

Question 4.7.14. The techniques used to build complex SPR subspaces are based on the work of Rudin. Can one find analogues of the techniques of Bourgain and his successors?

Question 4.7.15. Can one build "large" and/or natural subspaces of $L^2(\mathbb{R}^d)$ which do Pauli stable phase retrieval?

Question 4.7.16. Let $p \geq 2$ and let (r_n) be a mean zero iid sequence in $L^p(\mu;\mathbb{R})$ with non-constant moduli. Does the closed span of (r_n) do SPR in L^p? In Example 4.7.9, we showed that this is true when $p = 4$, and by the interpolation/extrapolation theory, it follows that this also holds for $p > 4$. We do not know if it holds for $p = 2$; it would also be interesting to prove variants of Example 4.7.9 for independent but not necessarily iid variables.

Question 4.7.17. Can one build other interesting nonlinear SPR subsets of Banach lattices?

Chapter 5

Bases of non-negative functions

5.1 Introduction

Both the Fourier basis and the Haar basis for $L_2([0,1])$ consist of the constant 1 function together with a sequence of mean zero functions. Likewise, the Calderon condition gives that every wavelet basis for $L_2(\mathbb{R})$ consists entirely of mean zero functions. We are interested in the problem of determining for which $1 \leq p < \infty$ does $L_p(\mathbb{R})$ have a Schauder basis $(f_j)_{j=1}^{\infty}$ consisting entirely of non-negative functions. A sequence $(f_j)_{j=1}^{\infty}$ in $L_p(\mathbb{R})$ is called a *Schauder basis* for $L_p(\mathbb{R})$ if for all $f \in L_p(\mathbb{R})$ there exists unique scalars $(a_j)_{j=1}^{\infty}$ such that

$$f = \sum_{j=1}^{\infty} a_j f_j. \tag{5.1.1}$$

A Schauder basis is called *unconditional* if the series in (5.1.1) converges in every order. Unconditionality is a desirable property, but it has been shown to be too strong to impose on coordinate systems of non-negative functions. Indeed, for all $1 \leq p < \infty$, $L_p(\mathbb{R})$ does not have an unconditional Schauder basis or even unconditional quasi-basis (Schauder frame) consisting of non-negative functions [285]. In particular, both the positive and negative parts of an unconditional Schauder basis must have infinite weight (see [257] for precise quantitative statements).

Though $L_p(\mathbb{R})$ does not have an unconditional Schauder basis of non-negative functions, it does contain subspaces which do. It is clear that any normalized sequence of non-negative functions with disjoint support will be a Schauder basis for its closed span and will be 1-equivalent to the unit vector basis of ℓ_p. This trivial method is essentially the only way to build an unconditional Schauder basic sequence of non-negative functions in $L_p(\mathbb{R})$, as every normalized unconditional Schauder basic sequence of non-negative functions in $L_p(\mathbb{R})$ is equivalent to the unit vector basis for ℓ_p [193]. Likewise, if $(f_j, g_j^*)_{j=1}^\infty$ is an unconditional quasi-basis for a closed subspace X of $L_p(\mathbb{R})$ and $(f_j)_{j=1}^\infty$ is a sequence of non-negative functions then X embeds into ℓ_p [193].

The results for coordinate systems formed by non-negative functions are very different when one allows for conditionality. Indeed, for all $1 \leq p < \infty$, $L_p(\mathbb{R})$ has a Markushevich basis consisting of non-negative functions [285], and the characteristic functions of dyadic intervals form a quasi-basis for $L_p(\mathbb{R})$ consisting of non-negative functions [285]. For the case of conditional Schauder bases, Johnson and Schechtman constructed a Schauder basis for $L_1(\mathbb{R})$ consisting of non-negative functions [193]. Their construction relies heavily on the structure of L_1, and the problem on the existence of conditional Schauder bases for $L_p(\mathbb{R})$ remained open for all $1 < p < \infty$. Our main result is to provide a construction for a Schauder basis of $L_2(\mathbb{R})$ consisting of non-negative functions. For the remaining cases $1 < p < \infty$ with $p \neq 2$, we are not able to build a Schauder basis for the whole space. However, we prove that for all $1 < p < \infty$ there exists a Schauder basic sequence $(f_j)_{j=1}^\infty$ of non-negative functions in $L_p(\mathbb{R})$ such that $L_p(\mathbb{R})$ embeds into the closed span of $(f_j)_{j=1}^\infty$. This chapter is based on [116], which is joint work with Dan Freeman and Alexander Powell.

There are interesting comparisons between results on coordinate systems of non-negative functions for $L_p(\mathbb{R})$ and results on coordinate systems of translations of a single function. As is the case for non-negative functions, there does not exist an unconditional Schauder basis for $L_p(\mathbb{R})$ consisting of translations of a single function ([268] for $p = 2$, [260] for $1 < p \leq 4$, and [114] for $4 < p$). On the other hand, for the range $2 < p < \infty$ there does exist a sequence $(f_j)_{j=1}^\infty$ of translations of a single function in $L_p(\mathbb{R})$ and a sequence of functionals $(g_j^*)_{j=1}^\infty$ in $L_p(\mathbb{R})^*$ such that $(f_j, g_j^*)_{j=1}^\infty$ is an unconditional Schauder frame for $L_p(\mathbb{R})$ [114]. The corresponding result for the range $1 < p < 2$ is unknown, but for $1 < p \leq 2$ the sequence of functionals $(g_j^*)_{j=1}^\infty$ in $L_p(\mathbb{R})^*$ cannot be chosen to be semi-normalized [50]. We take a unifying approach and prove that for all $1 \leq p < \infty$, there exists a Schauder frame $(f_j, g_j^*)_{j=1}^\infty$ of $L_p(\mathbb{R})$ such that $(f_j)_{j=1}^\infty$ is a sequence of translations of a single non-negative function. We obtain this result by first proving that if X is any separable Banach space with the bounded approximation property and $D \subseteq X$ has dense span in X then there exists a Schauder frame for X whose vectors are elements of D.

5.2 A non-negative Schauder basis for Hilbert lattices

Given a separable infinite dimensional Banach space X, a sequence of vectors $(x_j)_{j=1}^\infty$ in X is called a *Schauder basis* of X if for all $x \in X$ there exists a unique sequence of scalars $(a_j)_{j=1}^\infty$ such that

$$x = \sum_{j=1}^{\infty} a_j x_j. \tag{5.2.1}$$

A Schauder basis $(x_j)_{j=1}^\infty$ is called *unconditional* if the series in (5.2.1) converges in every order. If $(x_j)_{j=1}^\infty$ is a Schauder basis then there exists a unique sequence of bounded linear functionals $(x_j^*)_{j=1}^\infty$ called the *biorthogonal functionals* of $(x_j)_{j=1}^\infty$ such that $x_j^*(x_j) = 1$ for all $j \in \mathbb{N}$ and $x_j^*(x_i) = 0$ for all $j \neq i$. A sequence of vectors is called *basic* if it is a Schauder basis for its closed span. A basic sequence (x_j) is called C-basic for some constant $C > 0$ if for all $m \leq n$ we have that

$$\left\|\sum_{j=1}^{m} a_j x_j\right\| \leq C \left\|\sum_{j=1}^{n} a_j x_j\right\| \quad \text{for all sequences of scalars } (a_j)_{j=1}^n. \tag{5.2.2}$$

It follows from the uniform boundedness principle that every basic sequence is C-basic for some constant C. The least value C such that a sequence (x_j) is C-basic is called the *basis constant* of (x_j).

We will be interested in bases of $L_p(\mathbb{R})$ where each of the basis vectors x_j is a non-negative function. A basis of $L_p(\mathbb{R})$ (or more generally any Banach space) allows one to consider $L_p(\mathbb{R})$ as a sequence space via the identification $L_p(\mathbb{R}) \ni x \leftrightarrow (x_j^*(x))_{j=1}^\infty$. In applications, sequences are often easier to work with than functions, and the benefit of the basis vectors being non-negative is that whenever a sequence of non-negative numbers represents a function, then that function must be non-negative as well. For this reason, the study of non-negative coordinate systems in function spaces has seen recent attention. In particular, non-negative bases have been shown to be useful in non-negative matrix factorizations in neuro imaging [126] and in modeling mental functions [128].

Question 9.1 in [285] asked if given $1 \leq p < \infty$, does there exist a Schauder basis for $L_p(\mathbb{R})$ consisting of non-negative functions? This was recently solved for $L_1(\mathbb{R})$ [193], but all other cases remained open. Our goal in this section is to give a procedure for creating a Schauder basis for $L_2(\mathbb{R})$ formed of non-negative functions. We will be using the terms positive and non-negative interchangeably as the set of non-negative functions in $L_p(\mathbb{R})$ is the positive cone of $L_p(\mathbb{R})$ when viewed as a Banach lattice in the pointwise a.e. ordering.

There does not exist an unconditional positive Schauder basis for $L_p(\mathbb{R})$ for any $1 \leq p < \infty$ [285] (see also [193, Theorem 2]). Thus, any positive Schauder basis we create must necessarily be conditional, and the property of conditionality will factor heavily into our construction. The following lemma is our main tool, and it is based on a classical construction for a conditional Schauder basis for ℓ_2 (see for example pages 235-237 in [9]).

Lemma 5.2.1. Let $\varepsilon > 0$ and $1 \geq c > 0$. There exists $N \in \mathbb{N}$ and a sequence $(x_j)_{j=1}^{2N}$ in the positive cone of $\ell_2(\mathbb{Z}_{2N} \oplus \mathbb{Z}_{2N})$ such that

(i) $(x_j)_{j=1}^{2N}$ is $(1+\varepsilon)$-basic.

(ii) The orthogonal projection of $(0)_{j=1}^{2N} \oplus (\frac{1}{\sqrt{N}}, \frac{c}{\sqrt{N}}, \frac{1}{\sqrt{N}}, \frac{c}{\sqrt{N}} ...)_{j=1}^{2N}$ onto the span of $(x_j)_{j=1}^{2N}$ has norm at most ε.

(iii) The distance from $(0)_{j=1}^{2N} \oplus (\frac{-c}{\sqrt{N}}, \frac{1}{\sqrt{N}}, \frac{-c}{\sqrt{N}}, \frac{1}{\sqrt{N}} ...)_{j=1}^{2N}$ to the span of $(x_j)_{j=1}^{2N}$ is at most ε.

Proof. Fix $0 < \varepsilon < 1$. Let $N \in \mathbb{N}$ and $(a_j)_{j=1}^{N} \subseteq (0, \infty)$ be such that $\sum_{j=1}^{N} j a_j^2 < \varepsilon^2$ and $\sum_{j=1}^{N} a_j > \varepsilon^{-2} c^{-2}$. We prove that such a sequence exists later in Lemma 5.3.1. Let T_1 be the right shift operator in each coordinate of $\ell_2(\mathbb{Z}_{2N} \oplus \mathbb{Z}_{2N})$. That is, for $(\alpha_1, \ldots, \alpha_{2N}) \oplus (\beta_1, \ldots, \beta_{2N}) \in \ell_2(\mathbb{Z}_{2N} \oplus \mathbb{Z}_{2N})$ we have that

$$T_1(\alpha_1, \alpha_2, ..., \alpha_{2N}) \oplus (\beta_1, \beta_2, ..., \beta_{2N}) = (\alpha_{2N}, \alpha_1, \alpha_2, ..., \alpha_{2N-1}) \oplus (\beta_{2N}, \beta_1, \beta_2, ..., \beta_{2N-1}).$$

For $m \in \mathbb{N}$, we let $T_m = (T_1)^m$. We let $(e_j)_{j=1}^{2N}$ be the unit vector basis of $\ell_2(\mathbb{Z}_{2N} \oplus 0)$ and $(f_j)_{j=1}^{2N}$ be the unit vector basis of $\ell_2(0 \oplus \mathbb{Z}_{2N})$. We let $x_1 \in \ell_2(\mathbb{Z}_{2N} \oplus \mathbb{Z}_{2N})$ be the vector $x_1 = e_1 + \sum_{j=1}^{N} a_j e_{2j} + \sum_{j=1}^{N} \varepsilon a_j f_{2j}$ and $x_2 = e_2 + \varepsilon c f_1$. For all $1 \leq n < N$, we let $x_{2n+1} = T_{2n} x_1$ and $x_{2n+2} = T_{2n} x_2$. For $1 \leq j < N$ we have that,

$$x_{2j-1} = e_{2j-1} + \sum_{k=1}^{N} a_{k-j+1} e_{2k} + \sum_{k=1}^{N} \varepsilon a_{k-j+1} f_{2k} \qquad \text{and} \qquad x_{2j} = e_{2j} + \varepsilon c f_{2j-1}, \quad (5.2.3)$$

where $k - j + 1$ is considered in the set $\{1, ..., N\}\text{mod}N$. It will also be helpful to express the sequence $(x_j)_{j=1}^{2N}$ as

$$
\begin{array}{rl}
x_1 &= (1,\ a_1,\ 0, a_2, 0, a_3, ..., a_{N-1}, 0,\ a_N\ ) \oplus (\ 0,\ \ \varepsilon a_1,\ \ 0,\ \varepsilon a_2,\ 0, ...),\\
x_2 &= (0,\ \ 1,\ \ 0,\ 0,\ 0,\ 0,\ ...\ \ 0,\ \ 0\ \ 0\ ) \oplus (\varepsilon c,\ \ 0,\ \ 0,\ \ 0,\ \ 0, ...),\\
x_3 &= (0,\ a_N,\ 1, a_1, 0, a_2, ..., a_{N-2}, 0, a_{N-1}) \oplus (\ 0,\ \ \varepsilon a_N,\ \ 0,\ \varepsilon a_1,\ 0, ...),\\
x_4 &= (0,\ \ 0,\ \ 0,\ 1,\ 0,\ 0,\ ...\ \ 0,\ \ 0\ \ 0\ ) \oplus (\ 0,\ \ 0,\ \ \varepsilon c,\ \ 0,\ \ 0, ...),\\
x_5 &= (0, a_{N-1}, 0, a_N, 1, a_1, ..., a_{N-3}, 0, a_{N-2}) \oplus (\ 0,\ \varepsilon a_{N-1},\ 0,\ \varepsilon a_N,\ 0, ...),\\
x_6 &= (0,\ \ 0,\ \ 0,\ 0,\ 0,\ 1,\ ...\ \ 0,\ \ 0\ \ 0\ ) \oplus (\ 0,\ \ 0,\ \ 0,\ \ 0,\ \ \varepsilon c, ...),\\
& \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots\\
x_{2N-3} &= (0,\ a_3,\ 0, a_4, 0, a_5, ...\ \ a_1,\ 0,\ a_2\ ) \oplus (\ 0,\ \ \varepsilon a_3,\ \ 0,\ \varepsilon a_4,\ 0, ...),\\
x_{2N-2} &= (0,\ \ 0,\ \ 0,\ 0,\ 0,\ 0,\ ...\ \ 1,\ \ 0,\ \ 0\ ) \oplus (\ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0, ...),\\
x_{2N-1} &= (0,\ a_2,\ 0, a_3, 0, a_4, ...\ \ a_N,\ 1,\ a_1\ ) \oplus (\ 0,\ \ \varepsilon a_2,\ \ 0,\ \varepsilon a_3,\ 0, ...),\\
x_{2N} &= (0,\ \ 0,\ \ 0,\ 0,\ 0,\ 0,\ ...\ \ 0,\ \ 0,\ \ 1\ ) \oplus (\ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0, ...).
\end{array}
$$

Let $x = \sum_{j=1}^N \frac{1}{\sqrt{N}} f_{2j-1} + \sum_{j=1}^N \frac{c}{\sqrt{N}} f_{2j}$ and $y = \sum_{j=1}^N \frac{c}{\sqrt{N}} f_{2j-1} + \sum_{j=1}^N \frac{-1}{\sqrt{N}} f_{2j}$ We will prove that this sequence $(x_j)_{j=1}^{2N}$ satisfies:

(a) $(x_j)_{j=1}^{2N}$ is $(1+4\varepsilon)$-basic.

(b) The orthogonal projection of x onto the span of $(x_j)_{j=1}^{2N}$ has norm at most $3c\varepsilon$.

(c) The distance from y to the span of $(x_j)_{j=1}^{2N}$ is at most ε.

We first prove (b). We let Px be the orthogonal projection of x onto the span of $(x_j)_{j=1}^{2N}$. As x is uniformly distributed in both the odd coordinates and the even coordinates, Px will have the form $\sum_{j=1}^N ax_{2j-1} + \sum_{j=1}^N bx_{2j}$ for some $a, b \in \mathbb{R}$. One can check that if $a = 0$ then $\|Px\| = \langle x, Px\rangle^{1/2} = \varepsilon c(1+\varepsilon^2 c^2)^{-1/2} < 3\varepsilon c$. We now assume that $a \neq 0$. Thus, we have the following equality for $\beta = b/a$.

$$\|Px\| = \frac{\langle x, \sum_{j=1}^N ax_{2j-1} + \sum_{j=1}^N bx_{2j}\rangle}{\|\sum_{j=1}^N ax_{2j-1} + \sum_{j=1}^N bx_{2j}\|} = \max_{\beta\in\mathbb{R}} \frac{\langle x, \sum_{j=1}^N x_{2j-1} + \sum_{j=1}^N \beta x_{2j}\rangle}{\|\sum_{j=1}^N x_{2j-1} + \sum_{j=1}^N \beta x_{2j}\|}.$$

By taking the derivative with respect to β, the maximum will be obtained when

$$\begin{aligned}
&\frac{d}{d\beta}\Big\langle x, \sum_{j=1}^N x_{2j-1} + \sum_{j=1}^N \beta x_{2j}\Big\rangle \Big\| \sum_{j=1}^N x_{2j-1} + \sum_{j=1}^N \beta x_{2j}\Big\| \\
&\qquad\qquad = \frac{d}{d\beta}\Big\| \sum_{j=1}^N x_{2j-1} + \sum_{j=1}^N \beta x_{2j}\Big\| \Big\langle x, \sum_{j=1}^N x_{2j-1} + \sum_{j=1}^N \beta x_{2j}\Big\rangle.
\end{aligned} \tag{5.2.4}$$

Let $A = \sum_{j=1}^N a_j$. Then we get the following simplified expansion.

$$\begin{aligned}\sum_{j=1}^N x_{2j-1} + \sum_{j=1}^N \beta x_{2j} &= \sum_{j=1}^N e_{2j-1} + \sum_{j=1}^N \Big(\beta + \sum_{i=1}^N a_i\Big) e_{2j} + \sum_{j=1}^N \varepsilon c \beta f_{2j-1} + \sum_{j=1}^N \Big(\varepsilon \sum_{i=1}^N a_i\Big) f_{2j} \\ &= \sum_{j=1}^N e_{2j-1} + \sum_{j=1}^N (\beta + A) e_{2j} + \sum_{j=1}^N \varepsilon c \beta f_{2j-1} + \sum_{j=1}^N \varepsilon A f_{2j}.\end{aligned}$$

This gives,

$$\Big\| \sum_{j=1}^N x_{2j-1} + \sum_{j=1}^N \beta x_{2j} \Big\| = \big(N + N(\beta+A)^2 + N\varepsilon^2 c^2 \beta^2 + N\varepsilon^2 A^2\big)^{1/2}, \tag{5.2.5}$$

$$\frac{d}{d\beta} \Big\| \sum_{j=1}^N x_{2j-1} + \sum_{j=1}^N \beta x_{2j} \Big\| = \big(N + N(\beta+A)^2 + N\varepsilon^2 c^2 \beta^2 + N\varepsilon^2 A^2\big)^{-1/2} \big(N(\beta+A) + N\varepsilon^2 c^2 \beta\big), \tag{5.2.6}$$

$$\Big\langle x, \sum_{j=1}^N x_{2j-1} + \sum_{j=1}^N \beta x_{2j} \Big\rangle = N^{1/2} \varepsilon c \beta + N^{1/2} \varepsilon c A, \tag{5.2.7}$$

$$\frac{d}{d\beta} \Big\langle x, \sum_{j=1}^N x_{2j-1} + \sum_{j=1}^N \beta x_{2j} \Big\rangle = N^{1/2} \varepsilon c. \tag{5.2.8}$$

Substituting the above equalities into Equation (5.2.4) gives that

$$N^{1/2} \varepsilon c \big(N + N(\beta+A)^2 + N\varepsilon^2 c^2 \beta^2 + N\varepsilon^2 A^2\big)^{1/2} = \frac{(N^{1/2}\varepsilon c \beta + N^{1/2} \varepsilon c A)(N(\beta+A) + N\varepsilon^2 c^2 \beta)}{\big(N + N(\beta+A)^2 + N\varepsilon^2 c^2 \beta^2 + N\varepsilon^2 A^2\big)^{1/2}}.$$

Multiplying both sides by the denominator and dividing by $N^{3/2}\varepsilon c$ gives the following.

$$\begin{aligned}1 + (\beta+A)^2 + \varepsilon^2 c^2 \beta^2 + \varepsilon^2 A^2 &= (\beta+A)(\beta + A + \varepsilon^2 c^2 \beta), \\ 1 + (\beta+A)^2 + \varepsilon^2 c^2 \beta^2 + \varepsilon^2 A^2 &= (\beta+A)^2 + \varepsilon^2 c^2 \beta^2 + \varepsilon^2 c^2 \beta A, \\ 1 + \varepsilon^2 A^2 &= \varepsilon^2 c^2 \beta A.\end{aligned}$$

Thus, the critical point is at $\beta = \frac{1+\varepsilon^2 A^2}{\varepsilon^2 c^2 A}$. Hence, $\sum_{j=1}^N x_{2j-1} + \sum_{j=1}^N \frac{1+\varepsilon^2 A^2}{\varepsilon^2 c^2 A} x_{2j}$ will be a scalar multiple of the projection Px. We now use (5.2.5) to obtain a lower bound for the following.

$$\begin{aligned}
\Big\|\sum_{j=1}^{N}x_{2j-1}+\sum_{j=1}^{N}\frac{1+\varepsilon^2A^2}{\varepsilon^2c^2A}x_{2j}\Big\| &> \Big\|\sum_{j=1}^{N}x_{2j-1}+\sum_{j=1}^{N}\frac{\varepsilon^2A^2}{\varepsilon^2c^2A}x_{2j}\Big\| \\
&= \Big\|\sum_{j=1}^{N}x_{2j-1}+\sum_{j=1}^{N}c^{-2}Ax_{2j}\Big\| \\
&= (N+N(c^{-2}A+A)^2+N\varepsilon^2c^2(c^{-2}A)^2+N\varepsilon^2A^2)^{1/2} \qquad \text{by (5.2.5)} \\
&> N^{1/2}c^{-2}A \qquad \text{by the second term in the sum.}
\end{aligned}$$

We now use (5.2.7) to obtain an upper bound for the following.

$$\begin{aligned}
\Big\langle x,\sum_{j=1}^{N}x_{2j-1}+\sum_{j=1}^{N}\frac{1+\varepsilon^2A^2}{\varepsilon^2c^2A}x_{2j}\Big\rangle &< \Big\langle x,\sum_{j=1}^{N}x_{2j-1}+\sum_{j=1}^{N}\frac{2\varepsilon^2A^2}{\varepsilon^2c^2A}x_{2j}\Big\rangle \\
&= \Big\langle x,\sum_{j=1}^{N}x_{2j-1}+\sum_{j=1}^{N}2c^{-2}Ax_{2j}\Big\rangle \\
&= N^{1/2}\varepsilon c(2c^{-2}A)+N^{1/2}\varepsilon cA \qquad \text{by (5.2.7)} \\
&< 3c^{-1}N^{1/2}\varepsilon A.
\end{aligned}$$

We obtain an upper bound on $\|Px\|$ by

$$\begin{aligned}
\|Px\| &= \frac{\langle x,\sum_{j=1}^{N}x_{2j-1}+\sum_{j=1}^{N}\frac{1+\varepsilon^2A^2}{\varepsilon^2c^2A}x_{2j}\rangle}{\|\sum_{j=1}^{N}x_{2j-1}+\sum_{j=1}^{N}\frac{1+\varepsilon^2A^2}{\varepsilon^2c^2A}x_{2j}\|} \\
&< \frac{3c^{-1}N^{1/2}\varepsilon A}{c^{-2}N^{1/2}A} \\
&= 3c\varepsilon.
\end{aligned}$$

This proves (b). We will now prove (c).

We have that

$$\begin{aligned}
\Big\|\Big(\sum_{j=1}^{N}\frac{-1}{\varepsilon AN^{1/2}}x_{2j-1}+\frac{1}{\varepsilon N^{1/2}}x_{2j}\Big)-y\Big\| &= \Big\|\sum_{j=1}^{N}\frac{-1}{\varepsilon AN^{1/2}}e_{2j-1}\Big\| \\
&= \varepsilon^{-1}A^{-1} \\
&< \varepsilon \qquad \text{as } A=\sum_{j=1}^{N}a_j>\varepsilon^{-2}.
\end{aligned}$$

This proves that the distance from y to the span of $(x_j)_{j=1}^{2N}$ is at most ε and hence we have proven (c).

We now prove (a). Let $0 \leq M < N$ and $(b_j)_{j=1}^{2N} \in \ell_2(\mathbb{Z}_{2N})$. We will first prove that $\|\sum_{j=1}^{2M+1} b_j x_j\| \leq (1+4\varepsilon)\|\sum_{j=1}^{2N} b_j x_j\|$.

The series $\sum_{j=1}^{2N} b_j x_j$ is expressed in terms of the basis $(e_j)_{j=1}^{2N} \cup (f_j)_{j=1}^{2N}$ by

$$\begin{aligned}&\sum_{j=1}^{2N} b_j x_j\\&= \sum_{j=1}^{N} b_{2j-1}e_{2j-1} + \sum_{j=1}^{N}\Big(b_{2j} + \sum_{i=0}^{N-1} b_{2i+1}a_{j-i}\Big)e_{2j} + \sum_{j=1}^{N} \varepsilon b_{2j}cf_{2j-1} + \sum_{j=1}^{N}\Big(\varepsilon\sum_{i=0}^{N-1} b_{2i+1}a_{j-i}\Big)f_{2j}.\end{aligned} \tag{5.2.9}$$

The series $\sum_{j=1}^{2M+1} b_j x_j$ is expressed in terms of the basis $(e_j)_{j=1}^{2N} \cup (f_j)_{j=1}^{2N}$ by

$$\sum_{j=1}^{2M+1} b_j x_j = \sum_{j=1}^{M+1} b_{2j-1}e_{2j-1} + y_{1,1} + y_{1,2} + \sum_{j=1}^{M} \varepsilon b_{2j}cf_{2j-1} + y_{2,1} + y_{2,2}. \tag{5.2.10}$$

Where,

$$y_{1,1} = \sum_{j=1}^{M}\Big(b_{2j} + \sum_{i=0}^{M} b_{2i+1}a_{j-i}\Big)e_{2j} \quad \text{and} \quad y_{1,2} = \sum_{j=M+1}^{N}\Big(\sum_{i=0}^{M} b_{2i+1}a_{j-i}\Big)e_{2j}$$

$$y_{2,1} = \sum_{j=1}^{M}\Big(\varepsilon\sum_{i=0}^{M} b_{2i+1}a_{j-i}\Big)f_{2j} \quad \text{and} \quad y_{2,2} = \sum_{j=M+1}^{N}\Big(\varepsilon\sum_{i=0}^{M} b_{2i+1}a_{j-i}\Big)f_{2j}.$$

Note that

$$\Big\|\sum_{j=1}^{2N} b_j x_j\Big\|^2 \geq \Big\|\sum_{j=1}^{N} b_{2j-1}e_{2j-1}\Big\|^2 = \sum_{j=1}^{N} b_{2j-1}^2. \tag{5.2.11}$$

We first show that $\|y_{1,2}\| < \varepsilon \|\sum_{j=1}^{2N} b_j x_j\|$.

$$\begin{aligned}
\|y_{1,2}\|^2 &= \Big\| \sum_{j=M+1}^{N} \Big(\sum_{i=0}^{M} b_{2i+1} a_{j-i} \Big) e_{2j} \Big\|^2 \\
&= \sum_{j=M+1}^{N} \Big| \sum_{i=0}^{M} b_{2i+1} a_{j-i} \Big|^2 \\
&\leq \sum_{j=M+1}^{N} \Big(\sum_{i=0}^{M} b_{2i+1}^2 \Big) \Big(\sum_{i=0}^{M} a_{j-i}^2 \Big) \quad \text{by Cauchy-Schwartz} \\
&\leq \Big\| \sum_{j=1}^{2N} b_j x_j \Big\|^2 \sum_{j=M+1}^{N} \sum_{i=0}^{M} a_{j-i}^2 \quad \text{by (5.2.11)} \\
&\leq \Big\| \sum_{j=1}^{2N} b_j x_j \Big\|^2 \sum_{j=1}^{N} j a_j^2 < \varepsilon^2 \Big\| \sum_{j=1}^{2N} b_j x_j \Big\|^2.
\end{aligned}$$

Thus we have that,

$$\|y_{1,2}\| < \varepsilon \Big\| \sum_{j=1}^{2N} b_j x_j \Big\|. \tag{5.2.12}$$

The same argument as above gives the following inequality.

$$\Big\| \sum_{j=1}^{M} \Big(\sum_{i=M+1}^{N-1} b_{2i+1} a_{j-i} \Big) e_{2j} \Big\| < \varepsilon \Big\| \sum_{j=1}^{2N} b_j x_j \Big\|. \tag{5.2.13}$$

We can now estimate $\|y_{1,1}\|$.

$$\begin{aligned}
\|y_{1,1}\| &= \Big\| \sum_{j=1}^{M} \Big(b_{2j} + \sum_{i=0}^{M} b_{2i+1} a_{j-i} \Big) e_{2j} \Big\| \\
&< \Big\| \sum_{j=1}^{M} \Big(b_{2j} + \sum_{i=0}^{M} b_{2i+1} a_{j-i} \Big) e_{2j} \Big\| - \Big\| \sum_{j=1}^{M} \Big(\sum_{i=M+1}^{N-1} b_{2i+1} a_{j-i} \Big) e_{2j} \Big\| + \varepsilon \Big\| \sum_{j=1}^{2N} b_j x_j \Big\| \quad \text{by (5.2.13)} \\
&\leq \Big\| \sum_{j=1}^{M} \Big(b_{2j} + \sum_{i=0}^{N-1} b_{2i+1} a_{j-i} \Big) e_{2j} \Big\| + \varepsilon \Big\| \sum_{j=1}^{2N} b_j x_j \Big\| \\
&= \Big\| \sum_{j=1}^{M} \Big(b_{2j} + \sum_{i=1}^{N} b_{2j-2i-1} a_i \Big) e_{2j} \Big\| + \varepsilon \Big\| \sum_{j=1}^{2N} b_j x_j \Big\|.
\end{aligned}$$

Thus, we have that

$$\|y_{1,1}\| < \Big\| \sum_{j=1}^{M} \Big(b_{2j} + \sum_{i=1}^{N} b_{2j-2i-1} a_i\Big) e_{2j} \Big\| + \varepsilon \Big\| \sum_{j=1}^{2N} b_j x_j \Big\|. \tag{5.2.14}$$

The same technique for estimating $y_{1,1}$ and $y_{1,2}$ gives that

$$\|y_{2,1}\| < \Big\| \sum_{j=1}^{M} \Big(\varepsilon \sum_{i=1}^{N} b_{2j-2i-1} a_i\Big) f_{2j} \Big\| + \varepsilon \Big\| \sum_{j=1}^{2N} b_j x_j \Big\| \quad \text{and} \quad \|y_{2,2}\| < \varepsilon \Big\| \sum_{j=1}^{2N} b_j x_j \Big\|. \tag{5.2.15}$$

We consider (5.2.10) with the inequalities (5.2.12), (5.2.14), and (5.2.15) to get

$$\begin{aligned}
\Big\| \sum_{j=1}^{2M+1} b_j x_j \Big\| &< \Big\| \sum_{j=1}^{M+1} b_{2j-1} e_{2j-1} + \sum_{j=1}^{M} \Big(b_{2j} + \sum_{i=1}^{N} b_{2j-2i-1} a_i\Big) e_{2j} \\
&\qquad + \sum_{j=1}^{M} \varepsilon b_{2j} c f_{2j-1} + \sum_{j=1}^{M} \Big(\varepsilon \sum_{i=1}^{N} b_{2j-2i-1} a_i\Big) f_{2j} \Big\| + 4\varepsilon \Big\| \sum_{j=1}^{2N} b_j x_j \Big\| \\
&\leq \Big\| \sum_{j=1}^{N} b_{2j-1} e_{2j-1} + \sum_{j=1}^{N} \Big(b_{2j} + \sum_{i=1}^{N} b_{2j-2i-1} a_i\Big) e_{2j} \\
&\qquad + \sum_{j=1}^{N} \varepsilon b_{2j} c f_{2j-1} + \sum_{j=1}^{N} \Big(\varepsilon \sum_{i=1}^{N} b_{2j-2i-1} a_i\Big) f_{2j} \Big\| + 4\varepsilon \Big\| \sum_{j=1}^{2N} b_j x_j \Big\| \\
&= \Big\| \sum_{j=1}^{2N} b_j x_j \Big\| + 4\varepsilon \Big\| \sum_{j=1}^{2N} b_j x_j \Big\|.
\end{aligned}$$

This proves for all $0 \leq M < N$ that $\| \sum_{j=1}^{2M+1} b_j x_j \| \leq (1+4\varepsilon) \| \sum_{j=1}^{2N} b_j x_j \|$. The same argument proves that also $\| \sum_{j=1}^{2M} b_j x_j \| \leq (1+4\varepsilon) \| \sum_{j=1}^{2N} b_j x_j \|$. Thus, the sequence $(x_j)_{j=1}^{2N}$ has basis constant $(1+4\varepsilon)$ and we have proven (a).

□

Before presenting our main theorem, we discuss the central idea behind our construction and its relation to the construction of Johnson and Schechtman [193]. The conditional Schauder basis for $L_1(\mathbb{R})$ constructed by Johnson and Schechtman can be formed inductively where at each step they break up a Haar vector f into a positive part f^+ and a negative part f^- then append a vector $2 \cdot 1_{(n,n+1)}$ to both parts where $(n, n+1)$ is disjoint from the support of all vectors created so far in the induction process. The vectors $f^+ + 2 \cdot 1_{(n,n+1)}$ and $f^- + 2 \cdot 1_{(n,n+1)}$ are then both positive vectors. One can then recover the vector f by $f = (f^+ + 2 \cdot 1_{(n,n+1)}) - (f^- + 2 \cdot 1_{(n,n+1)})$. Furthermore, the zero vector is the closest vector to $f^+ + f^-$ in the span of $f^+ + 2 \cdot 1_{(n,n+1)}$ and $f^- + 2 \cdot 1_{(n,n+1)}$. This idea can be used to build a Schauder basis for $L_1(\mathbb{R})$, but it fails for $L_p(\mathbb{R})$ for all $1 < p < \infty$.

Our procedure for constructing a positive Schauder basis for $L_2(\mathbb{R})$ is also constructed inductively. However, at each step instead of breaking up a vector into 2 pieces, we break it up into many pieces. That is, given $\varepsilon > 0$ and $f \in L_2(\mathbb{R})$ we choose a suitably large $N \in \mathbb{N}$, and then we break up the positive part of f into N pieces $(f_n^+)_{n=1}^N$ with the same distribution and the negative part of f into N pieces $(f_n^-)_{n=1}^N$ with the same distribution. Here we mean that two functions $g, h : \mathbb{R} \to \mathbb{R}$ have *the same distribution* if for all $J \subseteq \mathbb{R}$ we have that $\lambda(g^{-1}(J)) = \lambda(h^{-1}(J))$ where λ is Lebesgue measure. Given $(f_n^+)_{n=1}^N$ and $(f_n^-)_{n=1}^N$, we use Lemma 5.2.1 to create a positive highly conditional basic sequence $(x_n)_{n=1}^{2N}$ with disjoint support from f and append $(x_{2n-1})_{n=1}^N$ onto $(f_n^-)_{n=1}^N$ and append $(x_{2n})_{n=1}^N$ onto $(f_n^+)_{n=1}^N$. The vectors $f_n^+ + x_{2n}$ and $f_n^- + x_{2n-1}$ are then both positive vectors for all $n \in \mathbb{N}$. The conditionality of $(x_n)_{n=1}^{2N}$ allows for f to be within ε of $(\sum_{n=1}^N f_n^+ + x_{2n}) - (\sum_{n=1}^N f_n^- + x_{2n-1})$ and for the orthogonal projection of $f^+ + f^-$ onto $span_{1 \le n \le N}\{f_n^+ + x_{2n}, f_n^- + x_{2n-1}\}$ to have norm smaller than ε.

Theorem 5.2.2. For all $\varepsilon > 0$, there exists a positive Schauder basis for $L_2(\mathbb{R})$ with basis constant at most $1 + \varepsilon$.

Proof. Let $0 < \varepsilon < 1/2$ and $\varepsilon_j \searrow 0$ such that $\sum \varepsilon_j < \varepsilon$ and $\prod(1 + \varepsilon_j) < 1 + \varepsilon$. Let $(h_j)_{j=1}^\infty$ be a Schauder basis for $L_2(\mathbb{R})$ which is an enumeration of the union of the Haar bases for $L_2([n, n+1])$ for all $n \in \mathbb{Z}$. We assume that $h_1 = 1_{[0,1]}$. We will inductively construct a sequence of nonnegative vectors $(z_j)_{j=1}^\infty$ and an increasing sequence of integers $(N_j)_{j=1}^\infty$ such that for all $n \in \mathbb{N}$,

(a) z_n is piecewise constant, i.e., z_n is a finite linear combination of characteristic functions of intervals in $\mathbb{R}$.

(b) $(z_j)_{j=1}^{N_n}$ is $\prod_{j \le n}(1 + \varepsilon_j)$ basic.

(c) $dist(h_n, span_{j \le N_n}(z_j)) < \varepsilon_n$.

We first claim that $(z_j)_{j=1}^\infty$ will be a Schauder basis for $L_2(\mathbb{R})$ with basis constant at most $1 + \varepsilon$. Indeed, by (b) the sequence $(z_j)_{j=1}^\infty$ is $\prod(1 + \varepsilon_j) < (1 + \varepsilon)$ basic. By (c) the span of $(z_j)_{j=1}^\infty$ contains a perturbation of an orthonormal basis and hence has dense span. Thus all that remains is to construct (z_j) by induction.

For the base case we take $z_1 = h_1 = 1_{[0,1]}$ and $N_1 = 1$. Thus all three conditions are trivially satisfied. Now let $k \in \mathbb{N}$ and assume that $(z_j)_{j=1}^{N_k}$ are given to satisfy the induction hypothesis. Without loss of generality we may assume that h_{k+1} is not contained in the span of $(z_j)_{j=1}^{N_k}$. This is because if $h_{k+1} \in span_{j\leq N_k}(z_j)$ we could just take $N_{k+1} = N_k + 1$ and $z_{N_{k+1}}$ to be the indicator function of an interval with support disjoint from the support of z_j for all $1 \leq j \leq N_k$. This would trivially satisfy (a), (b), and (c). Thus, we may assume that $P_{(span_{j\leq N_k}(z_j))^\perp} h_{k+1} \neq 0$. If $y \in L_2(\mathbb{R})$ we write $y = y^+ - y^-$ where y^+ and y^- are non-negative and disjoint. Let y be a multiple of $P_{(span_{j\leq N_k}(z_j))^\perp} h_{k+1}$ such that $\|y^-\| = 1$ and $c := \|y^+\| \leq 1$. Note that y is piecewise constant as h_{k+1} and $(z_j)_{j=1}^{N_k}$ are all piecewise constant. If $c = 0$ set $z_{k+1} = y^-$ and $N_{k+1} = N_k + 1$, else we proceed as follows:

Let $\varepsilon' > 0$. By Lemma 5.2.1 there exists $N \in \mathbb{N}$ and $(x_j)_{j=1}^{2N}$ in the positive cone of $\ell_2(\mathbb{Z}_{2N} \oplus \mathbb{Z}_{2N})$ such that

(i) $(x_j)_{j=1}^{2N}$ is $(1 + \varepsilon')$-basic.

(ii) The orthogonal projection of $(0, ..., 0) \oplus (\frac{1}{\sqrt{N}}, \frac{c}{\sqrt{N}}, ..., \frac{1}{\sqrt{N}}, \frac{c}{\sqrt{N}})$ onto the span of $(x_j)_{j=1}^{2N}$ has norm at most ε'.

(iii) The distance from $(0, ..., 0) \oplus (\frac{c}{\sqrt{N}}, \frac{-1}{\sqrt{N}}, ..., \frac{c}{\sqrt{N}}, \frac{-1}{\sqrt{N}})$ to the span of $(x_j)_{j=1}^{2N}$ is at most ε'.

Let X_k be the span of y and $(z_j)_{j=1}^{N_k}$. Note that X_k is a space of simple functions with finitely many discontinuities. We claim that there exists a sequence of finite unions of intervals $(G_j)_{j=1}^{2N}$ in $\mathbb{R}$ such that

(i) The sequence $(G_j)_{j=1}^{2N}$ is pairwise disjoint.

(ii) $\cup_{j=1}^{N} G_{2j-1}$ is the support of y^+ and $\cup_{j=1}^{N} G_{2j}$ is the support of y^-.

(iii) For all $x \in X_k$, the sequence of functions $(x|_{G_{2j-1}})_{j=1}^{N}$ all have the same distribution.

(iv) For all $x \in X_k$, the sequence of functions $(x|_{G_{2j}})_{j=1}^{N}$ all have the same distribution.

To prove this, we let $(E_j)_{j=1}^{M_1}$ be a partition of the support of y^+ into intervals such that for all $1 \leq j \leq M_1$ both y and z_i are constant on E_j for all $1 \leq i \leq N_k$. We know by (a) that such a partition exists. Likewise, let $(F_j)_{j=1}^{M_0}$ be a partition of the support of y^- into intervals such that for all $1 \leq j \leq M_0$ both y and z_i are constant on F_j for all $1 \leq i \leq N_k$. For all $1 \leq j \leq M_1$ let $(E_{i,j})_{i=1}^{N}$ be a partition of E_j into intervals of equal length, and for all $1 \leq j \leq M_0$ let $(F_{i,j})_{i=1}^{N}$ be a partition of F_j into intervals of equal length. For all $1 \leq i \leq N$

we let $G_{2i-1} = \cup_{j=1}^{M_1} E_{i,j}$ and let $G_{2i} = \cup_{j=1}^{M_0} F_{i,j}$. By construction, $(G_i)_{i=1}^{2N}$ satisfies (i),(ii),(iii), and (iv).

Let $(H_j)_{j=1}^{2N}$ be a sequence of unit length intervals in $\mathbb{R}$ with pairwise disjoint support which is disjoint from the support of y and the support of z_j for all $1 \leq j \leq N_k$. We now define a map $\Psi : \ell_2(\mathbb{Z}_{2N} \oplus \mathbb{Z}_{2N}) \to L_2(\mathbb{R})$ by

$$\Psi(\alpha_1, ..., \alpha_{2N}, \beta_1, ..., \beta_{2N}) = \sum_{j=1}^{N} c^{-1} N^{1/2} \beta_{2j-1} 1_{G_{2j-1}} y^+ + \sum_{j=1}^{N} N^{1/2} \beta_{2j} 1_{G_{2j}} y^- + \sum_{j=1}^{2N} \alpha_j 1_{H_j}.$$

By (i),(ii),(iii), and that $\|y^+\| = c$ we have that $\|1_{G_{2j-1}} y^+\| = cN^{-1/2}$ for all $1 \leq j \leq N$. Likewise, as $\|y^-\| = 1$ we have that $\|1_{G_{2j}} y^-\| = N^{-1/2}$ for all $1 \leq j \leq N$. Thus, Ψ is an isometric embedding and maps positive vectors in $\ell_2(\mathbb{Z}_{2N} \oplus \mathbb{Z}_{2N})$ to positive vectors in $L_2(\mathbb{R})$. We let $N_{k+1} = N_k + 2N$ and let $z_{N_k+j} = \Psi(x_j)$ for all $1 \leq j \leq 2N$. As y is piecewise constant, H_i is an interval, and G_i is a finite union of intervals for all $1 \leq i \leq 2N$, we have that z_j is piecewise constant for all $N_k < j \leq N_{k+1}$. Thus we have satisfied (a).

Note that $\Psi((0, ..., 0) \oplus (\frac{c}{\sqrt{N}}, \frac{-1}{\sqrt{N}}, ..., \frac{c}{\sqrt{N}}, \frac{-1}{\sqrt{N}})) = y$, thus by (3) the distance from y to the span of $(z_j)_{j=N_k+1}^{N_{k+1}}$ is at most ε' which proves (c) if ε' is small enough.

Let $x \in span_{j \leq N_k} z_j$. Let $(e_j)_{j=1}^{2N}$ denote the unit vector basis for the second coordinate of $\ell_2(\mathbb{Z}_{2N} \oplus \mathbb{Z}_{2N})$. Then by (iii), we have that $\langle \Psi(e_{2j-1}), x\rangle = \langle \Psi(e_{2i-1}), x\rangle$ for all $1 \leq i, j \leq N$, and by (iv) we have that $\langle \Psi(e_{2j}), x\rangle = \langle \Psi(e_{2i}), x\rangle$ for all $1 \leq i, j \leq N$. We have that x is orthogonal to y and $y = \frac{c}{\sqrt{N}}\Psi(e_1) - \frac{1}{\sqrt{N}}\Psi(e_2) + ... + \frac{c}{\sqrt{N}}\Psi(e_{2N-1}) - \frac{1}{\sqrt{N}}\Psi(e_{2N})$. Thus the orthogonal projection of x onto $\Psi(\ell_2(\mathbb{Z}_{2N} \oplus \mathbb{Z}_{2N}))$ is a multiple of $\Psi(e_1) + c\Psi(e_2) + ... + \Psi(e_{2N-1}) + c\Psi(e_{2N})$. Hence by (2) the orthogonal projection of x onto $span_{N_k < j \leq N_{k+1}} z_j = span_{1 \leq j \leq 2N} \Psi(x_j)$ has norm at most $2\varepsilon'\|x\|$. The sequence $(z_j)_{j=1}^{N_k}$ is $\prod_{j \leq k}(1 + \varepsilon_j)$ basic and $(z_j)_{j=N_k+1}^{N_{k+1}}$ is $(1 + \varepsilon')$ basic. The inner product between a unit vector in $span_{j \leq N_k} z_j$ and a unit vector in $span_{N_k < j \leq N_{k+1}} z_j$ is at most $2\varepsilon'$. Thus, if ε' is small enough then $(z_j)_{j=1}^{N_{k+1}}$ is $\prod_{j \leq k+1}(1 + \varepsilon_j)$ basic which proves (b). This completes the construction of (z_j) by induction.

□

Remark 5.2.3. Similar to [193], one can use classification theorems to extend the above result to all separable $L_2(\mu)$. See, for example, [218] or Section 2.7 of [244]. That is, if $L_2(\mu)$ is separable then for all $\varepsilon > 0$ there exists a positive Schauder basis for $L_2(\mu)$ with basis constant at most $1 + \varepsilon$.

5.3 A large non-negative basic sequence in Lebesgue spaces

Our method in Section 5.2 repeatedly makes use of orthogonal projections onto subspaces of $L_2(\mathbb{R})$. This prevents us from extending the construction to $L_p(\mathbb{R})$ for $p \neq 2$. However, we are able to obtain the result for large subspaces of $L_p(\mathbb{R})$. Indeed, for $\varepsilon > 0$ and $1 < p < \infty$, we will construct a positive $(2+\varepsilon)$-basic sequence $(z_j)_{j=1}^{\infty}$ in $L_p(\mathbb{R})$ such that $L_p(\mathbb{R})$ is isomorphic to a subspace of the closed span of $(z_j)_{j=1}^{\infty}$.

Lemma 5.3.1. For all $\varepsilon > 0$ and $1 < p < \infty$ there exists $N \in \mathbb{N}$ and $a_n \searrow 0$ such that $\sum_{n=1}^{N} a_n > \varepsilon^{-2}$ and $\sum_{n=1}^{N}(\sum_{j=n}^{N} a_j^q)^{p/q} < \varepsilon^p$ where $1/p + 1/q = 1$.

Proof. We consider the function $f : [1, \infty) \to \mathbb{R}$ given by $f(x) = ((x+1)\ln(x+1))^{-1}$. Then,

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} ((x+1)\ln(x+1))^{-1} dx = \infty.$$

We also have the following upper bound,

$$\begin{aligned}
\int_1^{\infty} \left(\int_x^{\infty} f(t)^q\,dt\right)^{p/q} dx &= \int_1^{\infty} \left(\int_x^{\infty} ((t+1)\ln(t+1))^{-q}\,dt\right)^{p/q} dx \\
&\leq \int_1^{\infty} \left(\int_x^{\infty} (t+1)^{-q}\,dt\right)^{p/q} \ln(x+1)^{-p}\,dx \quad \text{ as } \ln(t+1)^{-q} \leq \ln(x+1)^{-q} \\
&= (q-1)^{-p/q} \int_1^{\infty} (x+1)^{(1-q)p/q} \ln(x+1)^{-p}\,dx \\
&= (q-1)^{-p/q} \int_1^{\infty} (x+1)^{-1} \ln(x+1)^{-p}\,dx \quad \text{ as } p^{-1} + q^{-1} = 1 \\
&= (q-1)^{-p/q}(p-1)^{-1}\ln(2)^{1-p}.
\end{aligned}$$

As f is a decreasing function, we have that $\sum_{n=1}^{\infty} f(n) = \infty$ and $\sum_{n=1}^{\infty}(\sum_{j=n}^{\infty} f(j)^q)^{p/q} < \infty$. Hence, for all $\varepsilon > 0$ we may choose $N \in \mathbb{N}$ and $a_n \searrow 0$ such that $\sum_{n=1}^{N} a_n > \varepsilon^{-2}$ and $\sum_{n=1}^{N}(\sum_{j=n}^{N} a_j^q)^{p/q} < \varepsilon^p$. In particular, for all $\varepsilon > 0$ we may choose

$$a_n = \Big((n+2)\ln(n+2)\Big)^{-1}\Big((q-1)^{-p/q}(p-1)^{-1}\ln(2)^{1-p}\Big)^{-1/p}\varepsilon,$$

and then choose $N \in \mathbb{N}$ such that $\sum_{n=1}^{N} a_n > \varepsilon^{-2}$.

□

The following lemma is an extension of Lemma 5.2.1 to $\ell_p(\mathbb{Z}_{2N} \oplus \mathbb{Z}_{2N})$ where $1 < p < \infty$, and the proof will follow along the same lines. In the previous section we constructed a positive Schauder basis for all of $L_2(\mathbb{R})$ and this required a variable $0 < c \leq 1$ in Lemma 5.2.1. For $p \neq 2$, we will only be constructing a positive Schauder basis for a subspace of $L_p(\mathbb{R})$, and for this reason we will no longer need the variable c.

Lemma 5.3.2. Let $\varepsilon > 0$ and $1 < p, q < \infty$ with $1/p + 1/q = 1$. There exists $N \in \mathbb{N}$ and a sequence $(x_j)_{j=1}^{2N}$ in the positive cone of $\ell_p(\mathbb{Z}_{2N} \oplus \mathbb{Z}_{2N})$ such that

(i) $(x_j)_{j=1}^{2N}$ is $(1+\varepsilon)$-basic.

(ii) If $f^* = (0)_{j=1}^{2N} \oplus (N^{-1/q})_{j=1}^{2N} \in \ell_q(\mathbb{Z}_{2N} \oplus \mathbb{Z}_{2N})$ then $|f^*(x)| \leq \varepsilon \|x\|$ for all x in the span of $(x_j)_{j=1}^{2N}$.

(iii) The distance from $(0)_{j=1}^{2N} \oplus ((-1)^j N^{-1/p})$ to the span of $(x_j)_{j=1}^{2N}$ is at most ε.

Proof. The proof follows the same strategy as Lemma 5.2.1. Fix $0 < \varepsilon < 1$. By Lemma 5.3.1, there exists $N \in \mathbb{N}$ and $(a_j)_{j=1}^{N} \subseteq (0, \infty)$ such that

$$\sum_{n=1}^{N} a_n > \varepsilon^{-2} \quad \text{and} \quad \sum_{n=1}^{N} \Big(\sum_{j=n}^{N} a_j^q \Big)^{p/q} < \varepsilon^p. \tag{5.3.1}$$

Consider the space $\ell_p(\mathbb{Z}_{2N} \oplus \mathbb{Z}_{2N})$. Let T_1 be the cyclic right shift operator on this space. That is, for $(\alpha_1, \ldots, \alpha_{2N}) \oplus (\beta_1, \ldots, \beta_{2N}) \in \ell_p(\mathbb{Z}_{2N} \oplus \mathbb{Z}_{2N})$ let

$$T_1(\alpha_1, \alpha_2, ..., \alpha_{2N}) \oplus (\beta_1, \beta_2, ..., \beta_{2N}) = (\alpha_{2N}, \alpha_1, \alpha_2, ..., \alpha_{2N-1}) \oplus (\beta_{2N}, \beta_1, \beta_2, ..., \beta_{2N-1}).$$

For $m \in \mathbb{N}$, we let $T_m = (T_1)^m$. We let $(e_j)_{j=1}^{2N}$ be the unit vector basis of $\ell_p(\mathbb{Z}_{2N}) \oplus 0$ and $(f_j)_{j=1}^{2N}$ be the unit vector basis of $0 \oplus \ell_p(\mathbb{Z}_{2N})$. We denote $(e_j^*)_{j=1}^{2N}$ and $(f_j^*)_{j=1}^{2N}$ to be the biorthogonal functionals to $(e_j)_{j=1}^{2N}$ and $(f_j)_{j=1}^{2N}$. We let $x_1 \in \ell_p(\mathbb{Z}_{2N} \oplus \mathbb{Z}_{2N})$ be the vector $x_1 = e_1 + \sum_{j=1}^{N} a_j e_{2j} + \sum_{j=1}^{N} \varepsilon a_j f_{2j}$ and $x_2 = e_2 + \varepsilon f_1$. For all $1 \leq n < N$, we let $x_{2n+1} = T_{2n} x_1$ and $x_{2n+2} = T_{2n} x_2$. That is,

$$
\begin{array}{lllllllllllllllll}
x_1 & = (1, & a_1, & 0, a_2, 0, a_3, ..., a_{N-1}, 0, & a_N &) \oplus (0, & \varepsilon a_1, & 0, \varepsilon a_2, 0, ...), \\
x_2 & = (0, & 1, & 0, \ 0, \ 0, \ 0, \ ... \quad 0, & 0 \quad 0 &) \oplus (\varepsilon, & 0, & 0, \ 0, \ 0, ...), \\
x_3 & = (0, & a_N, & 1, a_1, 0, a_2, ..., a_{N-2}, 0, & a_{N-1} &) \oplus (0, & \varepsilon a_N, & 0, \varepsilon a_1, 0, ...), \\
x_4 & = (0, & 0, & 0, \ 1, \ 0, \ 0, \ ... \quad 0, & 0 \quad 0 &) \oplus (0, & 0, & \varepsilon, \ 0, \ 0, ...), \\
x_5 & = (0, & a_{N-1}, & 0, a_N, 1, a_1, ..., a_{N-3}, 0, & a_{N-2} &) \oplus (0, & \varepsilon a_{N-1}, & 0, \varepsilon a_N, 0, ...), \\
x_6 & = (0, & 0, & 0, \ 0, \ 0, \ 1, \ ... \quad 0, & 0 \quad 0 &) \oplus (0, & 0, & 0, \ 0, \ \varepsilon, ...). \\
 & & & \quad \vdots & & & & \quad \vdots \\
x_{2N-3} & = (0, & a_3, & 0, a_4, 0, a_5, ... \quad a_1, \ 0, & a_2 &) \oplus (0, & \varepsilon a_3, & 0, \varepsilon a_4, 0, ...) \\
x_{2N-2} & = (0, & 0, & 0, \ 0, \ 0, \ 0, \ ... \quad 1, \ 0, & 0 &) \oplus (0, & 0, & 0, \ 0, \ 0, ...) \\
x_{2N-1} & = (0, & a_2, & 0, a_3, 0, a_4, ... \quad a_N, \ 1, & a_1 &) \oplus (0, & \varepsilon a_2, & 0, \varepsilon a_3, 0, ...) \\
x_{2N} & = (0, & 0, & 0, \ 0, \ 0, \ 0, \ ... \quad 0, \ 0, & 1 &) \oplus (0, & 0, & 0, \ 0, \ 0, ...)
\end{array}
$$

Let $f^* = \sum_{j=1}^{2N} N^{-1/q} f_j^*$ and $y = \sum_{j=1}^{2N} (-1)^j N^{-1/p} f_j$ We will prove that the sequence $(x_j)_{j=1}^{2N}$ satisfies:

(a) $(x_j)_{j=1}^{2N}$ is $(1+4\varepsilon)$-basic.

(b) $f^*(z) \leq \varepsilon \|z\|$ for all z in the span of $(x_j)_{j=1}^{2N}$.

(c) The distance from y to the span of $(x_j)_{j=1}^{2N}$ is at most ε.

We first prove (b). As the unit ball of $\ell_p(\mathbb{Z}_{2N} \oplus \mathbb{Z}_{2N})$ is strictly convex, there exists a unique unit norm vector z in the span of $(x_j)_{j=1}^{2N}$ so that $f^*(z)$ is maximal. As f^* is uniformly distributed on $0 \oplus \mathbb{Z}_{2N}$, z will have the form $\sum_{j=1}^{N} a x_{2j-1} + \sum_{j=1}^{N} b x_{2j}$ for some $a, b \in \mathbb{R}$. One can check that if $a = 0$ then $f^*(z) = \varepsilon(1+\varepsilon^p)^{-1/p} < \varepsilon$. We now assume that $a \neq 0$. Thus,

$$
f^*(z) = \frac{f^*(\sum_{j=1}^{N} a x_{2j-1} + \sum_{j=1}^{N} b x_{2j})}{\| \sum_{j=1}^{N} a x_{2j-1} + \sum_{j=1}^{N} b x_{2j} \|} = \max_{\beta \in \mathbb{R}} \frac{|f^*(\sum_{j=1}^{N} x_{2j-1} + \sum_{j=1}^{N} \beta x_{2j})|}{\| \sum_{j=1}^{N} x_{2j-1} + \sum_{j=1}^{N} \beta x_{2j} \|}
$$

Let $A = \sum_{j=1}^{N} a_j$. Then we get the following simplified expansion.

$$
\begin{aligned}
\sum_{j=1}^{N} x_{2j-1} + \sum_{j=1}^{N} \beta x_{2j} &= \sum_{j=1}^{N} e_{2j-1} + \sum_{j=1}^{N} \big(\beta + \sum_{i=1}^{N} a_i\big) e_{2j} + \sum_{j=1}^{N} \varepsilon \beta f_{2j-1} + \sum_{j=1}^{N} \big(\varepsilon \sum_{i=1}^{N} a_i\big) f_{2j} \\
&= \sum_{j=1}^{N} e_{2j-1} + \sum_{j=1}^{N} (\beta + A) e_{2j} + \sum_{j=1}^{N} \varepsilon \beta f_{2j-1} + \sum_{j=1}^{N} \varepsilon A f_{2j}.
\end{aligned}
$$

This gives the following two equalities,

$$\Big\|\sum_{j=1}^{N} x_{2j-1}+\sum_{j=1}^{N}\beta x_{2j}\Big\| = \big(N+N|\beta+A|^p+N\varepsilon^p|\beta|^p+N\varepsilon^p A^p\big)^{1/p}, \tag{5.3.2}$$

$$f^*\Big(\sum_{j=1}^{N} x_{2j-1}+\sum_{j=1}^{N}\beta x_{2j}\Big) = N^{1/p}\varepsilon\beta+N^{1/p}\varepsilon A. \tag{5.3.3}$$

Let $\beta\in\mathbb{R}$ such that

$$f^*(z) = \frac{|f^*(\sum_{j=1}^N x_{2j-1}+\sum_{j=1}^N \beta x_{2j})|}{\|\sum_{j=1}^N x_{2j-1}+\sum_{j=1}^N \beta x_{2j}\|}.$$

For $\lambda := \beta/A$, we have the following two results.

$$\begin{aligned}\Big\|\sum_{j=1}^{N} x_{2j-1}+\sum_{j=1}^{N}\beta x_{2j}\Big\| &= \big(N+N|\lambda A+A|^p+N\varepsilon^p(|\lambda|A)^p+N\varepsilon^p A^p\big)^{1/p}\\ &> \big(N|\lambda A+A|^p\big)^{1/p} = |1+\lambda|AN^{1/p},\end{aligned}$$

$$f^*\Big(\sum_{j=1}^{N} x_{2j-1}+\sum_{j=1}^{N}\beta x_{2j}\Big) = N^{1/p}\varepsilon\lambda A+N^{1/p}\varepsilon A = \varepsilon(1+\lambda)AN^{1/p}.$$

If $\lambda=-1$ then by the above equality we would have $f^*(\sum_{j=1}^N x_{2j-1}+\sum_{j=1}^N \beta x_{2j}) = 0$. Otherwise, we have that,

$$|f^*(z)| < \varepsilon|1+\lambda|AN^{1/p}/(|1+\lambda|AN^{1/p}) = \varepsilon.$$

Thus, we have proven (b). We will now prove (c).

Recall that $y=\sum_{j=1}^{2N}(-1)^j N^{-1/p} f_j$. We have that

$$\begin{aligned}\Big\|\Big(\sum_{j=1}^{N}\frac{1}{\varepsilon AN^{1/p}}x_{2j-1}-\frac{1}{\varepsilon N^{1/p}}x_{2j}\Big)-y\Big\| &= \Big\|\sum_{j=1}^{N}\frac{1}{\varepsilon AN^{1/p}}e_{2j-1}\Big\|\\ &= \varepsilon^{-1}A^{-1}\\ &< \varepsilon \qquad \text{as } A=\sum_{j=1}^{N}a_j > \varepsilon^{-2}.\end{aligned}$$

This proves that the distance from y to the span of $(x_j)_{j=1}^{2N}$ is at most ε and hence we have proven (c).

We now prove (a). Let $0 \leq M < N$ and $(b_j)_{j=1}^{2N} \in \ell_p(\mathbb{Z}_{2N})$. We will prove that $\| \sum_{j=1}^{2M+1} b_j x_j \| \leq (1+4\varepsilon) \| \sum_{j=1}^{2N} b_j x_j \|$.

The series $\sum_{j=1}^{2N} b_j x_j$ is expressed in terms of the basis $(e_j)_{j=1}^{2N} \cup (f_j)_{j=1}^{2N}$ by

$$\sum_{j=1}^{N} b_{2j-1} e_{2j-1} + \sum_{j=1}^{N} \Big(b_{2j} + \sum_{i=0}^{N-1} b_{2i+1} a_{j-i} \Big) e_{2j} + \sum_{j=1}^{N} \varepsilon b_{2j} f_{2j-1} + \sum_{j=1}^{N} \Big(\varepsilon \sum_{i=0}^{N-1} b_{2i+1} a_{j-i} \Big) f_{2j}. \quad (5.3.4)$$

The series $\sum_{j=1}^{2M+1} b_j x_j$ is expressed in terms of the basis $(e_j)_{j=1}^{2N} \cup (f_j)_{j=1}^{2N}$ by

$$\sum_{j=1}^{2M+1} b_j x_j = \sum_{j=1}^{M+1} b_{2j-1} e_{2j-1} + y_{1,1} + y_{1,2} + \sum_{j=1}^{M} \varepsilon b_{2j} f_{2j-1} + y_{2,1} + y_{2,2}. \quad (5.3.5)$$

Where,

$$y_{1,1} = \sum_{j=1}^{M} \big(b_{2j} + \sum_{i=0}^{M} b_{2i+1} a_{j-i} \big) e_{2j} \quad \text{and} \quad y_{1,2} = \sum_{j=M+1}^{N} \big(\sum_{i=0}^{M} b_{2i+1} a_{j-i} \big) e_{2j},$$

$$y_{2,1} = \sum_{j=1}^{M} \big(\varepsilon \sum_{i=0}^{M} b_{2i+1} a_{j-i} \big) f_{2j} \quad \text{and} \quad y_{2,2} = \sum_{j=M+1}^{N} \big(\varepsilon \sum_{i=0}^{M} b_{2i+1} a_{j-i} \big) f_{2j}.$$

Note that

$$\Big\| \sum_{j=1}^{2N} b_j x_j \Big\|^p \geq \Big\| \sum_{j=1}^{N} b_{2j-1} e_{2j-1} \Big\|^p = \sum_{j=1}^{N} |b_{2j-1}|^p. \quad (5.3.6)$$

We first show that $\|y_{1,2}\| < \varepsilon \| \sum_{j=1}^{2N} b_j x_j \|$.

$$\begin{aligned}
\|y_{1,2}\|^p &= \Big\| \sum_{j=M+1}^{N} \Big(\sum_{i=0}^{M} b_{2i+1} a_{j-i} \Big) e_{2j} \Big\|^p \\
&= \sum_{j=M+1}^{N} \Big| \sum_{i=0}^{M} b_{2i+1} a_{j-i} \Big|^p \\
&\le \sum_{j=M+1}^{N} \Big(\sum_{i=0}^{M} |b_{2i+1}|^p \Big) \Big(\sum_{i=0}^{M} a_{j-i}^q \Big)^{p/q} \qquad \text{by Hölder's Inequality,} \\
&\le \Big\| \sum_{j=1}^{2N} b_j x_j \Big\|^p \sum_{j=M+1}^{N} \Big(\sum_{i=0}^{M} a_{j-i}^q \Big)^{p/q} \qquad \text{by (5.3.6),} \\
&\le \Big\| \sum_{j=1}^{2N} b_j x_j \Big\|^p \sum_{j=1}^{N} \Big(\sum_{i=j}^{N} a_i^q \Big)^{p/q} \\
&< \Big\| \sum_{j=1}^{2N} b_j x_j \Big\|^p \varepsilon^p \qquad \text{by (5.3.1).}
\end{aligned}$$

Thus we have that,

$$\|y_{1,2}\| < \varepsilon \Big\| \sum_{j=1}^{2N} b_j x_j \Big\|. \tag{5.3.7}$$

The same argument as above gives the following inequality.

$$\Big\| \sum_{j=1}^{M} \Big(\sum_{i=M+1}^{N-1} b_{2i+1} a_{j-i} \Big) e_{2j} \Big\| < \varepsilon \Big\| \sum_{j=1}^{2N} b_j x_j \Big\|. \tag{5.3.8}$$

We can now estimate $\|y_{1,1}\|$.

$$\begin{aligned}
\|y_{1,1}\| &= \Big\| \sum_{j=1}^{M} \Big(b_{2j} + \sum_{i=0}^{M} b_{2i+1} a_{j-i} \Big) e_{2j} \Big\| \\
&< \Big\| \sum_{j=1}^{M} \Big(b_{2j} + \sum_{i=0}^{M} b_{2i+1} a_{j-i} \Big) e_{2j} \Big\| - \Big\| \sum_{j=1}^{M} \Big(\sum_{i=M+1}^{N-1} b_{2i+1} a_{j-i} \Big) e_{2j} \Big\| + \varepsilon \Big\| \sum_{j=1}^{2N} b_j x_j \Big\| \quad \text{by (5.3.8)} \\
&\le \Big\| \sum_{j=1}^{M} \Big(b_{2j} + \sum_{i=0}^{N-1} b_{2i+1} a_{j-i} \Big) e_{2j} \Big\| + \varepsilon \Big\| \sum_{j=1}^{2N} b_j x_j \Big\| \\
&= \Big\| \sum_{j=1}^{M} \Big(b_{2j} + \sum_{i=1}^{N} b_{2j-2i-1} a_i \Big) e_{2j} \Big\| + \varepsilon \Big\| \sum_{j=1}^{2N} b_j x_j \Big\|.
\end{aligned}$$

Thus, we have that

$$\|y_{1,1}\| < \Big\|\sum_{j=1}^{M}\Big(b_{2j}+\sum_{i=1}^{N}b_{2j-2i-1}a_i\Big)e_{2j}\Big\|+\varepsilon\Big\|\sum_{j=1}^{2N}b_jx_j\Big\|. \tag{5.3.9}$$

The same technique for estimating $y_{1,1}$ and $y_{1,2}$ gives that

$$\|y_{2,1}\| < \Big\|\sum_{j=1}^{M}\Big(\varepsilon\sum_{i=1}^{N}b_{2j-2i-1}a_i\Big)f_{2j}\Big\|+\varepsilon\Big\|\sum_{j=1}^{2N}b_jx_j\Big\| \quad\text{and}\quad \|y_{2,2}\| < \varepsilon\Big\|\sum_{j=1}^{2N}b_jx_j\Big\|. \tag{5.3.10}$$

We consider (5.3.5) with the inequalities (5.3.7), (5.3.9), and (5.3.10) to get

$$\begin{aligned}\Big\|\sum_{j=1}^{2M+1}b_jx_j\Big\| &< \Big\|\sum_{j=1}^{M+1}b_{2j-1}e_{2j-1}+\sum_{j=1}^{M}\Big(b_{2j}+\sum_{i=1}^{N}b_{2j-2i-1}a_i\Big)e_{2j}\\ &\qquad+\sum_{j=1}^{M}\varepsilon b_{2j}f_{2j-1}+\sum_{j=1}^{M}\Big(\varepsilon\sum_{i=1}^{N}b_{2j-2i-1}a_i\Big)f_{2j}\Big\|+4\varepsilon\Big\|\sum_{j=1}^{2N}b_jx_j\Big\|\\ &\le \Big\|\sum_{j=1}^{N}b_{2j-1}e_{2j-1}+\sum_{j=1}^{N}\Big(b_{2j}+\sum_{i=1}^{N}b_{2j-2i-1}a_i\Big)e_{2j}\\ &\qquad+\sum_{j=1}^{N}\varepsilon b_{2j}f_{2j-1}+\sum_{j=1}^{N}\Big(\varepsilon\sum_{i=1}^{N}b_{2j-2i-1}a_i\Big)f_{2j}\Big\|+4\varepsilon\Big\|\sum_{j=1}^{2N}b_jx_j\Big\|\\ &= \Big\|\sum_{j=1}^{2N}b_jx_j\Big\|+4\varepsilon\Big\|\sum_{j=1}^{2N}b_jx_j\Big\|.\end{aligned}$$

This proves for all $0\le M<N$ that $\|\sum_{j=1}^{2M+1}b_jx_j\|\le(1+4\varepsilon)\|\sum_{j=1}^{2N}b_jx_j\|$. The same argument proves that also $\|\sum_{j=1}^{2M}b_jx_j\|\le(1+4\varepsilon)\|\sum_{j=1}^{2N}b_jx_j\|$. Thus, the sequence $(x_j)_{j=1}^{2N}$ has basic constant $(1+4\varepsilon)$ and we have proven (a). $\square$

We now show how the conditional positive basic sequence constructed in Lemma 5.3.2 can be inductively used to build a basic sequence in $L_p(\mathbb{R})$. We will construct a positive basic sequence in $L_p(\mathbb{R})$ which contains a perturbation of a Haar type system in $L_p([0,1])$. Recall that a sequence of vectors $(g_j)_{j=0}^\infty$ in $L_p([0,1])$ is called a *Haar type system* if there is a sequence of partitions $(\{E_{j,n}\}_{j=0}^{2^n-1})_{n=0}^\infty$ of $[0,1]$ such that $E_{0,0}=[0,1]$ and $g_0=1_{[0,1]}$ and for all $n\in\mathbb{N}$ and $0\le j\le 2^{n-1}-1$ we have that $\{E_{2j,n},E_{2j+1,n}\}$ is a partition of $E_{j,n-1}$ with $\lambda(E_{2j,n})=\lambda(E_{2j+1,n})=2^{-n}$ and $g_{2^{n-1}+j}=2^{(n-1)/p}(1_{E_{2j,n}}-1_{E_{2j+1,n}})$. Note that the Haar basis for $L_p([0,1])$ is a Haar type system, and every Haar type system in $L_p([0,1])$ is 1-equivalent to the Haar basis. Thus, if $(g_j)_{j=0}^\infty$ is a Haar type system in $L_p([0,1])$ then the closed span of $(g_j)_{j=0}^{\infty}$ is isometric to $L_p([0,1])$. We will denote the usual Haar basis for $L_p([0,1])$ by $(h_j)_{j=0}^{\infty}$, and denote its dual sequence by $(h_j^*)_{j=0}^{\infty}$ (which is just the Haar basis for $L_q([0,1])$ for $1/p+1/q=1$.)

Theorem 5.3.3. For all $1<p<\infty$, there exists a positive Schauder basic sequence $(z_j)_{j=0}^{\infty}$ in $L_p(\mathbb{R})$ such that $L_p(\mathbb{R})$ is isomorphic to a subspace of the closed span of $(z_j)_{j=0}^{\infty}$.

Proof. Let $0<\varepsilon<1$ and $\varepsilon_j \searrow 0$ such that $\sum 2\varepsilon_j<\varepsilon$ and $\prod(1+\varepsilon_j)<1+\varepsilon$. We will inductively construct a sequence of non-negative vectors $(z_j)_{j=0}^{\infty}$ in $L_p(\mathbb{R})$, increasing sequences of integers $(M_j)_{j=0}^{\infty}$ and $(N_j)_{j=0}^{\infty}$, and a Haar type system $(g_j)_{j=0}^{\infty}$ in $L_p([0,1])$ such that $M_0=N_0=0$, $z_0=g_0=1_{[0,1]}$, and for all $n\in\mathbb{N}$ we have that

(a) $g_n \in span(h_j)_{j=M_{n-1}+1}^{M_n}$.

(b) $span(z_j|_{[0,1]})_{j=0}^{N_n} \subseteq span(h_j)_{j=0}^{M_n}$ and each of the functions $(z_j|_{[0,1]^c})_{j=0}^{N_{n-1}}$ have disjoint support from each of the functions $(z_j|_{[0,1]^c})_{j=N_{n-1}+1}^{N_n}$.

(c) If $P_{M_{n-1}}$ is the basis projection onto $span(h_j)_{j=0}^{M_{n-1}}$ then $\|P_{M_{n-1}}x\| \leq \varepsilon_n\|x\|$ for all $x\in span(z_j)_{j=N_{n-1}+1}^{N_n}$.

(d) $(z_j)_{j=N_{n-1}+1}^{N_n}$ is $(1+\varepsilon)-$basic.

(e) $dist(g_n, span_{N_{n-1}<j\leq N_n}(z_j))<\varepsilon_n$.

Before proving that this is possible, we show that building such a sequence $(z_j)_{j=0}^{\infty}$ will prove our theorem. By (e), the span of $(z_j)_{j=0}^{\infty}$ contains a perturbation of a Haar type system for $L_p([0,1])$ and hence $L_p([0,1])$ is isomorphic to a subspace of the closed span of $(z_j)_{j=0}^{\infty}$. We now show that $(z_j)_{j=0}^{\infty}$ is a basic sequence. Let $x=\sum_{j=0}^{\infty} a_j z_j \in span(z_j)_{j=0}^{\infty}$ and let $N\in\mathbb{N}$. We will prove that $\|\sum_{j=0}^{\infty} a_j z_j\| \geq \frac{1}{2(1+\varepsilon)^2}\|\sum_{j=0}^{N} a_j z_j\|$.

We denote $x_0=a_0z_0$ and $x_n=\sum_{j=N_{n-1}+1}^{N_n} a_j z_j$ for all $n\in\mathbb{N}$. We denote $y_0=x_0$ and $y_n=x_n-P_{M_{n-1}}x_n$ for all $n\in\mathbb{N}$. By (c), we have that $\|y_n-x_n\|\leq\varepsilon_n\|x_n\|$. We have by (b) that $(y_n|_{[0,1]})_{n=1}^{\infty}$ is a block sequence of the Haar basis and that $(y_n|_{[0,1]^c})_{n=1}^{\infty}$ is a sequence of vectors with disjoint support. Thus $(y_n)_{n=0}^{\infty}$ is 1-basic as the Haar sequence is 1-basic. As $(x_n)_{n=0}^{\infty}$ is a perturbation of $(y_n)_{n=0}^{\infty}$, we have that $(x_n)_{n=0}^{\infty}$ is $(1+\varepsilon)$-basic. Let $K\in\mathbb{N}\cup\{0\}$ such that $N_K<N\leq N_{K+1}$. Thus,

$$\|x\| \geq (1+\varepsilon)^{-1}\|\sum_{n=0}^{K} x_n\| \text{ and } \|x\| \geq (1+\varepsilon)^{-1}\|x_{K+1}\|.$$

By (d), we have that $\|x_{K+1}\| \geq (1+\varepsilon)^{-1}\|\sum_{j=N_K+1}^{N} a_j z_j\|$. Thus, we have that

$$\begin{aligned}\Big\|\sum_{j=0}^{\infty} a_j z_j\Big\| &\geq (1+\varepsilon)^{-1}\max\Big(\Big\|\sum_{n=0}^{K} x_n\Big\|, \|x_{K+1}\|\Big)\\ &\geq (1+\varepsilon)^{-1}\max\Big(\Big\|\sum_{n=0}^{K} x_n\Big\|, (1+\varepsilon)^{-1}\Big\|\sum_{j=N_K+1}^{N} a_j z_j\Big\|\Big)\\ &\geq 2^{-1}(1+\varepsilon)^{-2}\Big\|\sum_{n=0}^{K} x_n + \sum_{j=N_K+1}^{N} a_j z_j\Big\|\\ &= 2^{-1}(1+\varepsilon)^{-2}\Big\|\sum_{j=0}^{N} a_j z_j\Big\|.\end{aligned}$$

This proves that $(z_j)_{j=0}^{\infty}$ is $2(1+\varepsilon)^2$-basic. Thus all that remains is to construct $(z_j)_{j=0}^{\infty}$ and $(g_j)_{j=0}^{\infty}$ by induction.

For the base case we take $z_0 = g_0 = 1_{[0,1]}$, $M_0 = N_0 = 0$, $M_{-1} = N_{-1} = -1$, and we formally define $P_{-1} = 0$ as the projection onto the zero vector. Thus all five conditions are trivially satisfied for $n = 0$. Now let $k \in \mathbb{N}_0$ and assume that $(g_m)_{m=0}^{k}$ and $(z_m)_{m=0}^{N_k}$ have been chosen to satisfy conditions (a),(b),(c),(d), and (e). For each $m \in \mathbb{N}$ we let $m = 2^{n_m-1} + j_m$ where $n_m \in \mathbb{N}$ and $0 \leq j_m < 2^{n_m-1}$. For $1 \leq m \leq k$, we denote $E_{2j_m,n_m} \subseteq [0,1]$ to be the support of g_m^+ and $E_{2j_m+1,n_m} \subseteq [0,1]$ to be the support of g_m^-. Being an initial segment of a Haar type system, $E_{2j_m,n_m} \cup E_{2j_m+1,n_m} = E_{j_m,n_m-1}$ for $1 \leq m \leq k$, and for the induction we must find an appropriate partition of $E_{j_{k+1},n_{k+1}-1}$. Note that if $j_k + 1 < 2^{n_k-1}$ then $j_{k+1} = j_k + 1$ and $n_{k+1} = n_k$; if $j_k + 1 = 2^{n_k-1}$ then $j_{k+1} = 0$ and $n_{k+1} = n_k + 1$.

As $(g_m)_{m=0}^{k}$ is contained in the span of the initial segment of the Haar basis $(h_j)_{j=0}^{M_k}$, we may partition $E_{j_{k+1},n_{k+1}-1}$ into two sets of equal measure $E_{2j_{k+1},n_{k+1}}$ and $E_{2j_{k+1}+1,n_{k+1}}$ such that both sets are a finite union of disjoint dyadic intervals and for all $x \in span(h_j)_{j=0}^{M_k}$, the distribution of $x|_{E_{2j_{k+1},n_{k+1}}}$ is the same as the distribution of $x|_{E_{2j_{k+1}+1,n_{k+1}}}$. We let $g_{k+1} = 2^{(n_{k+1}-1)/p}(1_{E_{2j_{k+1},n_{k+1}}} - 1_{E_{2j_{k+1}+1,n_{k+1}}})$. As the support of g_{k+1}^+ and the support of g_{k+1}^- are both finite unions of disjoint dyadic intervals, we have that $g_{k+1} \in span(h_j)_{j=1}^{\infty}$. Let $0 \leq m \leq M_k$. As the distribution of $h_m|_{E_{2j_{k+1},n_{k+1}}}$ is the same as the distribution of $h_m|_{E_{2j_{k+1}+1,n_{k+1}}}$, we have that $h_m^*(g_{k+1}) = 0$. Thus, $g_{k+1} \in span(h_j)_{j=M_k+1}^{\infty}$.

Thus, we have the following three properties.

(α) $(g_j)_{j=0}^{k+1}$ is the initial segment of a Haar type system in $L_p([0,1])$,

(β) $g_{k+1} \in span(h_j)_{j=M_k+1}^{\infty}$,

(γ) For all $x \in span(h_j)_{j=0}^{M_k}$, the distribution of $x|_{supp(g_{k+1}^+)}$ is the same as the distribution of $x|_{supp(g_{k+1}^-)}$.

By Lemma 5.3.2 there exists $N \in \mathbb{N}$ and $(x_j)_{j=1}^{2N}$ in the positive cone of $\ell_p(\mathbb{Z}_{2N} \oplus \mathbb{Z}_{2N})$ such that

(i) $(x_j)_{j=1}^{2N}$ is $(1+\varepsilon)$-basic.

(ii) If $f = (0)_{j=1}^{2N} \oplus ((2N)^{-1/q})_{j=1}^{2N} \in \ell_q(\mathbb{Z}_{2N}) \oplus \ell_q(\mathbb{Z}_{2N})$ then $(2N)^{1/q}|f(x)| \leq \frac{\varepsilon_{k+1}}{M_k+1}\|x\|$ for all x in the span of $(x_j)_{j=1}^{2N}$.

(iii) The distance from $(0)_{j=1}^{2N} \oplus ((-1)^j(2N)^{-1/p})$ to the span of $(x_j)_{j=1}^{2N}$ is at most ε_{k+1}.

As in the proof of Theorem 5.2.2, there exists a sequence of finite unions of disjoint dyadic intervals $(G_j)_{j=1}^{2N}$ in $[0,1]$ such that

(i) The sequence $(G_j)_{j=1}^{2N}$ is pairwise disjoint and $\lambda(G_j) = \lambda(G_i)$ for all i, j.

(ii) $\cup_{j=1}^{N} G_{2j-1}$ is the support of g_{k+1}^+ and $\cup_{j=1}^{N} G_{2j}$ is the support of g_{k+1}^-.

(iii) For all $x \in span(h_j)_{j=0}^{M_k}$, the sequence of functions $(x|_{G_j})_{j=1}^{2N}$ all have the same distribution.

Let $(H_j)_{j=1}^{2N}$ be a sequence of unit length intervals in $\mathbb{R} \setminus [0,1]$ with pairwise disjoint support which is disjoint from the support of z_j for all $0 \leq j \leq N_k$. We now define a map $\Psi : \ell_p(\mathbb{Z}_{2N} \oplus \mathbb{Z}_{2N}) \to L_p(\mathbb{R})$ by

$$\Psi(\alpha_1, ..., \alpha_{2N}, \beta_1, ..., \beta_{2N}) = \sum_{j=1}^{N}(2N)^{1/p}\beta_{2j-1}1_{G_{2j-1}}g_{k+1}^+ + \sum_{j=1}^{N}(2N)^{1/p}\beta_{2j}1_{G_{2j}}g_{k+1}^- + \sum_{j=1}^{2N}\alpha_j 1_{H_j}.$$

By (i), (ii), and that $\|g_{k+1}^-\| = \|g_{k+1}^+\| = 2^{-1/p}$ we have that $\|1_{G_{2j-1}}g_{k+1}^+\| = (2N)^{-1/p}$ and $\|1_{G_{2j}}g_{k+1}^-\| = (2N)^{-1/p}$ for all $1 \leq j \leq N$. Thus, Ψ is an isometric embedding and maps positive elements of $\ell_p(\mathbb{Z}_{2N} \oplus \mathbb{Z}_{2N})$ to positive functions in $L_p(\mathbb{R})$. We let $N_{k+1} = N_k + 2N$ and let $z_{N_k+j} = \Psi(x_j)$ for all $1 \leq j \leq 2N$. Thus, (d) is clearly satisfied.

Note that $\Psi((0,...,0) \oplus (\frac{1}{(2N)^{1/p}}, \frac{-1}{(2N)^{1/p}}, ..., \frac{1}{(2N)^{1/p}}, \frac{-1}{(2N)^{1/p}})) = g_{k+1}$, thus by (3) the distance from g_{k+1} to the span of $(z_j)_{j=N_k+1}^{N_{k+1}}$ is at most ε_{k+1} which proves (e).

Let $z \in span(z_j)_{j=N_k+1}^{N_{k+1}}$ with $\|z\| = 1$. We now prove that $\|P_{M_k} z\| \leq \varepsilon_{k+1}$. Note that $P_{M_k}(z) = \sum_{j=0}^{M_k} h_j^*(z) h_j$. Let $1 \leq m \leq M_k$. We have that the functions $(h_m|_{G_j})_{j=1}^{2N}$ all have equal distribution and $g_{k+1} \in span(h_j)_{j=M_k+1}^{\infty}$. Hence, $h_m^*(1_{G_j})$ is independent of j. Let $x = (\alpha_1, ..., \alpha_{2N}, \beta_1, ..., \beta_{2N}) \in span(x_j)_{j=1}^{2N}$ such that $\Psi(x) = z$. Let $f = (0)_{j=1}^{2N} \oplus ((2N)^{-1/q})_{j=1}^{2N} \in \ell_q(\mathbb{Z}_{2N}) \oplus \ell_q(\mathbb{Z}_{2N})$. By (2), we have that $(2N)^{1/q}|f(x)| \leq \frac{\varepsilon_{k+1}}{M_k+1}$. Since the biorthogonal functionals $(h_j^*)_{j=0}^{\infty}$ form the standard Haar basis in $L_q([0,1])$, h_m^* is a multiple of h_m, and we denote this multiple by $C_{p,m}$. We now have that

$$
\begin{aligned}
|h_m^*(z)| &= C_{p,m}|\int_0^1 h_m z \, dt| \\
&= C_{p,m}\Big|\int_0^1 h_m \Psi(x) dt\Big| \\
&= C_{p,m}\Big|\int_0^1 h_m \sum_{j=1}^{N} (2N)^{1/p} \beta_{2j-1} 1_{G_{2j-1}} g_{k+1}^+ + \sum_{j=1}^{N} (2N)^{1/p} \beta_{2j} 1_{G_{2j}} g_{k+1}^- dt\Big| \\
&= \Big|\sum_{j=1}^{N} \beta_{2j-1} + \sum_{j=1}^{N} \beta_{2j}\Big| (2N)^{1/p} 2^{(n_{k+1}-1)/p} \Big| h_m^*(1_{G_1})\Big| \\
&\leq (2N)^{1/q}|f(x)| \leq \frac{\varepsilon_{k+1}}{M_k+1}.
\end{aligned}
$$

Thus we have that $\|P_{M_k} z\| = \|\sum_{j=0}^{M_k} h_j^*(z) h_j\| \leq \sum_{j=0}^{M_k} \|h_j^*(z) h_j\| \leq \varepsilon_{k+1}$. This proves (c). For all $1 \leq j \leq 2N$, we have that G_j is a finite union of disjoint dyadic intervals. Thus, $span(z_j|_{[0,1]})_{j=N_k+1}^{N_{k+1}} \subseteq span(h_j)_{j=0}^{\infty}$. By ($\beta$), we also have that $g_{k+1} \in span(h_j)_{j=M_k+1}^{\infty}$. We now choose $M_{k+1} \in \mathbb{N}$ such that $span(z_j|_{[0,1]})_{j=N_k+1}^{N_{k+1}} \subseteq span(h_j)_{j=0}^{M_{k+1}}$ and $g_{k+1} \in span(h_j)_{j=0}^{M_{k+1}}$. Thus, (a) holds and our proof is complete.

$\square$

5.4 Schauder frames

Previously, we have considered Schauder bases for Banach spaces, which give unique representations for vectors. Given a Banach space X with dual X^*, a sequence of pairs $(x_j, f_j)_{j=1}^{\infty}$

in $X \times X^*$ is called a *Schauder frame* or *quasi-basis* of X if

$$x = \sum_{j=1}^{\infty} f_j(x)x_j \qquad \text{for all } x \in X. \tag{5.4.1}$$

A Schauder frame is called *unconditional* if the above series converges in every order. Schauder frames are a possibly redundant coordinate system in that the sequence of coefficients $(f_j(x))_{j=1}^{\infty}$ which can be used to reconstruct x in (5.4.1) may not be unique. Note that if $(x_j)_{j=1}^{\infty}$ is a Schauder basis of X with biorthogonal functionals $(x_j^*)_{j=1}^{\infty}$ then $(x_j, x_j^*)_{j=1}^{\infty}$ is a Schauder frame of X. Thus, Schauder frames are a generalization of Schauder bases.

For all $1 \leq p < \infty$, there does not exist an unconditional Schauder frame $(x_j, f_j)_{j=1}^{\infty}$ for $L_p(\mathbb{R})$ such that $(x_j)_{j=1}^{\infty}$ is a sequence of non-negative functions [285]. However, for all $1 \leq p < \infty$, there does exist a conditional Schauder frame $(x_j, f_j)_{j=1}^{\infty}$ for $L_p(\mathbb{R})$ such that $(x_j)_{j=1}^{\infty}$ is a sequence of non-negative functions [285]. Indeed, if $(e_j)_{j=1}^{\infty}$ is a Schauder basis for $L_p(\mathbb{R})$ with biorthogonal functionals $(e_j^*)_{j=1}^{\infty}$ then we may define a Schauder frame $(x_j, f_j)_{j=1}^{\infty}$ for $L_p(\mathbb{R})$ by $x_{2j} = e_j^+$, $x_{2j-1} = e_j^-$, $f_{2j} = e_j^*$, and $f_{2j-1} = -e_j^*$ for all $j \in \mathbb{N}$.

For each $1 \leq p < \infty$ and $\lambda \in \mathbb{R}$, we may define the right translation operator $T_\lambda : L_p(\mathbb{R}) \to L_p(\mathbb{R})$ by $T_\lambda f(t) = f(t - \lambda)$. Given $1 \leq p < \infty$, $f \in L_p(\mathbb{R})$, and $(\lambda_j)_{j=1}^{\infty} \subseteq \mathbb{R}$, there have been many interesting results on the possible structure of $(T_{\lambda_j} f)_{j=1}^{\infty}$, and the relation on the values $(\lambda_j)_{j=1}^{\infty}$ can be very subtle. For example, if $1 \leq p \leq 2$ then a simple Fourier transform argument gives that $(T_j f)_{j\in\mathbb{Z}}$ does not have dense span in $L_p(\mathbb{R})$ [21, 254, 255, 256]. On the other hand, if $2 < p < \infty$ then there does exist $f \in L_p(\mathbb{R})$ such that the span of $(T_j f)_{j\in\mathbb{Z}}$ is dense in $L_p(\mathbb{R})$ [21, 254, 255, 256]. Surprisingly, if $\varepsilon_j \neq 0$ for all $j \in \mathbb{Z}$ and $\varepsilon_j \to 0$ for $|j| \to \infty$ then there does exist $f \in L_2(\mathbb{R})$ such that $(T_{j+\varepsilon_j} f)_{j\in\mathbb{Z}}$ has dense span in $L_2(\mathbb{R})$ [267]. For any $(\lambda_j)_{j=1}^{\infty} \subseteq \mathbb{R}$, $1 \leq p < \infty$, and $f \in L_p(\mathbb{R})$ the sequence $(T_{\lambda_j} f)_{j=1}^{\infty}$ is not an unconditional Schauder basis for $L_p(\mathbb{R})$ ([268] for $p = 2$, [260] for $1 < p \leq 4$, and [114] for $4 < p$). However, if $2 < p$ and $(\lambda_j)_{j=1}^{\infty}$ is unbounded then there exists $f \in L_p(\mathbb{R})$ and a sequence of functionals $(g_j)_{j=1}^{\infty}$ such that $(T_{\lambda_j} f, g_j)_{j=1}^{\infty}$ is an unconditional Schauder frame of $L_p(\mathbb{R})$ [114]. It was not known for $1 \leq p < 2$ if there exists $(\lambda_j)_{j=1}^{\infty} \subseteq \mathbb{R}$, $f \in L_p(\mathbb{R})$, and a sequence of functionals $(g_j)_{j=1}^{\infty}$ such that $(T_{\lambda_j} f, g_j)_{j=1}^{\infty}$ is an unconditional Schauder frame or even conditional Schauder frame for $L_p(\mathbb{R})$. However, if the sequence $(g_j)_{j=1}^{\infty}$ is semi-normalized (in particular $(\|g_j\|^{-1})_{j=1}^{\infty}$ is bounded) then $(T_{\lambda_j} f, g_j)_{j=1}^{\infty}$ cannot be an unconditional Schauder frame for $L_p(\mathbb{R})$ for $1 \leq p \leq 2$ [50].

We will prove for all $1 \leq p < \infty$ that there exists a single non-negative function $f \in L_p(\mathbb{R})$ such that $(T_{\lambda_j} f, g_j)_{j=1}^\infty$ is a Schauder frame for $L_p(\mathbb{R})$ for some sequence of constants $(\lambda_j)_{j=1}^\infty$ and some sequence of functionals $(g_j)_{j=1}^\infty$. We will obtain this as a corollary from the following general result about the existence of certain Schauder frames, which we believe to be of independent interest. The proof of the following theorem is inspired by Pelczynski's proof that every separable Banach space with the bounded approximation property is isomorphic to a complemented subspace of a Banach space with a Schauder basis [273].

Theorem 5.4.1. Let X be a Banach space with a Schauder basis $(e_j)_{j=1}^\infty$. Suppose that $D \subseteq X$ is a subset whose span is dense in X. Then there exists a Schauder frame (quasi-basis) for X whose vectors are elements of D.

Proof. As $(e_j)_{j=1}^\infty$ is a Schauder basis of X, there exists $\varepsilon_j \searrow 0$ such that if $(u_j)_{j=1}^\infty \subseteq X$ and $\|e_j - u_j\| < \varepsilon_j$ for all $j \in \mathbb{N}$ then $(u_j)_{j=1}^\infty$ is a Schauder basis of X (see Theorem 1.3.19 [9]). As the span of D is dense in X we may choose $(u_j)_{j=1}^\infty \subseteq span(D)$ such that $\|e_j - u_j\| < \varepsilon_j$ for all $j \in \mathbb{N}$. Let $(u_j^*)_{j=1}^\infty$ be the sequence of biorthogonal functionals to $(u_j)_{j=1}^\infty$. For each $n \in \mathbb{N}$, we may choose a linearly independent and finite ordered set $(x_{j,n})_{j=1}^{J_n}$ in D such that u_n can be expressed as the finite sum $u_n = \sum_{j=1}^{J_n} a_{j,n} x_{j,n}$ where $a_{j,n}$ are non-zero scalars.

Let C_n be the basis constant of $(x_{j,n})_{j=1}^{J_n}$ and choose $N_n \in \mathbb{N}$ such that $C_n \leq N_n$. We currently have that u_n may be uniquely expressed as $u_n = \sum_{j=1}^{J_n} a_{j,n} x_{j,n}$, but to make a Schauder frame we will use the redundant expansion $u_n = \sum_{i=1}^{N_n} \sum_{j=1}^{J_n} N_n^{-1} a_{j,n} x_{j,n}$. We claim that $((x_{j,n}, N_n^{-1} a_{j,n} u_n^*))_{n\in\mathbb{N}, 1\leq i\leq N_n, 1\leq j\leq J_n}$ is a Schauder frame of X where we order $\{(n,i,j)\}_{n\in\mathbb{N},1\leq i\leq N_n,1\leq j\leq J_n}$ lexicographically. That is, $(n_1, i_1, j_1) \leq (n_2, i_2, j_2)$ if and only if

(i) $n_1 < n_2$, or

(ii) $n_1 = n_2$ and $i_1 < i_2$, or

(iii) $n_1 = n_2$ and $i_1 = i_2$ and $j_1 \leq j_2$.

Let $x \in X$ and $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\| \sum_{n=m_1}^{m_2} u_n^*(x) u_n \| < \varepsilon$ for all $m_2 \geq m_1 \geq N$. Consider a fixed (n_0, i_0, j_0) with $n_0 > N$, $1 \leq j_0 \leq J_{n_0}$, and $1 \leq i_0 \leq N_{n_0}$. We now have that,

$$
\begin{aligned}
&\Big\| x - \sum_{(n,i,j)\leq(n_0,i_0,j_0)} N_n^{-1} a_{j,n} u_n^*(x) x_{j,n} \Big\| \\
&\leq \Big\| x - \sum_{n=1}^{n_0-1} \sum_{i=1}^{N_n} \sum_{j=1}^{J_n} N_n^{-1} a_{j,n} u_n^*(x) x_{j,n} \Big\| + \Big\| \sum_{i=1}^{i_0-1} \sum_{j=1}^{J_{n_0}} N_{n_0}^{-1} a_{j,n_0} u_{n_0}^*(x) x_{j,n_0} \Big\| \\
&+ \Big\| \sum_{j=1}^{j_0} N_{n_0}^{-1} a_{j,n_0} u_{n_0}^*(x) x_{j,n_0} \Big\| \\
&\leq \Big\| x - \sum_{n=1}^{n_0-1} u_n^*(x) u_n \Big\| + \sum_{i=1}^{i_0-1} N_{n_0}^{-1} \| u_{n_0}^*(x) u_{n_0} \Big\| + \Big\| \sum_{j=1}^{j_0} N_{n_0}^{-1} a_{j,n_0} u_{n_0}^*(x) x_{j,n_0} \Big\| \\
&< \varepsilon + \varepsilon + C_{n_0} \Big\| \sum_{j=1}^{J_{n_0}} N_{n_0}^{-1} a_{j,n_0} u_{n_0}^*(x) x_{j,n_0} \Big\| \\
&= \varepsilon + \varepsilon + C_{n_0} N_{n_0}^{-1} \| u_{n_0}^*(x) u_{n_0} \| \\
&< \varepsilon + \varepsilon + \varepsilon \qquad\qquad\qquad\qquad \text{as } C_{n_0} \leq N_{n_0}.
\end{aligned}
$$

We have that $\sum_{(n,i,j)} N_n^{-1} a_{j,n} u_n^*(x) x_{j,n}$ converges to x, and hence the sequence of pairs $((x_{j,n}, N_n^{-1} a_{j,n} u_n^*))_{n\in\mathbb{N}, 1\leq i\leq N_n, 1\leq j\leq J_n}$ is a Schauder frame of X.

□

The previous theorem applied to Banach spaces with a Schauder basis, and we now show that the same conclusion can be obtained for separable Banach spaces with the bounded approximation property.

Corollary 5.4.2. Let X be a separable Banach space with the bounded approximation property (i.e. X has a quasi-basis). Suppose that $D \subseteq X$ is a subset whose span is dense in X. Then there exists a Schauder frame (quasi-basis) for X whose vectors are elements of D.

Proof. As X is separable and has the bounded approximation property there exists a Banach space Y with a basis such that $X \subseteq Y$ and there is a bounded projection $P : Y \to X$. As the span of D is dense in X, the span of $D \cup (I_Y - P)Y$ is dense in Y, where I_Y is the identity operator on Y. By Theorem 5.4.1, there exists a Schauder frame $(x_j, f_j)_{j=1}^\infty \cup (y_j, g_j)_{j=1}^\infty$ for Y, where $x_j \in D$ and $y_j \in (I_Y - P)Y$ for all $j \in \mathbb{N}$. The projection of a Schauder frame onto a complemented subspace is a Schauder frame for that subspace. Thus, $(Px_j, f_j|_X)_{j=1}^\infty \cup$

$(Py_j, g_j|_X)_{j=1}^\infty$ is a Schauder frame for X. This is the same as, $(x_j, f_j|_X)_{j=1}^\infty \cup (0, g_j|_X)_{j=1}^\infty$. Hence, $(x_j, f_j|_X)_{j=1}^\infty$ is a Schauder frame of X whose vectors are in D. □

We now give the following application to translations of a single positive vector.

Corollary 5.4.3. For all $1 \le p < \infty$, the Banach space $L_p(\mathbb{R})$ has a Schauder frame of the form $(x_j, f_j)_{j=1}^\infty$ where $(x_j)_{j=1}^\infty$ is a sequence of translates of a single non-negative function. In the range $1 < p < \infty$ this function can be taken to be the indicator function of a bounded interval in $\mathbb{R}$, and for $p = 1$ the function can be any non-negative function whose Fourier transform has no real zeroes.

Proof. We first consider the case $p = 1$. Let $f \in L_1(\mathbb{R})$. By Wiener's tauberian theorem, the set of translations of f has dense span in $L_1(\mathbb{R})$ if and only if the Fourier transform of f has no real zeroes [328]. Thus by Theorem 5.4.1 if the Fourier transform of f has no real zeroes then there exists a sequence of translations $(x_j)_{j=1}^\infty$ of f and a sequence of linear functionals $(f_j)_{j=1}^\infty$ such that $(x_j, f_j)_{j=1}^\infty$ is a Schauder frame of $L_1(\mathbb{R})$. As an example of a function $f \in L_1(\mathbb{R})$ such that $\hat{f}$ has no real zeroes, one can take $f(t) = e^{-t^2}$ for all $t \in \mathbb{R}$.

We now fix $1 < p < \infty$ and consider the interval $(0, 1] \subseteq \mathbb{R}$. Note that the span of the indicator functions of bounded intervals in $\mathbb{R}$ is dense in $L_p(\mathbb{R})$. Thus we just need to prove that every indicator function of a bounded interval is in the closed span of the translates of $(0, 1]$ and then apply Theorem 5.4.1 to get a Schauder frame of translates of the indicator function of $(0, 1]$. Let $D \subseteq L_p(\mathbb{R})$ be the span of the set of translates of $1_{(0,1]}$.

Let $1 > \varepsilon > 0$. For each $\lambda \in \mathbb{R}$, we denote $T_\lambda : L_p(\mathbb{R}) \to L_p(\mathbb{R})$ to be the operator which shifts functions λ to the right. That is, for all $f \in L_p(\mathbb{R})$, $T_\lambda f(t) = f(t - \lambda)$ for all $t \in \mathbb{R}$. Let $x_1 = 1_{(0,1]} - T_\varepsilon 1_{(0,1]} = 1_{(0,\varepsilon]} - 1_{(1,1+\varepsilon]}$. Thus, $x_1 \in D$. For $n \in \mathbb{N}$, we define $x_{n+1} \in D$ by

$$x_{n+1} = \sum_{j=0}^{n} T_j x_1 = \sum_{j=0}^{n} 1_{(j,j+\varepsilon]} - 1_{(j+1,j+1+\varepsilon]} = 1_{(0,\varepsilon]} - 1_{(n+1,n+1+\varepsilon]}.$$

As $1 < p < \infty$, the sequence $(x_n)_{n=1}^\infty$ converges weakly to $1_{(0,\varepsilon]}$. Thus, $1_{(0,\varepsilon]}$ is in the weak-closure and hence norm-closure of D as D is convex. This proves that every indicator function of an interval of length at most 1 is contained in $\overline{D}$. As every bounded interval is the disjoint union of finitely many intervals of length at most 1, we have that the indicator function of any bounded interval is contained in $\overline{D}$. □

When using a Schauder basis or Schauder frame to reconstruct a vector in a Banach space, we have that the partial sums of the series in (5.2.1) and (5.4.1) converge in norm. A Banach lattice is a Banach space endowed with an appropriate partial order. For example $L_p(\mathbb{R})$ is a Banach lattice with the partial order given for $f, g \in L_p(\mathbb{R})$ by $f \leq g$ if and only if $f(t) \leq g(t)$ for a.e. $t \in \mathbb{R}$. When considering Banach lattices, one cares about both the norm structure of the Banach space as well as the endowed order structure. This leads us to consider Schauder bases and Schauder frames where the partial sums of the reconstruction formula converge in order as well as in norm.

Let $(y_n)_{n=1}^\infty$ be a sequence in a Banach lattice X. We say that $(y_n)_{n=1}^\infty$ *converges uniformly* to y and write $y_n \xrightarrow{u} y$ if there exists a positive vector $w \in X$ such that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|y - y_n| \leq \varepsilon w$ for all $n \geq N$. The vector w is called a *regulator* of the sequence $(y_n)_{n=1}^\infty$. Let $(x_j)_{j=1}^\infty$ be a Schauder basis for a subspace E of Banach lattice X with biorthogonal functionals $(x_j^*)_{j=1}^\infty$. We say that the sequence $(x_j)_{j=1}^\infty$ is *bibasic* if for all $x \in E$ we have that $\sum_{j=1}^n x_j^*(x)x_j \xrightarrow{u} x$. Similarly, let $(x_j, f_j)_{j=1}^\infty$ be a Schauder frame for a subspace E of a Banach lattice X. We say that $(x_j, f_j)_{j=1}^\infty$ is a *u-frame* if for all $x \in E$ we have that $\sum_{j=1}^n f_j(x)x_j \xrightarrow{u} x$. The difference between the two names (bibasis and u-frame) is that the bibasis condition is equivalent to multiple different properties [314, Theorem 3.1] or [315, Theorem 20.1], whereas this is not the case in the context of frames.

We now extend Theorem 5.4.1 to the setting of Banach lattices with a bibasis.

Theorem 5.4.4. *Let X be a Banach lattice with a bibasis $(e_j)_{j=1}^\infty$. Suppose that $D \subseteq X$ is a subset whose span is dense in X. Then there exists a u-frame for X whose vectors are elements of D.*

Proof. The proof begins analogously to Theorem 5.4.1, noting that small perturbations of bibases are bibases ([314, Theorem 4.2]).

We construct $(u_n)_{n=1}^\infty$ and $((x_{j,n})_{j\in J_n})_{n=1}^\infty$ as in the proof of Theorem 5.4.1. We currently have that $((x_{j,n}, N_n^{-1}a_{j,n}u_n^*))_{n\in\mathbb{N},1\leq i\leq N_n,j\in J_n}$ is a Schauder frame of X in the lexicographical order whenever the N_n are sufficiently large. We now need to show that it is a u-frame. For each $n \in \mathbb{N}$, we define $v_n = \sum_{j\in J_n} |x_{j,n}|$. Let $v = \sum_{n=1}^\infty \frac{1}{2^n}\frac{v_n}{\|v_n\|}$ and choose $N_n \in \mathbb{N}$ such that $N_n \geq 4^n\|u_n^*\|\|v_n\| \max_{j\in J_n} |a_{j,n}|$. Then for each $x \in X$ and each subset I_n of J_n, $|\sum_{j\in I_n} N_n^{-1}a_{j,n}u_n^*(x)x_{j,n}| \leq \frac{1}{2^n}v\|x\|$. We claim that $((x_{j,n}, N_n^{-1}a_{j,n}u_n^*))_{n\in\mathbb{N},1\leq i\leq N_n,j\in J_n}$ is a u-frame of X where, again, we order $\{(n,i,j)\}_{n\in\mathbb{N},1\le i\le N_n,j\in J_n}$ lexicographically.

Let $x \in X$ and let $w \in X^+$ be a regulator for $\sum_{j=1}^n u_n^*(x)u_n \xrightarrow{u} x$. In particular, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\frac{1}{2^N}\|x\| < \varepsilon$ and $|\sum_{n=m_1}^{m_2} u_n^*(x)u_n| \le \varepsilon w$ for all $m_2 \ge m_1 \ge N$. Consider a fixed (n_0, i_0, j_0) with $n_0 > N$, $j_0 \in J_{n_0}$, and $1 \le i_0 \le N_{n_0}$. By analogous estimates one shows that

$$\left| x - \sum_{(n,i,j)\le(n_0,i_0,j_0)} N_n^{-1} a_{j,n} u_n^*(x) x_{j,n} \right| \le 3\varepsilon(v \vee w).$$

Hence $\sum_{(n,i,j)} N_n^{-1} a_{j,n} u_n^*(x) x_{j,n}$ converges to x uniformly with regulator $v \vee w$, proving that $((x_{j,n}, N_n^{-1} a_{j,n} u_n^*))_{n\in\mathbb{N},1\le i\le N_n,j\in J_n}$ is a u-frame of X. $\square$

The Haar system is not a bibasis for $L_1(\mathbb{R})$, but the Haar system is a bibasis for $L_p(\mathbb{R})$ for the range $1 < p < \infty$ [144]. Thus, the following corollary follows from Theorem 5.4.4 and Corollary 5.4.3.

Corollary 5.4.5. *For all $1 < p < \infty$, the Banach space $L_p(\mathbb{R})$ has a u-frame of the form $(x_j, f_j)_{j=1}^\infty$ where $(x_j)_{j=1}^\infty$ is a sequence of translates of a single non-negative function. Furthermore, this function can be taken to be the indicator function of a bounded interval in $\mathbb{R}$.*

As in Corollary 5.4.2, it is possible to weaken the assumptions in Theorem 5.4.4. However, a direct application of the proof in Corollary 5.4.2 does not apply in the lattice case. Instead, we notice that the bibasis assumption was only used to justify a small perturbation argument. The next proposition establishes a stability for u-frames that likely can be improved, but is sufficient for our purposes.

Proposition 5.4.6. *Suppose (x_j, f_j) is a u-frame for a closed subspace E of a Banach lattice X, and let $0 < \varepsilon < 1$. Then if (y_j) is a sequence in E with the property that*

$$\|x_j - y_j\| \le \frac{\varepsilon}{2^{2j+1}\|f_j\|},$$

then there exists $g_j \in E^*$ such that (y_j, g_j) is a u-frame for E.

Proof. For $x \in E$ define $S(x) = \sum_{j=1}^\infty f_j(x)(x_j - y_j)$. It is shown in [285, Lemma 2.3] that S is well-defined, and $\|S\| < 1$. In fact, it is easy to see that the sum is uniformly Cauchy, and hence uniformly converges. One then defines $T = I - S$, so that $T(x) = u - \sum_{j=1}^\infty f_j(x) y_j$ is an invertible operator on E. Replacing x with $T^{-1}x$, we see that $x = u - \sum_{j=1}^{\infty} f_j(T^{-1}x)y_j$, so that $(y_j, (T^{-1})^* f_j)$ is a u-frame for E. $\square$

Incorporating this small perturbation result into the proof of Theorem 5.4.4, we obtain the following.

Corollary 5.4.7. *Let E be a closed subspace of a Banach lattice X and let $D \subseteq E$ have dense span. If E admits a u-frame, then there is a u-frame (x_j, f_j) for E with each $x_j \in D$.*

Remark 5.4.8. Although Corollary 5.4.7 involves notions from Banach lattice theory, it actually implies Corollary 5.4.2. Indeed, let X be a Banach space with a Schauder frame and let $D \subseteq X$ be a subset whose span is dense in X. Since X is separable, we may view X as a subspace of $C[0,1]$. However, in $C[0,1]$ it is easy to see that norm convergence and u-convergence coincide. Hence, we deduce that X has a u-frame when viewed as a subspace of $C[0,1]$. We then apply Corollary 5.4.7 to find a u-frame (which, in particular, will be a frame) (x_j, f_j) for X with each $x_j \in D$. Recovering Corollary 5.4.2 is one reason we chose to present Corollary 5.4.7 for u-frames instead of other types of order convergence.

5.5 Open problems

Johnson and Schechtman constructed a Schauder basis for $L_1(\mathbb{R})$ consisting of non-negative functions [193], and in Theorem 5.2.2 we construct a Schauder basis for $L_2(\mathbb{R})$ consisting of non-negative functions. The following remaining cases are still open.

Question 5.5.1. Let $1 < p < \infty$ with $p \neq 2$. Does $L_p(\mathbb{R})$ have a Schauder basis consisting of non-negative functions?

In Theorem 5.3.3, we showed that $L_p(\mathbb{R})$ contains a basic sequence $(f_n)_{n=1}^{\infty}$ of non-negative functions such that $L_p(\mathbb{R})$ embeds into the closed span of $(f_n)_{n=1}^{\infty}$. Furthermore, the proof gives that for all $\varepsilon > 0$, $(f_n)_{n=1}^{\infty}$ can be chosen to be $(2+\varepsilon)$-basic.

Question 5.5.2. Let $1 \leq p < \infty$ with $p \neq 2$. For all $\varepsilon > 0$, does $L_p(\mathbb{R})$ contain a $(1+\varepsilon)$-basic sequence $(f_n)_{n=1}^{\infty}$ of non-negative functions such that $L_p(\mathbb{R})$ embeds into the closed span of $(f_n)_{n=1}^{\infty}$? What is the infimum of the set of all basis constants of non-negative bases in $L_1(\mathbb{R})$?

The questions about non-negative bases in $L_p(\mathbb{R})$ that are considered here and in [285] naturally extend to general Banach lattices. We say that a Schauder basis $(x_n)_{n=1}^{\infty}$ of a

Banach lattice is *positive* if $x_n \geq 0$ for all $n \in \mathbb{N}$. We say that a Schauder basis $(x_n)_{n=1}^\infty$ has *positive biorthogonal functionals* if the biothorgonal functionals $(x_n^*)_{n=1}^\infty$ satisfy $x_n^* \geq 0$ for all $n \in \mathbb{N}$. In the case of $L_p(\mu)$ or $C([0,1])$, Schauder bases of non-negative functions correspond exactly with Schauder bases of positive vectors. The unit vector basis for ℓ_p is a positive Schauder basis for all $1 \leq p < \infty$, and the Faber-Schauder system in $C([0,1])$ is a Schauder basis of non-negative functions [107].

The existence of positive bases in L_1 has the following application to the general theory of Banach lattices:

Proposition 5.5.3. Every separable Banach lattice embeds lattice isometrically into a Banach lattice with a positive Schauder basis.

Proof. It was shown in [223] that every separable Banach lattice embeds lattice isometrically into $C(\Delta, L_1)$, where Δ denotes the Cantor set and $C(\Delta, L_1)$ denotes the Banach space of continuous functions from Δ to L_1. Hence, it suffices to show that $C(\Delta, L_1)$ has a positive Schauder basis.

By [193], L_1 has a basis (f_j) of positive vectors, and by the proof of [300, Proposition 2.5.1], $C(\Delta)$ has a basis (d_i) of positive vectors. For each $i, j \in \mathbb{N}$, define $d_i \otimes f_j \in C(\Delta, L_1)$ via $(d_i \otimes f_j)(t) = d_i(t) f_j$ for all $t \in \Delta$. Clearly, $d_i \otimes f_j \geq 0$ in $C(\Delta, L_1)$.

Now note that $C(\Delta, L_1)$ is lattice isometric to $C(\Delta) \otimes_\lambda L_1$, the injective tensor product of $C(\Delta)$ and L_1. We order the collection $(d_i \otimes f_j)_{i,j\in\mathbb{N}}$ into the sequence $(z_k)_{k=1}^\infty$ by $z_1 = d_1 \otimes f_1$ and for $k > 1$ we let

$$z_k = \begin{cases} d_i \otimes f_{n+1} & \text{for } k = n^2 + i \text{ where } i, n \in \mathbb{N} \text{ and } 1 \leq i \leq n+1, \\ d_{n+1} \otimes f_{n+1-i} & \text{for } k = n^2 + n + 1 + i \text{ where } i, n \in \mathbb{N} \text{ and } 1 \leq i \leq n. \end{cases}$$

Then [303, Theorem 18.1 and Corollary 18.3] guarantee that $(z_k)_{k=1}^\infty$ is a Schauder basis. $\square$

We have given several examples of Banach lattices with positive bases, including $L_1(\mathbb{R})$, $L_2(\mathbb{R}), C([0,1])$, ℓ_p, $C(\Delta, L_1)$ and others. By duality it is easy to see that $L_2(\mathbb{R})$ has a basis with positive biorthogonal functionals, and using [303, Proposition 10.1, p. 321] one sees that if K is compact, Hausdorff and $C(K)$ is infinite-dimensional then $C(K)$ cannot have a basis with positive biorthogonal functionals. Obviously, the spaces ℓ_p have a basis with positive biorthogonal functionals whenever $1 \le p < \infty$. A general question to pose is:

Question 5.5.4. Give further examples of Banach lattices possessing positive bases and/or bases with positive biorthogonal functionals. Of particular interest are Banach lattices possessing bases but lacking positive bases.

There are other weaker forms of coordinate systems for which one can impose positivity conditions. For example, we refer the reader to [314, Remark 7.13] for questions regarding the structure of Banach lattices possessing FDDs with positivity properties on their associated projections. Recall that a *Markushevich basis* of a Banach space X is a biorthogonal system $(x_n, x_n^*)_{n=1}^\infty$ such that the closed span of $(x_n)_{n=1}^\infty$ is X and the collection of functionals $(x_n^*)_{n=1}^\infty$ separates the points of X. Obviously, when X is a Banach lattice one can put positivity conditions on x_n and x_n^*, and in [285] it is shown that for all $1 \le p < \infty$, $L_p(\mathbb{R})$ has a Markushevich basis consisting of non-negative functions. This leaves another general question:

Question 5.5.5. Which separable Banach lattices have Markushevich bases consisting of positive vectors? Which separable Banach lattices have Markushevich bases consisting of positive functionals?

As noted above, if K is compact, Hausdorff and $C(K)$ is infinite-dimensional then $C(K)$ cannot have a basis with positive biorthogonal functionals. The basis assumption can be weakened. Indeed, we have the following simple observation.

Proposition 5.5.6. No infinite dimensional $C(K)$ admits a biorthogonal system (x_k, x_k^*) with $\overline{\text{span}}(x_k) = C(K)$, and (x_k^*) positive.

Proof. Since scaling doesn't affect positivity, we may assume x_k^* is normalized by sending $(x_k, x_k^*) \mapsto (y_k, y_k^*) := (\|x_k^*\| x_k, \frac{x_k^*}{\|x_k^*\|})$. Since (x_k^*) is now normalized, for each $x \in C(K)$ we have $x_k^*(x) \to 0$. Indeed, for $\varepsilon > 0$ find $a_1, \dots, a_M$ such that $\|x - \sum_{i=1}^M a_i x_i\| < \varepsilon$. If $k > M$ then $|x_k^*(x)| = |x_k^*(x - \sum_{i=1}^M a_i x_i)| < \|x_k^*\| \varepsilon = \varepsilon$. However, since x_k^* is positive, $1 = \|x_k^*\| = x_k^*(\mathbb{1})$, a contradiction. $\square$

Suppose that X is a Banach lattice with a Schauder frame $(x_j, f_j)_{j=1}^\infty$. By splitting up each vector into its positive and negative parts, we obtain that the sequence of pairs $(x_1^+, f_1), (x_1^-, -f_1), (x_2^+, f_2), (x_2^-, -f_2), \ldots$ is a Schauder frame of X consisting of positive vectors. Thus, every Banach lattice with a Schauder frame has a Schauder frame with positive vectors. Similarly, every Banach lattice with a Schauder frame has a Schauder frame with positive functionals. On the other hand, in [285] it is proven for all $1 \leq p < \infty$ that $L_p(\mathbb{R})$ does not have an unconditional Schauder frame consisting of positive vectors. As with the other types of coordinate systems mentioned above, we pose the general question of constructing unconditional Schauder frames with desirable positivity conditions.

Question 5.5.7. Which separable Banach lattices have an unconditional Schauder frame with positive vectors? Which separable Banach lattices have an unconditional Schauder frame with positive functionals?

Chapter 6

Free Banach lattices

6.1 Introduction and preliminaries

This chapter is based on the memoir [262], joint with Timur Oikhberg, Pedro Tradacete and Vladimir Troitsky. The goal is to investigate free Banach lattices generated by Banach spaces. The history of this notion is quite recent: While free vector lattices were already present in the literature in the 1960's [40, 55], the corresponding normed version had been completely overlooked until B. de Pagter and A. Wickstead in [269] first considered the free Banach lattice generated by a set. This can be considered as a natural precursor of the construction, due to A. Avilés, J. Rodríguez and P. Tradacete in [26], of the free Banach lattice generated by a Banach space.

Given a Banach space E, the free Banach lattice generated by E is a Banach lattice $\mathrm{FBL}[E]$ together with a linear isometric embedding $\phi_E : E \to \mathrm{FBL}[E]$ such that for every bounded linear operator $T : E \to X$ into a Banach lattice X, there is a unique lattice homomorphism $\widehat{T} : \mathrm{FBL}[E] \to X$ such that $\widehat{T} \circ \phi_E = T$ and $\|\widehat{T}\| = \|T\|$. From a categorical point of view, this can be seen as a functor from the category of Banach spaces and bounded linear operators into the smaller category of Banach lattices and lattice homomorphisms. It is, in a certain sense, analogous to well studied functors in analysis, such as the one from compact Hausdorff spaces K into spaces of continuous functions $C(K)$, or the functor creating the Lipschitz free space generated by a (pointed) metric space (cf. [129]).

It will soon become clear that understanding the correspondence $E \mapsto \mathrm{FBL}[E]$ is a key to properly understand the interplay between Banach space and Banach lattice properties, a goal that has been pursued ever since the first developments of these theories (see, e.g., [189, 196]). In particular, our investigation will be far from categorical, focusing mainly on the fine structure of $\mathrm{FBL}[E]$ and the correspondence $E \mapsto \mathrm{FBL}[E]$.

For several reasons, it will be convenient to also work with free Banach lattices satisfying some convexity conditions, as considered in [185]. For a Banach space E and $p \in [1, \infty]$, we define the free p-convex Banach lattice over E as follows: $\mathrm{FBL}^{(p)}[E]$ is a p-convex Banach lattice with p-convexity constant 1 together with a linear isometric embedding $\phi_E : E \to \mathrm{FBL}^{(p)}[E]$ such that for every bounded linear operator $T : E \to X$ into a p-convex Banach lattice X, there is a unique lattice homomorphism $\widehat{T} : \mathrm{FBL}^{(p)}[E] \to X$ such that $\widehat{T} \circ \phi_E = T$, and $\|\widehat{T}\| \leq M^{(p)}(X)\|T\|$. Here, $M^{(p)}(X)$ denotes the p-convexity constant of X. It is clear that $\mathrm{FBL}^{(1)}[E]$ coincides with $\mathrm{FBL}[E]$ (and we will stick to the latter notation).

Much of our investigation relies on the following explicit functional representation of $\mathrm{FBL}^{(p)}[E]$, first established in [185]. For this, denote by $H[E]$ the linear subspace of $\mathbb{R}^{E^*}$ consisting of all positively homogeneous functions $f : E^* \to \mathbb{R}$. For $1 \leq p < \infty$ and $f \in H[E]$ we define

$$\|f\|_{\mathrm{FBL}^{(p)}[E]} = \sup \left\{ \left(\sum_{k=1}^{n} |f(x_k^*)|^p \right)^{1/p} : n \in \mathbb{N},\, x_1^*, \ldots, x_n^* \in E^*,\, \sup_{x \in B_E} \sum_{k=1}^{n} |x_k^*(x)|^p \leq 1 \right\}. \tag{6.1.1}$$

Note that, for $(x_k^*)_{k=1}^n \subseteq E^*$, by considering the operator $T : E \to \ell_p^n$ given by $Tx = (x_k^*(x))_{k=1}^n$ and using the fact that $\|T\| = \|T^{**}\|$, it follows that

$$\sup_{x \in B_E} \sum_{k=1}^{n} |x_k^*(x)|^p = \sup_{x^{**} \in B_{E^{**}}} \sum_{k=1}^{n} |x^{**}(x_k^*)|^p. \tag{6.1.2}$$

Given any $x \in E$, let $\delta_x \in H[E]$ be defined by

$$\delta_x(x^*) := x^*(x) \quad \text{for all } x^* \in E^*.$$

Then, $\mathrm{FBL}^{(p)}[E]$ coincides with the closed sublattice of $H[E]$ generated by $\{\delta_x : x \in E\}$ with respect to $\|\cdot\|_{\mathrm{FBL}^{(p)}[E]}$, together with the map $x \mapsto \delta_x$. As mentioned, this explicit representation of $\mathrm{FBL}^{(p)}[E]$ was originally proven in [185]; the proof will be recalled below. As we will see, it is the interplay between the universal property of $\mathrm{FBL}^{(p)}[E]$ and the explicit functional representation that allows us to discern the fine structure of these spaces.

When $p = \infty$, $\mathrm{FBL}^{(\infty)}[E]$ is nothing but the closed sublattice of $C(B_{E^*})$ generated by the point evaluations. Here, $C(B_{E^*})$ denotes the space of continuous functions on the dual ball of E, which is equipped with the relative w^*-topology. In particular, we have

$$\|f\|_{\mathrm{FBL}^{(\infty)}[E]} = \sup\{|f(x^*)| : x^* \in E^*, \|x^*\| \leq 1\}. \tag{6.1.3}$$

As we will show in Proposition 6.2.2, the closure of the point evaluations in $C(B_{E^*})$ coincides with the space of *all* positively homogeneous weak* continuous functions on B_{E^*}. This gives a very concrete description of $\mathrm{FBL}^{(\infty)}[E]$ - this space often behaves differently from $\mathrm{FBL}^{(p)}[E]$ when $1 \leq p < \infty$.

A word on free objects

The universal property defining $\mathrm{FBL}[E]$ (or analogously, $\mathrm{FBL}^{(p)}[E]$) can be visualized by means of the following diagram:

$$\begin{tikzcd} \mathrm{FBL}[E] \arrow[dr, "\widehat{T}"] & \\ E \arrow[u, "\phi_E"] \arrow[r, "T"'] & X \end{tikzcd}$$

meaning that for every *object* X (a Banach lattice) and every *linear operator* $T : E \to X$ there is a unique *morphism* $\widehat{T}$ (lattice homomorphism) making the diagram commutative.

The idea of the free object in a certain category (Banach lattices with lattice homomorphisms) generated by an object in a supercategory (Banach spaces with bounded linear operators) is certainly not new, and has been central in several developments in algebra, topology and analysis. We will not attempt here to address the fruitful developments of this idea in universal algebra, but let us just recall that these include many well-known concepts such as free groups, free modules, free algebras or free lattices.

The study of free objects in Banach space theory can be considered as a more recent development. However, some classical facts can be reworded in this language too. Consider, for instance, the subcategory of *dual* Banach spaces together with *dual operators* (equivalently, weak* continuous linear maps). Given a Banach space E, let $J_E : E \to E^{**}$ denote the canonical embedding; it is clear that every bounded linear operator $T : E \to X^*$ can be uniquely extended to a dual operator $\overset{*}{T} : E^{**} \to X^*$, given by $\overset{*}{T} = (T^* \circ J_X)^*$, in such a way that the following diagram commutes:

$$\begin{tikzcd} E^{**} \arrow[dr, "\overset{*}{T}"] & \\ E \arrow[u, "J_E"] \arrow[r, "T"] & X^* \end{tikzcd}$$

Thus, we can consider E^{**} as the *free dual Banach space generated by E.*

Lipschitz free spaces (also known as Arens-Eells or transportation cost spaces) have recently attracted considerable attention from researchers interested in Banach space theory and metric geometry (see, for instance, the survey paper [131]). These spaces can be defined as follows: Given a metric space M with a distinguished point 0, $\mathcal{F}(M)$ is a Banach space equipped with an isometric map $\delta : M \to \mathcal{F}(M)$ with the property that for every Banach space X and every Lipschitz map $f : M \to X$ with $f(0) = 0$, there is a unique linear operator $\widehat{f} : \mathcal{F}(M) \to X$ making the following diagram commute:

$$\begin{tikzcd} \mathcal{F}(M) \arrow[dr, "\widehat{f}"] & \\ M \arrow[u, "\delta"] \arrow[r, "f"] & X \end{tikzcd}$$

A very fruitful line of research is devoted to analyzing the interplay between Banach space properties of $\mathcal{F}(M)$ versus metric properties of M. Results from this line have deeply inspired our research on free Banach lattices.

Free objects also arise in the theory of group C^*-algebras. Suppose, for simplicity, that G is a discrete group. We say that a map $\pi : G \to A$ (A is a unital C^*-algebra) is a *unitary representation* if it takes G into the unitary group of A, and, for any $g, h \in G$, we have $\pi(gh) = \pi(g)\pi(h)$ (which implies $\pi(g^{-1}) = \pi(g)^*$). Then one defines the *full* (or *universal*) C^*-algebra $C^*(G)$ over G as follows: $C^*(G)$ is a C^*-algebra, together with a unitary representation $\psi : G \to C^*(G)$, with the property that for every C^*-algebra A, and every unitary representation $\pi : G \to A$, there exists a unique $*$-representation $\widehat{\pi} : C^*(G) \to A$, making the following diagram commute:

$$\begin{tikzcd} C^*(G) \arrow[dr, "\widehat{\pi}"] & \\ G \arrow[u, "\psi"] \arrow[r, "\pi"] & A \end{tikzcd}$$

For the construction and basic properties of $C^*(G)$, see [63, Section 2.5] or [281, Chapter 3]. One can also consider full C^*-algebras of a more general class of locally compact groups; in this case, certain continuity properties of representations need to be assumed. The reader is referred to [92, Chapter VII] for details.

The investigation of free C^*-algebras was motivated by two related questions.

1) Finding connections between properties of a group G and those of $C^*(G)$ (with the reduced group C^*-algebra $C^*_r(G)$ often also added to the mix). A sample result is [63, Theorem 2.6.8]: a discrete group G is amenable if and only if $C^*(G)$ is nuclear.

2) The famous Kirchberg's QWEP conjecture is equivalent to $C^*(\mathbb{F})$ having the Weak Expectation Property (a relaxation of injectivity) for any free group $\mathbb{F}$ [281, Chapter 13]. By [281, Chapter 14], this is also equivalent to Connes' Embedding Problem, which has recently been resolved in the negative in [188].

We refer the reader to [276] and references therein for several other constructions of free topological objects, as well as their connections to various universal constructions, such as free and tensor products [325]. We also note that a more "axiomatic" approach to freeness has been pursued by A. Ya. Helemskii in, e.g., [167], [168], [169], [170], and [205] (see also [19] for a different take on the same approach). Specifically, suppose $\mathcal{K}$ and $\mathcal{L}$ are categories, and $\Box$ is a faithful covariant functor $\mathcal{K} \to \mathcal{L}$ (usually, a "forgetful functor"). $\mathcal{K}$ is called a *rigged category*, and, with respect to appropriate rigs, the works cited above construct free objects in various situations. This includes quantum spaces [205], normed operator modules [169], normed modules over sequence algebras [167], matricially normed spaces [168], as well as multinormed spaces and their generalizations [170]. This allows one to construct projective objects in these categories as well. A similar approach has been recently used in [24].

Historical perspectives

Although free Banach lattices were not introduced until 2015, their inception triggered a rapid development of the theory. Here, we briefly summarize the literature. With one exception, the articles below focus on FBL; however, we work with the full scale of spaces $\mathrm{FBL}^{(p)}$, $1 \leq p \leq \infty$. That being said, most of the results in this chapter are either new for FBL, or require significantly different proofs in order to generalize to $\mathrm{FBL}^{(p)}$.

The theory of free Banach lattices began with [269], which introduced the concept of a free Banach lattice over a set A. In our terminology, this is simply the space $\mathrm{FBL}[\ell_1(A)]$. In [269], the authors proved several structural results, and showed that this new class of spaces differs significantly from the classical Banach lattices. After this, free Banach lattices over general Banach spaces were introduced in [26]. Among other things, this allowed the authors of [26] to answer some questions left open in [269], as well as a question of J. Diestel on weakly compactly generated Banach lattices, and opened the door for a deeper study of the relationship between Banach spaces and Banach lattices.

After the above two seminal works, the theory expanded in several directions. One interesting direction - that we will not pursue here - centers around the free Banach lattice generated by a lattice. Recall that a Banach lattice combines two distinct structures: The Banach space structure, and a lattice structure. In analogy with the free Banach lattice $\mathrm{FBL}[E]$ generated by a Banach space E, one can consider the free Banach lattice $\mathrm{FBL}\langle\mathbb{L}\rangle$ generated by a lattice $\mathbb{L}$. This latter construction is also quite rich, and the two theories parallel each other to some extent. However, there are also some interesting differences. For example, $\mathrm{FBL}\langle\mathbb{L}\rangle$ is always lattice isomorphic to an AM-space [34], whereas $\mathrm{FBL}[E]$ is lattice isomorphic to an AM-space if and only if E is finite dimensional. We refer the interested reader to [27, 28, 33, 34] for literature on $\mathrm{FBL}\langle\mathbb{L}\rangle$. For literature on free lattices (without involving norms), we mention [236] for free α-order complete vector lattices, [1] for free σ-order complete truncated vector lattices, [209] for free lattice-ordered Lie algebras, [320] for projective vector lattices, [210] for free lattice-ordered groups and free products of lattice-ordered groups, [117] for free products of Boolean algebras and measure algebras with applications to tensor products of universally complete vector lattices, and references therein.

As a second extension of the concept of free Banach lattices, free Banach lattices satisfying convexity conditions were constructed in [185]. Recall that the defining property of $\mathrm{FBL}[E]$ is that any linear operator from E to a Banach lattice X extends uniquely to $\mathrm{FBL}[E]$ as a lattice homomorphism of the same norm. If instead of looking at *all* Banach lattices X, one only requires that the above property hold for p-convex spaces, then one can construct a free object $\mathrm{FBL}^{(p)}[E]$ that is p-convex in its own right. This is of interest because p-convexity is a classical Banach lattice property (see, e.g., [231]), and because having a scale of spaces $\mathrm{FBL}^{(p)}[E]$, $1 \leq p \leq \infty$, adds an additional dimension to the theory, similar to how the L_p

scale enriches the study of L_1. Moreover, the paper [185] constructs various free lattices satisfying convexity conditions $\mathcal{D}$, by placing a maximal $\mathcal{D}$-convex norm on the free vector lattice, and then completing the resulting space. Such a construction builds on ideas from [321].

There have also been several papers focusing on FBL$[E]$, and its applications. For example, [90, 91] focus on the isometric theory of free Banach lattices. As a brief overview, [91] studies, among other things, when the norm of FBL$[E]$ is octahedral; [90] is able to use free Banach lattices to produce the first example of a lattice homomorphism that does not attain its norm. In a different direction, [30] studies free Banach lattices generated by the classical sequence spaces $\ell_p(\Gamma)$. In particular, the authors of [30] are able to precisely describe the moduli of the canonical unit vector bases, and when these spaces are weakly compactly generated. [31] studies when a Banach lattice X is lattice isomorphic to a lattice-complemented sublattice of FBL$[X]$. As it turns out, any Banach lattice X ordered by a 1-unconditional basis has this property.

Pure vector lattice properties of FBL$[E]$ have also been studied. For example, [25] is able to prove that FBL$[E]$ satisfies the countable chain condition, i.e., that any collection of pairwise disjoint vectors in FBL$[E]$ must be countable. Finally, there are many interesting applications of free Banach lattices. For example, [35] is able to classify the separable Banach lattices X such that whenever a Banach lattice Y contains a subspace isomorphic to X then it also contains a sublattice isomorphic to X. It is classical that $X = c_0$ has this property. However, it is shown in [35] that a separable Banach lattice has this property if and only if it lattice embeds into $C[0,1]$. In particular, whenever $C[0,1]$ embeds linearly into a Banach lattice, it also embeds as a sublattice. This feature is not shared by $C(\Delta)$, with Δ the Cantor set, even though $C[0,1]$ and $C(\Delta)$ are isomorphic as Banach spaces. As noted in [35], it is not known if every Banach lattice for which linear embeddability implies lattice embeddability is necessarily separable, but free Banach lattices put several constraints on how such a supposed space would look.

Free Banach lattices also play an important role in the study of projective Banach lattices. Projectivity for Banach lattices was also first considered by B. de Pagter and A. Wickstead in [269]. Informally, a Banach lattice P is projective if every lattice homomorphism from P into the quotient of a Banach lattice X lifts to a lattice homomorphism into X, with control of the norm. As a consequence of the fact that $\ell_1(A)$ is a projective Banach space for any nonempty set A, it easily follows that $\mathrm{FBL}[\ell_1(A)]$ is a projective Banach lattice. This is the first connection between projectivity and FBL, but the topics interlace in much deeper ways. We refer the reader to [22, 23] for some of the connections between projectivity and free Banach lattices; however, plenty more results are scattered throughout the literature we have cited in this subsection. Free Banach lattices also have interesting applications to amalgamation and injectivity; in particular, they can be used to define push-outs and thus play a role in the construction of Banach lattices of universal disposition and separably injective Banach lattices, see [29].

Very recent results on free Banach lattices - building off the work here - include [32, 123, 124, 264].

A brief outline of the results

We begin this chapter by giving, in Section 6.2, a "natural" functional representation of free Banach lattices (Theorem 6.2.1, Proposition 6.2.2). The proof of Theorem 6.2.1 is a slight simplification of the one from [185], but the identification $\mathrm{FBL}^{(\infty)}[E] = C_{ph}(B_{E^*})$ given in Proposition 6.2.2 is entirely new. Section 6.2 also contains various comments on the functional representation. The most important of these is Theorem 6.2.9, where we show that the free vector lattice over E is not only norm dense, but order dense, in $\mathrm{FBL}^{(p)}[E]$.

Section 6.3 studies the relationship between an operator $T : F \to E$, and its induced operator $\overline{T} : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$. In Proposition 6.3.2 we show that several properties – injectivity, surjectivity, density of the range, etc., – pass freely between T and $\overline{T}$. We then look at the way $\mathrm{FBL}^{(p)}[F]$ sits inside of $\mathrm{FBL}^{(p)}[E]$ when F is a subspace of E. Theorem 6.3.4 shows that, if $\iota\colon F \hookrightarrow E$ is the inclusion map, then $\bar{\iota}\colon \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ is order continuous – in other words, $\mathrm{FBL}^{(p)}[F]$ is a regular sublattice of $\mathrm{FBL}^{(p)}[E]$. Examples from Section 6.3 (built on "low-tech" Banach lattice techniques) show that, in the above setting, $\mathrm{FBL}^{(p)}[F]$ need not be closed in $\mathrm{FBL}^{(p)}[E]$ – that is, $\bar{\iota}$ need not be an isomorphic embedding. This leads us to study the "subspace problem": under what conditions does the embedding $\iota : F \hookrightarrow E$ induce a lattice isomorphic embedding $\bar{\iota} : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$?

In Section 6.3 we reduce this question to certain extension properties of (pairs of) Banach spaces. More specifically, in Theorem 6.3.7 we establish that $\bar{\iota}$ is bounded below if, and only if, any operator $T : F \to L_p$ extends to E. In this case, we say that the pair (F, E) has the POE-p (Property of operator extension into L_p – Definition 6.3.8). Although the POE-p is defined by extension properties of a family of operators, the fact that it is equivalent to the single operator $\bar{\iota}$ being an embedding gives some stability. More precisely, if, for every $\varepsilon > 0$, (F, E) has the POE-p with constant $C + \varepsilon$, then Proposition 6.3.9 shows that (F, E) has the POE-p with constant C. Section 6.3 finishes with a discussion of when $\bar{\iota}(\mathrm{FBL}^{(p)}[F])$ is complemented in $\mathrm{FBL}^{(p)}[E]$.

Returning to the question of when $\bar{\iota}$ is an embedding, in Section 6.4, we gather general facts about the POE-p. This includes a reformulation in terms of ℓ_∞-factorable operators, various criteria in terms of 2-summing operators, and a relation with $\mathcal{L}_p$-spaces. Section 6.4 explores the connections between the POE-p, passing to the double dual, and taking ultrapowers. This allows us to give several examples of spaces having, or failing, the POE-p.

Section 6.4 discusses several more properties of the POE-p. In particular, a push-out argument shows that one does not need to require a uniform constant independent of embeddings; it comes for free by Proposition 6.4.16. Similarly, to check whether a Banach space F has the POE-p, it suffices to only consider embeddings into spaces of the same density character, see Proposition 6.4.17. In Proposition 6.4.19 we use our results on the POE-p to examine connections between the POE-p for different values of p. In particular, we prove that ℓ_1 has the POE-p if and only if $2 \leq p \leq \infty$ (cf. Proposition 6.4.18 for a more general result on $\mathcal{L}_{1,\mu}$-spaces). On the other hand, a space with a normalized unconditional basis has the POE-1 if and only if that basis is equivalent to the c_0 basis (Proposition 6.4.20). We finish by showing, in Proposition 6.4.25, that (F, L_1) can never have the POE-p when $F \subseteq L_1$ is an infinite dimensional Hilbertian subspace.

In Section 6.5, we investigate the properties of the sequence $\left(|\delta_{x_k}|\right)_k \subseteq \mathrm{FBL}^{(p)}[E]$, where (x_k) is a basic (and often, unconditional) sequence in E. We show that every weakly null semi-normalized sequence (x_n) in a Banach space E has a subsequence so that $(|\delta_{x_{n_k}}|)$ is basic in $\mathrm{FBL}^{(p)}[E]$ (Proposition 6.5.8). Proposition 6.5.14 shows that, if a normalized basis (x_k) satisfies a lower 2-estimate, then $\left(|\delta_{x_k}|\right)_k$ is equivalent to the ℓ_1 basis. By Proposition 6.5.17, the converse is true for unconditional bases. We also examine whether $\left(|\delta_{x_k}|\right)_k$ is necessarily unconditional (Example 6.5.16, Proposition 6.5.18). In Section 6.5, we compute the moduli of branches of the Haar in $\mathrm{FBL}[L_1]$.

Part of the motivation to study the sequence $\left(|\delta_{x_k}|\right)_k \subseteq \mathrm{FBL}^{(p)}[E]$ comes from the universal property of free Banach lattices. Suppose $(x_k) \subseteq E$ is as above, and X is a p-convex Banach lattice. Then any operator $T : E \to X$ extends canonically to a lattice homomorphism $\widehat{T} : \mathrm{FBL}^{(p)}[E] \to X$, with $\widehat{T}|\delta_{x_k}| = |Tx_k|$. Consequently, the sequence $(|\delta_{x_k}|)$ "dominates" $(|Tx_k|)$. In particular, if $(|\delta_{x_k}|)$ is weakly null, then so is $(|Tx_k|)$; see Proposition 6.5.1, and the subsequent discussion.

We continue our work on $\big(|\delta_{x_k}|\big)_k$ in Section 6.6. In particular, in Proposition 6.6.1 and Corollary 6.6.2 we express, for $a_1, \ldots, a_n \geq 0$, the norm $\big\| \sum_{k=1}^n a_k |\delta_{x_k}| \big\|_{\mathrm{FBL}[E]}$ as a 1-summing norm of a certain operator. This is very useful for computations, and, in particular, allows us to recover some of the main results of [30].

Armed with this knowledge, we attempt to reconstruct properties of (x_k) from those of $\big(|\delta_{x_k}|\big)_k \subseteq \mathrm{FBL}^{(p)}[E]$. Our first task is to describe sequences $(x_k) \subseteq E$ which are equivalent to $\big(|\delta_{x_k}|\big)_k \subseteq \mathrm{FBL}^{(p)}[E]$, $p < \infty$. It turns out that, if this holds, and (x_k) is a normalized basis of E, then it has to be equivalent to the ℓ_1 basis (Proposition 6.6.3). However, in general, a normalized basic sequence (x_k) may be equivalent to $\big(|\delta_{x_k}|\big)$, but not to the ℓ_1 basis. Indeed, Proposition 6.6.5 shows that, if $(x_k) \subseteq C(\Omega)$ is a sequence equivalent to the ℓ_2 basis, then $\big(|\delta_{x_k}|\big) \subseteq \mathrm{FBL}^{(p)}[C(\Omega)]$ $(1 \leq p \leq \infty)$ is equivalent to the same basis. Moreover, in $\mathrm{FBL}^{(\infty)}[E]$, every unconditional basic sequence (x_k) is equivalent to $\big(|\delta_{x_k}|\big)$, in stark contrast to the case $p < \infty$.

In Corollary 6.6.9–Proposition 6.6.11 we characterize the normalized unconditional bases (x_k) of E for which $\overline{\mathrm{span}}[|\delta_{x_k}| : k \in \mathbb{N}]$ is complemented in $\mathrm{FBL}^{(p)}[E]$. For $p = 1$, this happens only for the ℓ_1 basis, for $p \in (1, \infty)$ this never happens, and for $p = \infty$ this happens only for the c_0 basis. In Corollary 6.6.14 and Corollary 6.6.15, we give examples of sequences $(x_k) \subseteq L_1$ for which $\big(|\delta_{x_k}|\big)_k \subseteq \mathrm{FBL}[L_1]$ is equivalent to the ℓ_1 basis.

We finish Section 6.6 by proving the following rigidity result: If (x_k) is an unconditional basis of E, and $\big(|\delta_{x_k}|\big) \subseteq \mathrm{FBL}^{(p)}[E]$ is equivalent to the ℓ_2 basis, then (x_k) must be equivalent to the c_0 basis (Theorem 6.6.17).

In Section 6.7, we use free Banach lattices to construct the first example of a subspace of a Banach lattice without a bibasic sequence (Theorem 6.7.5, Remark 6.7.6). This answers a question from [314]. In Section 6.7, we discuss connections with majorizing maps, and prove some results akin to the Bibasis Theorem 6.7.1. In particular, in Proposition 6.7.10, we show that the class of sequentially uniformly continuous operators (defined originally in [314]) coincides with the class of (∞, ∞)-regular operators (as defined in [295]). Moreover, in Section 6.7 we provide a converse to [314, Proposition 7.8]: The ℓ_1 basis is the only normalized basis that is absolute in any Banach lattice where it linearly embeds.

Section 6.8 examines (p-convex) Banach lattices E which embed into $\mathrm{FBL}^{(p)}[E]$ as a sublattice. Theorem 6.8.3 shows that, if the order on E is determined by its 1-unconditional basis, then E embeds into $\mathrm{FBL}^{(p)}[E]$ as a sublattice, complemented by a contractive lattice homomorphic projection. These results partially overlap with those in [31], though the proofs are very different.

In Section 6.9 we develop a dictionary between Banach space properties of E and Banach lattice properties of $\mathrm{FBL}^{(p)}[E]$. To begin, we prove that $\mathrm{FBL}^{(p)}[E]$ has a strong unit if and only if E is finite dimensional (Proposition 6.9.1), and $\mathrm{FBL}^{(p)}[E]$ has a quasi-interior point if and only if E is separable (Proposition 6.9.4). We further elaborate on this topic in Section 6.9, by showing (Proposition 6.9.6) that E is finite dimensional if and only if $\mathrm{FBL}^{(p)}[E]$ is finitely generated (and, in this case, $\dim E$ equals the smallest number of generators).

Section 6.9 considers the connection between E being weakly compactly generated and $\mathrm{FBL}^{(p)}[E]$ being lattice weakly compactly generated. This is a topic that has been explored before, and the implications E WCG $\Rightarrow$ $\mathrm{FBL}[E]$ LWCG $\not\Rightarrow$ $\mathrm{FBL}[E]$ WCG were used to solve a problem which was raised by J. Diestel in a conference in La Manga (Spain) in 2011. Our main contribution is to prove that if $\mathrm{FBL}^{(p)}[E]$ is LWCG then E is a subspace of a WCG space. This makes significant progress towards the conjecture that $\mathrm{FBL}[E]$ is LWCG if and only if E is WCG.

In Section 6.9 we consider the existence of complemented copies of ℓ_1. Theorem 6.9.20 shows that E contains a complemented copy of ℓ_1 if and only if $\mathrm{FBL}[E]$ contains a lattice complemented sublattice isomorphic to ℓ_1 if and only if $\mathrm{FBL}[E]$ contains a complemented copy of ℓ_1 (a few other equivalent conditions on $\mathrm{FBL}[E]$ are also given).

In Section 6.9 we characterize when $\mathrm{FBL}[E]$ satisfies an upper p-estimate, and deduce various corollaries. The main result is Theorem 6.9.21 which shows that id_{E^*} is $(q, 1)$-summing if and only if $\mathrm{FBL}[E]$ satisfies an upper p-estimate ($\frac{1}{p} + \frac{1}{q} = 1$). In particular, this shows that $\mathrm{FBL}[E]$ can never be more than 2-convex when E has infinite dimension. Theorem 6.9.21

also leads to a "local" version of Theorem 6.9.20: E contains uniformly complemented copies of ℓ_1^n if and only if $\mathrm{FBL}[E]$ contains uniformly lattice complemented sublattice copies of ℓ_1^n (Corollary 6.9.25). Further, it allows us to generalize some classical theorems on p-convex Banach lattices to Banach lattices with an upper p-estimate. Specifically, Corollary 6.9.29 shows that if a Banach lattice F embeds POE-1 into a Banach lattice E with an upper p-estimate $(1 < p < 2)$, then F must also have an upper p-estimate. This result with p-convexity in place of an upper p-estimate and POE-1 replaced by complementation is classical, see [231, Theorem 1.d.7]. Finally, we ask if $\mathrm{FBL}^{(p)}[E]$ can be q-convex for some $q > p$. This leads to an interesting dichotomy at $p = 2$, and the (sharp) estimate $q \leq \max\{2, p\}$ (Proposition 6.9.30). In particular, although there are examples of infinite dimensional E such that $\mathrm{FBL}[E]$ is 2-convex, it is impossible for $\mathrm{FBL}^{(2)}[E]$ to be more than 2-convex, unless E is finite dimensional, in which case it is ∞-convex.

In Section 6.9 we further pursue the automatic convexity of free Banach lattices, and use this to study the connections between $\mathrm{FBL}^{(p)}[E]$ and $\mathrm{FBL}^{(q)}[E]$ for various values of p and q. In the previous section, a characterization of when $\mathrm{FBL}[E]$ satisfies an upper p-estimate was given, and in this section a p-convex variant is proven. Specifically, Proposition 6.9.38 shows that $\mathrm{FBL}^{(p)}[E]$ and $\mathrm{FBL}^{(q)}[E]$ are lattice isomorphic if and only if every operator $T : E \to L_q$ factors strongly through L_p. This, of course, connects deeply with the Maurey-Nikishin factorization theory, and allows us to give new perspectives on this classical topic. One corollary (Corollary 6.9.41) is that when $q \geq 1$ and $p > \max\{2, q\}$ every infinite dimensional Banach space E admits an operator $T : E \to L_q$ which does not strongly factor through L_p. Moreover, we are able to prove the extrapolation Theorem 6.9.40: If $\mathrm{FBL}^{(p)}[E]$ has convexity $q > p$, then $\mathrm{FBL}^{(p)}[E]$ is lattice isomorphic to $\mathrm{FBL}[E]$. This complements the characterization that $\mathrm{FBL}[E]$ has non-trivial convexity if and only if E^* has non-trivial cotype given in Corollary 6.9.25. It also allows us to present various situations where $\mathrm{FBL}^{(p)}[E]$ and $\mathrm{FBL}^{(q)}[E]$ are lattice isomorphic, and gives us the ability to distinguish $\mathrm{FBL}^{(p)}[E]$ from the p-convexification of $\mathrm{FBL}[E]$.

In Section 6.9 we also elaborate on our study of upper p-estimates. One interesting fact about the free p-convex Banach lattice is that L_p is sufficient to witness its universal property, i.e., uniform extension of operators into L_p implies uniform extension of operators into an arbitrary p-convex Banach lattice. We prove a similar upper p-estimate version of this theorem. Specifically, we show in Proposition 6.9.36 that weak-L_p is sufficient to verify the universal property of being the free Banach lattice satisfying an upper p-estimate. Morally, this means that if a Banach lattice Z contains E as a generating set and allows uniform lattice homomorphic extension of maps from E into $L_{p,\infty}$, then the same is true with $L_{p,\infty}$ replaced by an arbitrary Banach lattice with an upper p-estimate. This then allows us to characterize the class of (p,∞)-convex operators in Corollary 6.9.37: An operator is (p,∞)-convex if and only if it strongly factors through a Banach lattice with an upper p-estimate.

Section 6.10 is devoted to determining whether $\mathrm{FBL}^{(p)}[E]$ and $\mathrm{FBL}^{(p)}[F]$ can be lattice isomorphic, even when the underlying spaces E and F are not. We begin, in Section 6.10, by representing lattice homomorphisms between free lattices as composition operators (Proposition 6.10.1). For $p=\infty$, we show that the lattices $\mathrm{FBL}^{(\infty)}[E]$ are lattice isometric to each other, for a wide class of spaces E (Theorem 6.10.24). On the other hand, for $p<\infty$, we show that $\mathrm{FBL}^{(p)}[E]$ will not be lattice isomorphic to a lattice quotient of $\mathrm{FBL}^{(p)}[F]$, provided E and F are "sufficiently different" (Proposition 6.10.12). Moreover, under fairly general conditions Theorem 6.10.18 shows that a lattice isometry between $\mathrm{FBL}^{(p)}[E]$ and $\mathrm{FBL}^{(p)}[F]$ ($p<\infty$) descends to an isometry between E and F. Along the way, we discover various properties of lattice homomorphisms between free Banach lattices.

Section 6.11 is based on the work [185], which was a precursor to [262]. Section 6.11 initiates a study of the various Banach lattices of homogeneous functions associated to a Banach space, focusing primarily on those defined via nonlinear (p,q)-summing maps. This direction will be be explored more comprehensively in [220].

Conventions

We use the standard Banach space and Banach lattice notation throughout the chapter. The reader can consult [9] and [230] for Banach spaces, [12], [231] and [244] for Banach lattices. We work with real spaces, though we refer the reader to [171] for information on free complex Banach lattices. The closed unit ball of a normed space E shall be denoted by B_E. We assume, without mention, that all measures involved are σ-finite. In particular, this convention is in place when we state that $L_\infty(\mu)$ is injective. We use the shorthand "L_p-space" for $L_p(\mu)$. When speaking of bases, (e_k) will be our notation for the standard unit vector basis of ℓ_r or c_0, and (x_k) will denote a generic basic sequence. From now on, "subspace" will be synonymous with "closed non-zero subspace", unless mentioned otherwise.

Throughout, operators are assumed to be linear and bounded. For extensions of operators, we adopt the following convention. For an operator T, we denote by $\widehat{T}$ its lattice homomorphic extension. Extensions which are merely linear and bounded are denoted by $\widetilde{T}$. We write $\overline{T} : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ for the canonical extension of $T : F \to E$; that is, $\overline{T} = \widehat{\phi_E \circ T}$. Also, when the Banach space E is unambiguous, we will write ϕ instead of ϕ_E for the canonical inclusion.

We shall use the term "lattice isomorphism" to mean "lattice homomorphic isomorphism"; "lattice isometry" is defined in a similar way. Further, we use the shorthand "lattice projection" to mean "idempotent lattice homomorphism." If there is a lattice projection from X to its sublattice Y, we say that Y is "lattice-complemented" in X.

6.2 Construction of free spaces and basic properties

In this section, for the convenience of the reader, we recall the explicit construction of $\mathrm{FBL}^{(p)}[E]$, and some of its basic properties. We first do the case $p < \infty$, and then provide a new, concrete description of $\mathrm{FBL}^{(\infty)}[E]$.

Recall that a Banach lattice X is p-convex for $1 \leq p \leq \infty$ if there is a constant $M \geq 1$ such that for every choice of $(x_k)_{k=1}^n \subseteq X$ we have

$$\left\| \left(\sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} \right\| \leq M \left(\sum_{k=1}^n \|x_k\|^p \right)^{\frac{1}{p}},$$

if $p < \infty$, or

$$\left\| \bigvee_{k=1}^n |x_k| \right\| \leq M \max_{1 \leq k \leq n} \|x_k\|,$$

if $p = \infty$. Let $M^{(p)}(X)$ denote the p-convexity constant of X; that is, the smallest possible value of M in the inequalities above. Note in particular that every Banach lattice X is 1-convex with $M^{(1)}(X) = 1$. We refer to [231, Section 1.d] for general background on p-convexity.

Let $H[E]$ denote the linear subspace of $\mathbb{R}^{E^*}$ consisting of all positively homogeneous functions $f : E^* \to \mathbb{R}$; i.e., functions satisfying $f(\lambda x^*) = \lambda f(x^*)$ for $\lambda \geq 0$ and $x^* \in E^*$.

Given $f \in H[E]$, set

$$\|f\|_{\mathrm{FBL}^{(p)}[E]} = \sup\left\{ \left(\sum_{k=1}^n |f(x_k^*)|^p\right)^{1/p} : n \in \mathbb{N},\, x_1^*, \dots, x_n^* \in E^*,\, \sup_{x \in B_E} \sum_{k=1}^n |x_k^*(x)|^p \le 1 \right\}.$$

It is easy to see that

$$H_p[E] := \left\{ f \in H[E] : \|f\|_{\mathrm{FBL}^{(p)}[E]} < \infty \right\}$$

is a sublattice of $H[E]$ and that $\|\cdot\|_{\mathrm{FBL}^{(p)}[E]}$ defines a complete p-convex lattice norm on $H_p[E]$ with p-convexity constant one. Moreover, for $x \in E$, we define $\delta_x \in H[E]$ by $\delta_x(x^*) = x^*(x)$ for $x^* \in E^*$. Note that $\|\delta_x\|_{\mathrm{FBL}^{(p)}[E]} = \|x\|$ for every $x \in E$.

Let FVL$[E]$ denote the sublattice generated by $\{\delta_x\}_{x \in E}$ in $H[E]$. FVL$[E]$ consists of all possible expressions which can be written with finitely many elements of the form δ_x and finitely many linear and lattice operations. In fact, by [12, p. 204, Exercise 8(b)] the sublattice generated by a subset W of a vector lattice is given by

$$\left\{ \bigvee_{k=1}^n u_k - \bigvee_{k=1}^n w_k : n \in \mathbb{N},\, u_1, \dots, u_n, w_1, \dots, w_n \in \operatorname{span} W \right\}. \tag{6.2.1}$$

As we will show in the proof of Theorem 6.2.1 below, FVL$[E]$ has the universal property of the free vector lattice over E. More specifically, every linear map $T : E \to X$ into an (Archimedean) vector lattice X uniquely extends to FVL$[E]$ as a lattice homomorphism. This justifies our notation, FVL$[E]$, for this space. We define FBL$^{(p)}[E]$ as the closure of FVL$[E]$ in $H_p[E]$, and note that the map $\phi_E \colon E \to \mathrm{FBL}^{(p)}[E]$ given by $\phi_E(x) = \delta_x$ is a linear isometry. The goal now is to show that this space satisfies the universal property of the free p-convex Banach lattice:

Theorem 6.2.1. *Let X be a p-convex Banach lattice* ($1 \le p < \infty$) *and $T \colon E \to X$ an operator. There is a unique lattice homomorphism $\widehat{T} \colon \mathrm{FBL}^{(p)}[E] \to X$ such that $\widehat{T} \circ \phi_E = T$, and $\|\widehat{T}\| \leqslant M^{(p)}(X)\, \|T\|$, where $M^{(p)}(X)$ denotes the p-convexity constant of X.*

Proof. We first want to show that there is a unique lattice homomorphism $\widehat{T} \colon$ FVL$[E] \to X$ such that $\widehat{T}\delta_x = Tx$ for every $x \in E$. To those familiar with the construction of the free vector lattice, this should be relatively clear, but we provide an explicit construction nonetheless.

Let $f \in$ FVL$[E]$. By definition, f is a lattice-linear combination of $\delta_{x_1}, \dots, \delta_{x_n}$ for some $x_1, \dots, x_n \in E$. We define $\widehat{T}f$ to be the same lattice-linear combination of $Tx_1, \dots, Tx_n$ in

X. That is, suppose that $f = F(\delta_{x_1}, \dots, \delta_{x_n})$ for some lattice-linear expression $F(t_1, \dots, t_n)$; we then define $\widehat{T}f = F(Tx_1, \dots, Tx_n)$. To show that $\widehat{T}$ is well-defined, suppose $f = G(\delta_{y_1}, \dots, \delta_{y_m})$ where $G(t_1, \dots, t_m)$ is another lattice-linear expression. Choose a maximal linearly independent subset of $x_1, \dots, x_n, y_1, \dots, y_m$; denote these variables by $z_1, \dots, z_k$. Then write $F(\delta_{x_1}, \dots, \delta_{x_n}) = \tilde{F}(\delta_{z_1}, \dots, \delta_{z_k})$ and $G(\delta_{y_1}, \dots, \delta_{y_m}) = \tilde{G}(\delta_{z_1}, \dots, \delta_{z_k})$, by replacing those elements of $\{x_1, \dots, x_n, y_1, \dots, y_m\} \setminus \{z_1, \dots, z_k\}$ by their representation as a linear combination of $z_1, \dots, z_k$. Since $\mathrm{FVL}[E]$ is a sublattice of $H[E]$, the lattice operations are pointwise, hence $f(x^*) = \tilde{F}(\delta_{z_1}(x^*), \dots, \delta_{z_k}(x^*)) = \tilde{F}(x^*(z_1), \dots, x^*(z_k))$ in $\mathbb{R}$ for each $x^* \in E^*$. Similarly, $f(x^*) = \tilde{G}(x^*(z_1), \dots, x^*(z_k))$. Since $z_1, \dots, z_k$ are linearly independent, by picking an appropriate x^* we deduce that $\tilde{F}(t_1, \dots, t_k) = \tilde{G}(t_1, \dots, t_k)$ for all $t_1, \dots, t_k \in \mathbb{R}$. Now by lattice-linear function calculus (cf. [231, p. 1.d]) we have that $\tilde{F}(Tz_1, \dots, Tz_k) = \tilde{G}(Tz_1, \dots, Tz_k)$ in X, and by linearity of T it follows that

$$F(Tx_1, \dots, Tx_n) = G(Ty_1, \dots, Ty_m).$$

Hence, $\widehat{T}$ is well-defined. Moreover, it is clear that $\widehat{T}$ is the unique lattice homomorphism extending T in the sense that $\widehat{T}\delta_x = Tx$ for every $x \in E$.

Our next objective is to show that

$$\left\|\widehat{T}f\right\|_X \leqslant M^{(p)}(X)\ \|T\|\ \|f\|_{\mathrm{FBL}^{(p)}[E]} \tag{6.2.2}$$

for every $f \in \mathrm{FVL}[E]$, as this will ensure that $\widehat{T}$ extends uniquely to a lattice homomorphism defined on all of $\mathrm{FBL}^{(p)}[E]$, and the extension has norm at most $M^{(p)}(X)\ \|T\|$. Without loss of generality, $\|T\| = 1$. We split the proof of the inequality (6.2.2) in two parts: First we establish it in the special case where $X = L_p(\mu)$ for some σ-finite measure space (Ω, Σ, μ), and then we show how to deduce the general version from the special case.

Thus, suppose first that $X = L_p(\mu)$ for some σ-finite measure space (Ω, Σ, μ); one could even assume that μ is a probability measure. Let $f \in \mathrm{FVL}[E]$. As explained above, f can be written as a lattice-linear expression $f = F(\delta_{x_1}, \dots, \delta_{x_m})$ for some $x_1, \dots, x_m \in E$ and $\widehat{T}f = F(Tx_1, \dots, Tx_m)$ in $L_p(\mu)$. Let $\varepsilon > 0$ and fix $\delta > 0$ (to be determined later). For each $i = 1, \dots, m$, find a simple function y_i such that $\|Tx_i - y_i\| < \delta$. Let $\mathcal{G}$ be the (finite) sub-σ-algebra generated by $y_1, \dots, y_m$. Let $P\colon L_p(\mu) \to L_p(\mathcal{G}, \mu)$ be the conditional expectation. Consider the lattice homomorphism $\widehat{PT}\colon \mathrm{FVL}[E] \to L_p(\mathcal{G}, \mu)$. It follows from $Py_i = y_i$ that

$$\|PTx_i - Tx_i\| \leqslant \|PTx_i - Py_i\| + \|y_i - Tx_i\| < 2\delta$$

for every $i = 1, \dots, m$. Since function calculus is norm continuous,

$$\left\|\widehat{T}f - \widehat{PT}f\right\| = \Big\|F(Tx_1, \dots, Tx_m) - F(PTx_1, \dots, PTx_m)\Big\| < \varepsilon$$

provided that δ is sufficiently small. It follows that $\left\|\widehat{T}f\right\| \leqslant \left\|\widehat{PT}f\right\| + \varepsilon$. Now note that $L_p(\mathcal{G}, \mu)$ is lattice isometric to ℓ_p^n for some n; let $U\colon L_p(\mathcal{G}, \mu) \to \ell_p^n$ be a lattice isometry. Let $R = UPT$; then $\widehat{R} = U\widehat{PT}$ because both are lattice homomorphisms agreeing on the generators.

Being an operator into ℓ_p^n, R can be represented as $Rx = \sum_{k=1}^n x_k^*(x)e_k$. for some $x_1^*, \dots, x_n^*$ in E^*. We then have

$$\sup_{x \in B_E} \Big(\sum_{k=1}^n |x_k^*(x)|^p\Big)^{\frac{1}{p}} = \|R\| \leqslant 1.$$

It follows from

$$\left\|\widehat{PT}f\right\| = \left\|U\widehat{PT}f\right\| = \left\|\widehat{R}f\right\| = \Big(\sum_{k=1}^n |f(x_k^*)|^p\Big)^{\frac{1}{p}} \leq \|f\|_{\mathrm{FBL}^{(p)}[E]}$$

that $\left\|\widehat{T}f\right\| \leq \|f\|_{\mathrm{FBL}^{(p)}[E]} + \varepsilon$. Since ε was arbitrary, we get $\left\|\widehat{T}f\right\| \leqslant \|f\|_{\mathrm{FBL}^{(p)}[E]}$.

We are now ready to tackle the general case where X is an arbitrary p-convex Banach lattice. Given $f \in \mathrm{FVL}[E]$, choose $x^* \in X_+^*$ with $\|x^*\| = 1$ and $x^*\big(|\widehat{T}f|\big) = \|\widehat{T}f\|_X$. Let N_{x^*} denote the null ideal generated by x^*, that is, $N_{x^*} = \big\{x \in X : x^*\big(|x|\big) = 0\big\}$, and let Y be the completion of the quotient lattice X/N_{x^*} with respect to the norm $\|x + N_{x^*}\| := x^*\big(|x|\big)$. Since this is an abstract L_1-norm, Y is lattice isometric to $L_1(\Omega, \Sigma, \mu)$ for some measure space (Ω, Σ, μ) (see, e.g., [231, Theorem 1.b.2]). The canonical quotient map of X onto X/N_{x^*} induces a lattice homomorphism $Q\colon X \to L_1(\Omega, \Sigma, \mu)$ with $\|Q\| = 1$. We may without loss of generality assume that (Ω, Σ, μ) is σ-finite, passing for instance to the band generated by $Q(\widehat{T}f)$.

Since Q is a lattice homomorphism and X is p-convex, we have

$$\left\|\Big(\sum_{k=1}^n \big|Q(x_k)\big|^p\Big)^{\frac{1}{p}}\right\|_{L_1(\mu)} \leqslant \left\|\Big(\sum_{k=1}^n |x_k|^p\Big)^{\frac{1}{p}}\right\|_X \leqslant M^{(p)}(X)\,\Big(\sum_{k=1}^n \|x_k\|_X^p\Big)^{\frac{1}{p}}$$

for every $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$. Hence the Maurey–Nikishin Factorization Theorem (see, e.g, [9, Theorem 7.1.2.], and recall that $p < \infty$) yields a positive function $h \in L_1(\Omega, \Sigma, \mu)$

with $\int_\Omega h\,d\mu = 1$ such that Q is bounded if we regard it as an operator into $L_p(h\,d\mu)$. More precisely, we have a factorization diagram

$$\begin{tikzcd} X \arrow[r, "Q"] \arrow[d, "S"'] & L_1(\mu) \\ L_p(h\,d\mu) \arrow[r, hook] & L_1(h\,d\mu), \arrow[u, "j_h"'] \end{tikzcd}$$

where $Sx = h^{-1}Qx$ satisfies $\|S\| \leqslant M^{(p)}(X)$ and $j_h(g) = gh$ is an isometric embedding. Note in particular that S is also a lattice homomorphism.

Let us now consider the composite operator $R = S \circ T \colon E \to L_p(h\,d\mu)$. By the first part of the proof, we know that there is a unique lattice homomorphism $\widehat{R} \colon \mathrm{FBL}^{(p)}[E] \to L_p(h\,d\mu)$ such that $\widehat{R}(\delta_x) = Rx$ for every $x \in E$, and $\left\|\widehat{R}\right\| = \|R\| \leqslant M^{(p)}(X)$. Since $S \circ \widehat{T}$ and $\widehat{R}$ are lattice homomorphisms which agree on the set $\{\delta_x : x \in E\}$, it follows that $S \circ \widehat{T}|_{\mathrm{FVL}[E]} = \widehat{R}|_{\mathrm{FVL}[E]}$. Hence we have

$$\begin{aligned} \left\|\widehat{T}f\right\|_X &= x^*\left(\left|\widehat{T}f\right|\right) = \left\|Q(\widehat{T}f)\right\|_{L_1(\mu)} \leqslant \left\|S(\widehat{T}f)\right\|_{L_p(hd\mu)} \\ &= \left\|\widehat{R}f\right\|_{L_p(h\,d\mu)} \leqslant M^{(p)}(X)\ \|f\|_{\mathrm{FBL}^{(p)}[E]}, \end{aligned}$$

as desired. □

We now consider the case $p = \infty$. By [74, Lemma 3], an ∞-convex Banach lattice X admits an equivalent norm (with equivalence constant equal to $M^{(\infty)}(X)$) under which it becomes an AM-space. In [185], it was shown that $\mathrm{FBL}^{(\infty)}[E]$ coincides with the closed sublattice generated by the point evaluations $\{\delta_x : x \in E\}$ in $C(B_{E^*})$. Here $C(B_{E^*})$ denotes the space of continuous functions on the dual ball of E, which is equipped with the relative w^*-topology. In particular, we have, per (6.1.3) above,

$$\|f\|_{\mathrm{FBL}^{(\infty)}[E]} = \sup\{|f(x^*)| : x^* \in E^*,\ \|x^*\| \leq 1\}.$$

In the case that E is finite dimensional, therefore, one can identify $\mathrm{FBL}^{(\infty)}[E]$ with either the space $C(S_{E^*})$ of continuous functions on the unit sphere of E^*, or the space $C_{ph}(B_{E^*})$ of continuous positively homogeneous functions on B_{E^*}. We now give an explicit description of $\mathrm{FBL}^{(\infty)}[E]$ - for general E - by showing that every positively homogeneous weak* continuous function on B_{E^*} lies in $\mathrm{FBL}^{(\infty)}[E]$:

Proposition 6.2.2. *Suppose E is a Banach space. Then $\mathrm{FBL}^{(\infty)}[E]$ coincides with the lattice $C_{ph}(B_{E^*})$ of positively homogeneous weak* continuous functions on B_{E^*}.*

Proof. We begin by reviewing the aforementioned identification of $\mathrm{FBL}^{(\infty)}[E]$ as a lattice of weak* continuous positively homogeneous functions on B_{E^*}, with the norm being the sup norm on the unit ball of E^*.

For this, recall that $\mathrm{FVL}[E]$ denotes the sublattice generated by $\{\delta_x\}_{x\in E}$ in $H[E]$. Since all the functions in this sublattice are positively homogeneous, we can, by restriction, identify this space with the sublattice of $\mathbb{R}^{B_{E^*}}$ generated by $\{\delta_x\}_{x\in E}$. It is clear that, under this identification, $\overline{\mathrm{FVL}[E]}^{\|\cdot\|_\infty} \subseteq C(B_{E^*})$, when B_{E^*} is equipped with the w^*-topology. We claim that the closed sublattice of $C(B_{E^*})$ generated by $\{\delta_x : x \in E\}$ (or, more specifically, $\{\|x\|_E \delta_{\frac{x}{\|x\|_E}} : x \in E\setminus\{0\}\}\cup\{0\}$, which will be our canonical copy of E) satisfies the universal property of $\mathrm{FBL}^{(\infty)}[E]$.

Indeed, let $T : E \to X$ be a bounded linear operator into an AM-space X, and assume without loss of generality that $\|T\| = 1$. We may view T as a map into X^{**}, and, since X^{**} is the dual of an AL-space, we can identify it lattice isometrically with $C(K)$ for some compact Hausdorff space K. As in the proof of Theorem 6.2.1, we can extend T to $\widehat{T} : \mathrm{FVL}[E] \to X^{**} = C(K)$ in a unique manner. It is clear that the range of $\widehat{T}$ is contained in X.

Fix $t_0 \in K$, let ϕ_{t_0} be the evaluation functional at t_0, and define $x^* = \phi_{t_0} \circ T$. Since $\|T\| = 1$, $x^* \in B_{E^*}$.

Let $f \in \mathrm{FVL}[E]$. Then $f = h(\delta_{x_1}, \dots, \delta_{x_m})$ for some $x_1, \dots, x_m \in E$ and some lattice-linear function h. By definition of the extension, $\widehat{T}f = h(Tx_1, \dots, Tx_m)$, which we can evaluate point-wise in $C(K)$ to get

$$\begin{aligned} |(\widehat{T}f)(t_0)| &= |h(Tx_1(t_0), \dots, Tx_m(t_0))| = |h(x^*(x_1), \dots, x^*(x_m))| \\ &= |h(\delta_{x_1}, \dots, \delta_{x_m})(x^*)| \le \|h(\delta_{x_1}, \dots, \delta_{x_m})\|_\infty. \end{aligned}$$

Since t_0 was arbitrary, $\|\widehat{T}f\|_X = \|\widehat{T}f\|_{C(K)} \le \|f\|_\infty$, so $\|\widehat{T}\| \le 1$. Hence, T extends uniquely to a norm one lattice homomorphism on $\overline{\mathrm{FVL}[E]}^{\|\cdot\|_\infty}$. This verifies the universal property of the free AM-space.

We will now show that $\overline{\mathrm{FVL}[E]}^{\|\cdot\|_\infty} = C_{ph}(B_{E^*})$. Let $\mathfrak{M}$ be the set of all triples (x^*, y^*, λ) where $x^*, y^* \in B_{E^*}$ and $0 \leqslant \lambda \leqslant 1$ are such that $f(x^*) = \lambda f(y^*)$ for all $f \in \overline{\mathrm{FVL}[E]}^{\|\cdot\|_\infty}$. By [195, Theorem 3], $\overline{\mathrm{FVL}[E]}^{\|\cdot\|_\infty} = C(B_{E^*}; \mathfrak{M})$, where $C(B_{E^*}; \mathfrak{M})$ consists of all functions f in $C(B_{E^*})$ such that $f(x^*) = \lambda f(y^*)$ whenever $(x^*, y^*, \lambda) \in \mathfrak{M}$. If $(x^*, y^*, \lambda) \in \mathfrak{M}$ then $\delta_x(x^*) = \lambda\delta_x(y^*)$ for every $x \in E$, that is, $x^*(x) = \lambda y^*(x)$ and, therefore, $x^* = \lambda y^*$. Since $\overline{\mathrm{FVL}[E]}^{\|\cdot\|_\infty}$ consists of positively homogeneous functions, we have $(x^*, y^*, \lambda) \in \mathfrak{M}$ if and only if $x^* = \lambda y^*$. It follows that $C(B_{E^*}; \mathfrak{M}) = C_{ph}(B_{E^*})$. □

Remark 6.2.3. Along similar lines, it is easy to check that $C(B_{E^*})$ together with the map $\phi_E(x) = \delta_x$ define the free $C(K)$-space (or free unital AM-space) generated by E (see [185, Theorem 5.4]).

Theorem 6.2.4. *Let E be a Banach space. For every compact Hausdorff space K and every norm one operator $T\colon E \to C(K)$, there exists a unique lattice homomorphism $\widehat{T}\colon C(B_{E^*}) \to C(K)$ such that $\widehat{T} \circ \phi_E = T$ and $\widehat{T}\mathbb{1} = \mathbb{1}$, where $\mathbb{1}$ denotes the constant function 1. Moreover, $\widehat{T}$ is an algebra homomorphism with $\big\|\widehat{T}\big\| = 1$.*

Proof. Since $\|T\| = 1$, the map $t \mapsto \eta_t \circ T$, where η_t is the evaluation functional at t, maps K into B_{E^*}, and it is continuous with respect to the relative weak* topology on B_{E^*}, so we can define a map $\widehat{T}\colon C(B_{E^*}) \to C(K)$ by $\widehat{T}(f)(t) = f(\eta_t \circ T)$ for $f \in C(B_{E^*})$ and $t \in K$. Since the algebraic and lattice operations in both $C(B_{E^*})$ and $C(K)$ are defined pointwise, it is easy to check that $\widehat{T}$ is a lattice and algebra homomorphism with $\widehat{T}\mathbb{1} = \mathbb{1}$ (see also [244, Theorem 3.2.12] for a more global picture of these maps). Moreover, we have

$$\big(\widehat{T} \circ \phi_E\big)(x)(t) = \delta_x(\eta_t \circ T) = (\eta_t \circ T)(x) = (Tx)(t)$$

for every $x \in E$ and $t \in K$, so that $\widehat{T} \circ \phi_E = T$. This implies in particular that $\big\|\widehat{T}\big\| \geqslant \|T\| = 1$. On the other hand, $\big|(\widehat{T}f)(t)\big| = \big|f(\eta_t \circ T)\big| \leqslant \|f\|_\infty$ for every $t \in K$ and $f \in C(B_{E^*})$, so that $\big\|\widehat{T}f\big\|_\infty \leqslant \|f\|_\infty$, and therefore $\big\|\widehat{T}\big\| = 1$.

Finally, to prove uniqueness, suppose that $U\colon C(B_{E^*}) \to C(K)$ is any lattice homomorphism satisfying $U \circ \phi_E = T$ and $U\mathbb{1} = \mathbb{1}$. Then $\widehat{T}$ and U agree on the sublattice of $C(B_{E^*})$ generated by $\{\delta_x : x \in E\} \cup \{\mathbb{1}\}$. The Stone–Weierstrass Theorem implies that this sublattice is dense in $C(B_{E^*})$, and therefore, being bounded, $\widehat{T}$ and U are equal. □

Remark 6.2.5. The universal property of $\mathrm{FBL}^{(p)}[E]$ can, in a sense, be extended. Indeed, recall that an operator $T : E \to X$ from a Banach space E to a Banach lattice X is called

p-convex ($1 \leq p \leq \infty$) if there is a constant M such that

$$\left\| \left(\sum_{k=1}^{n} |Tx_k|^p \right)^{\frac{1}{p}} \right\| \leq M \left(\sum_{k=1}^{n} \|x_k\|^p \right)^{\frac{1}{p}} \tag{6.2.3}$$

for every choice of vectors $(x_k)_{k=1}^n$ in E. In [287, Theorem 3] it was shown that an operator $T : E \to X$ is p-convex if and only if it strongly factors through a p-convex Banach lattice, i.e., there exists a p-convex Banach lattice Z, a lattice homomorphism $\varphi : Z \to X$ and a linear operator $R : E \to Z$ such that $T = \varphi R$. Using this fact, we see that an operator $T : E \to X$ is p-convex if and only if it strongly factors through $\mathrm{FBL}^{(p)}[E]$. In this case, one can choose the first operator $E \to \mathrm{FBL}^{(p)}[E]$ in the factorization to be the canonical embedding, and then the induced lattice homomorphism $\mathrm{FBL}^{(p)}[E] \to X$ is unique as $\phi_E(E)$ generates $\mathrm{FBL}^{(p)}[E]$ as a Banach lattice.

Remark 6.2.6. The above results allow one to identify elements of $\mathrm{FBL}^{(p)}[E]$ as functions on E^*. By positive homogeneity, such functions will be continuous on bounded subsets of E^*, in the weak* topology. Moreover, if E is finite dimensional, then norm and weak* convergence coincide, so every weak* convergent net in E^* is eventually bounded. This implies that elements of $\mathrm{FBL}^{(p)}[E]$ are weak* continuous on the whole of E^*.

Although the elements in $\mathrm{FVL}[E]$ are weak* continuous on the whole of E^*, by contrast, if E is infinite dimensional, then there exist $\phi \in \mathrm{FBL}[E]$ (hence also $\phi \in \mathrm{FBL}^{(p)}[E]$ for any p) which are not weak* continuous on E^*. To this end, find a normalized basic sequence $(x_k) \subseteq E$, and let $\phi = \sum_{k=1}^{\infty} 2^{-k}|\delta_{x_k}|$. We will construct an unbounded net $(z_\alpha^*) \subseteq E^*$ so that w*-$\lim_\alpha z_\alpha^* = 0$, yet $\inf_\alpha |\phi(z_\alpha^*)| \geq 1$. Indeed, let $\mathcal{A}$ be the set of all finite subsets of E, ordered by inclusion. For each $\alpha = (a_1, \dots, a_n) \in \mathcal{A}$, let k_α be the smallest k for which $x_k \notin \operatorname{span}[a_1, \dots, a_n]$. Find $z_\alpha^* \in E^*$ so that $z_\alpha^*(a_j) = 0$ for $1 \leq j \leq n$, and $z_\alpha^*(x_{k_\alpha}) = 2^{k_\alpha}$. The net (z_α^*) has the desired properties.

An issue similar to the above occurs in [269, Lemma 5.1 and Example 5.2]. Our explicit function space representation allows us to mostly bypass this technicality. On the other hand, we note that there *is* a topology on the whole of E^* that encodes the continuity of elements of $\mathrm{FBL}^{(p)}[E]$. More precisely, we leave it as an exercise to show that if f is in $\mathrm{FBL}^{(p)}[E]$, then it is continuous as a map $f : (E^*, bw^*) \to \mathbb{R}$, where bw^* is the bounded weak*-topology. Here, the bounded weak*-topology is the topology on E^* for which a set C is closed if and only if $C \cap A$ is w^*-closed in A whenever A is a norm bounded subset of E^*

(by [93, p. 49], it suffices to take A to be the closed unit ball). The bw^*-topology is in many ways similar to the w^*-topology, but it is also more subtle. See [93, Chapter 2] for a study of this topology.

Next, we mention two other features of the above construction of $\mathrm{FBL}^{(p)}[E]$. The first notes that one cannot restrict to extreme points of the ball to evaluate the $\mathrm{FBL}^{(\infty)}$-norm. The latter notes that the $\mathrm{FBL}^{(p)}$ norms are "nested" on $\mathrm{FVL}[E]$; the ability to compare these norms will be useful in various circumstances.

Remark 6.2.7. In (6.1.3), we can restrict the supremum of $x^* \in B_{E^*}$ to x^* in the unit sphere (this is due to homogeneity). However, we cannot restrict our attention to extreme points of the unit ball. For instance, let E be the space c_0, equipped with the equivalent norm $\|(x_1, x_2, \ldots)\| = \max_n \left\{|x_{2n-1}| + |x_{2n}|\right\}$. In other words, we have $E = c_0(\ell_1^2)$, which implies that $E^* = \ell_1(\ell_\infty^2)$, and hence the extreme points of the unit ball of E^* are of the form $(0, \ldots, 0, \pm 1, \pm 1, 0, \ldots)$. Here, the sequence starts with $2n$ zeros, $n \in \mathbb{N} \cup \{0\}$. Now denote by (e_k) the canonical basis of E (or c_0). Let $f = \left|\delta_{e_1}\right| - \left|\delta_{e_2}\right|$. Clearly, $\|f\|_{\mathrm{FBL}^{(\infty)}[E]} = 1$. However, if x^* is an extreme point of the unit ball of E^*, then $f(x^*) = 0$.

Remark 6.2.8. On $\mathrm{FVL}[E]$ all the $\mathrm{FBL}^{(p)}[E]$-norms can be evaluated. It is easy to see that the FBL-norm is the greatest, and the $\mathrm{FBL}^{(\infty)}$-norm is the smallest (to confirm this, note that, for $p < q$, $\mathrm{FBL}^{(q)}[E]$ is p-convex, hence the canonical embedding $\phi^{(q)} : E \to \mathrm{FBL}^{(q)}[E]$ extends to a contractive lattice homomorphism $\widehat{\phi^{(q)}} : \mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(q)}[E]$). This observation will be useful for describing the behaviour of the moduli of sequences in the various free spaces. Indeed, suppose (x_k) is a sequence in E, so that $(|\delta_{x_k}|)$ is equivalent to the unit vector basis of ℓ_1 when viewed in $\mathrm{FBL}^{(\infty)}[E]$. Then $(|\delta_{x_k}|)$ is equivalent to the unit vector basis of ℓ_1 no matter which $\mathrm{FBL}^{(p)}[E]$ we view it in.

We will see next that $\mathrm{FVL}[E]$ is always order dense in $\mathrm{FBL}^{(p)}[E]$. As was essentially shown in the first part of the proof of Theorem 6.2.1, $\mathrm{FVL}[E]$ has the universal property of being the free (Archimedean) vector lattice generated by the vector space E. Namely, every linear map $T : E \to X$ to an (Archimedean) vector lattice X extends uniquely to a lattice homomorphism $\widehat{T} : \mathrm{FVL}[E] \to X$ such that $\widehat{T}\delta_x = Tx$ for all $x \in E$. In other words, $\mathrm{FBL}^{(p)}[E]$ is simply the completion of the free vector lattice $\mathrm{FVL}[E]$ over E, under the maximal lattice norm with p-convexity constant 1, which agrees with the norm of E on the span of the generators; see [185]. Note that in this construction we are viewing E as a vector space. If A is a Hamel basis of E, then $\mathrm{FVL}[E]$ can be identified with $\mathrm{FVL}(A)$ (the free vector lattice over the set A, as constructed in [269, Section 3]).

Theorem 6.2.9. $\mathrm{FVL}[E]$ *is order dense in* $\mathrm{FBL}^{(p)}[E]$.

Recall that a sublattice A is *order dense* in a vector lattice Z if for any $z \in Z_+\backslash\{0\}$ there exists $a \in A\backslash\{0\}$ so that $0 \leq a \leq z$. As explained in [11, Section 5.3], a normed lattice is order dense in its norm completion if and only if it is regular in this completion. Moreover, this property admits an intrinsic characterization known as the *pseudo σ-Lebesgue property.* In [208], an order complete normed lattice X was constructed in such a way that its norm completion $\widehat{X}$ fails to be σ-order complete. It follows from [11, Theorem 5.32] that the inclusion $X \subseteq \widehat{X}$ cannot be order dense.

The proof of Theorem 6.2.9 requires the following:

Lemma 6.2.10. *If F is finite-dimensional, then for any open cone $C \subseteq F^*$, any $y_0^* \in C$, and any $\varepsilon > 0$, there exists $g \in \mathrm{FVL}[F]_+$ such that $g(y_0^*) > 0$, $g \leq \varepsilon$ on $B_{F^*} \cap C$, and g vanishes outside C.*

Proof. By renorming, we can, and do, assume that $F = \ell_1^n$. We represent elements of $\mathrm{FVL}[F]$ as piecewise affine functions on $F^* = \ell_\infty^n$; further, it suffices to consider the restrictions of such functions on the unit sphere.

Note that the function $s(t_1, \dots, t_n) = |t_1| \vee \dots \vee |t_n|$ is in $\mathrm{FVL}[F]$ and its restriction the unit sphere $S_{\ell_\infty^n}$ is $\mathbb{1}$. Without loss of generality, $y_0^* \in S_{\ell_\infty^n}$. Restricting functions in $C_{ph}(B_{\ell_\infty^n})$ to $S_{\ell_\infty^n}$, we may identify $C_{ph}(B_{\ell_\infty^n})$ with $C(S_{\ell_\infty^n})$; $\mathrm{FVL}[F]$ then becomes a dense sublattice of $C(S_{\ell_\infty^n})$ containing $\mathbb{1}$. Note that $C \cap S_{\ell_\infty^n}$ is an open subset of $S_{\ell_\infty^n}$ containing y_0^*. By Urysohn's Lemma, we can find $v \in C(S_{\ell_\infty^n})$ such that $0 \leqslant v \leqslant \mathbb{1}$, $v(y_0^*) = 1$, and v vanishes outside $C \cap S_{\ell_\infty^n}$. Since $\mathrm{FVL}[F]$ is dense in $C(S_{\ell_\infty^n})$, there exists $u \in \mathrm{FVL}[F]$ such that $\|v - u\|_{C(S_{\ell_\infty^n})} < \frac{1}{3}$. Put $w = (u - \frac{1}{3}\mathbb{1})^+$; then $w \in \mathrm{FVL}[F]$, w vanishes outside $C \cap S_{\ell_\infty^n}$, and $w(y_0^*) \geqslant \frac{1}{3} \neq 0$. Now put $g = \varepsilon w$ and extend g to $B_{\ell_\infty^n}$ by homogeneity; it is clear that g satisfies the required conditions. $\square$

Proof of Theorem 6.2.9. Since $\mathrm{FVL}[E] \subseteq \mathrm{FBL}^{(p)}[E] \subseteq \mathrm{FBL}^{(\infty)}[E]$, it suffices to prove the theorem for $p = \infty$; recall $\mathrm{FBL}^{(\infty)}[E] = C_{ph}(B_{E^*})$ (by Proposition 6.2.2). Take a non-zero $f \in \mathrm{FBL}^{(\infty)}[E]_+$; our goal is to show the existence of $h \in \mathrm{FVL}[E]\backslash\{0\}$ with $0 \leq h \leq f$.

Since $f \neq 0$, there exists $0 \neq x_0^* \in B_{E^*}$ with $f(x_0^*) > 0$. Hence there exists $\varepsilon > 0$ and a weak* open neighbourhood U of x_0^* in E^* such that f is greater than ε on $U \cap B_{E^*}$. Furthermore, we may assume that there exist $x_1, \dots, x_n \in E$ such that $x^* \in U$ if and only if

$\big|x^*(x_i)-x_0^*(x_i)\big| < 1$ for all $i = 1, \dots, n$. Since $x_0^* \neq 0$, by adding an extra point, if necessary, we may also assume that $x_0^*(x_i) \neq 0$ for some i.

Let F be the subspace of E spanned by $x_1, \dots, x_n$; let $\iota\colon F \hookrightarrow E$ be the inclusion map. Put $y_0^* = \iota^* x_0^* \in B_{F^*}$ and $V = \iota^*(U)$. Note that $y^* \in V$ if and only if $\big|y^*(x_i) - y_0^*(x_i)\big| < 1$ for all $i = 1, \dots, n$. Hence, V is a (weak*) open neighbourhood of y_0^* in F^*. We write $\mathrm{cone}(U) = \bigcup_{\lambda>0} \lambda U$. Clearly, $\mathrm{cone}(V) = \iota^*\big(\mathrm{cone}(U)\big)$.

By Lemma 6.2.10, there exists $g \in \mathrm{FVL}[F]_+$ which vanishes on the complement of $\mathrm{cone}(V)$, and satisfies $0 \leq g \leq \varepsilon$ on $B_{F^*} \cap \mathrm{cone}(V)$, as well as $g(y_0^*) > 0$. This g may be written as a lattice-linear expression of $\delta_{x_1}, \dots, \delta_{x_n}$. Let h be the same lattice-linear expression of $\delta_{x_1}, \dots, \delta_{x_n}$ in $\mathrm{FVL}[E]$. Then $h(x^*) = g(\iota^* x^*)$ for every $x^* \in E^*$. It follows that $h \geqslant 0$ and $h(x_0^*) = g(y_0^*) > 0$, hence $h \neq 0$.

We claim that $h \leqslant f$. Fix $x^* \in B_{E^*}$; we need to show that $h(x^*) \leqslant f(x^*)$. If $x^* \in U$ then $\iota^* x^* \in V \subseteq \mathrm{cone}(V)$, hence $h(x^*) = g(\iota^* x^*) \leq \varepsilon < f(x^*)$. Since both h and f are positively homogeneous, it follows that $h(x^*) \leqslant f(x^*)$ whenever $x^* \in \mathrm{cone}(U)$. On the other hand, if $x^* \notin \mathrm{cone}(U)$ then $\iota^* x^* \notin \mathrm{cone}(V)$ and, therefore, $h(x^*) = g(\iota^* x^*) = 0$. In either case, $h(x^*) \leqslant f(x^*)$. □

Theorem 6.2.9 allows us to recover an important result from [25]:

Corollary 6.2.11. *For* $1 \leq p \leq \infty$, *every disjoint collection of elements of* $\mathrm{FBL}^{(p)}[E]$ *is at most countable.*

Proof. Let (x_α) be a collection of pairwise disjoint elements of $\mathrm{FBL}^{(p)}[E]$. Without loss of generality, all the elements x_α are positive and non-zero. Use Theorem 6.2.9 to find $0 < y_\alpha \leq x_\alpha$, with $y_\alpha \in \mathrm{FVL}[E]$. Now, identify $\mathrm{FVL}[E]$ with $\mathrm{FVL}(A)$, where A is a Hamel basis of E, and $\mathrm{FVL}(A)$ is the free vector lattice over the *set* A. Finally, use the classical fact that pairwise disjoint collections in $\mathrm{FVL}(A)$ are at most countable. See, for example, [40, Theorem 2.5]. □

We conclude this section by noting some elementary facts about $\mathrm{FBL}^{(p)}[E]$. As mentioned, most of the literature on $\mathrm{FBL}[E]$ discussed in Section 6.1 generalizes to $\mathrm{FBL}^{(p)}[E]$ with relative ease, so we only collect here three of the most basic facts. Indeed, the following can be proved exactly as in [269, Section 6] by looking at the non-vanishing sets of weak* continuous functions on B_{E^*}:

Proposition 6.2.12. *Let E be a Banach space.*

(i) For every $x \in E$, $x \neq 0$, $|\delta_x|$ is a weak order unit in $\mathrm{FBL}^{(p)}[E]$.

(ii) If E has dimension strictly greater than one, then the only projection bands in $\mathrm{FBL}^{(p)}[E]$ *are $\{0\}$ and* $\mathrm{FBL}^{(p)}[E]$.

(iii) If E has dimension strictly greater than one, then $\mathrm{FBL}^{(p)}[E]$ *is not σ-order complete and contains no atoms.*

6.3 Properties of the extended operator: Injectivity, surjectivity, regularity, and the subspace problem

Recall that the universal property of $\mathrm{FBL}^{(p)}$ yields, in particular, that every bounded linear operator $T : F \to E$ between Banach spaces extends uniquely to a lattice homomorphism $\overline{T} : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ making the following diagram commute:

$$\begin{tikzcd}
\mathrm{FBL}^{(p)}[F] \arrow[r, "\overline{T}"] & \mathrm{FBL}^{(p)}[E] \\
F \arrow[u, "\phi_F"'] \arrow[r, "T"] & E \arrow[u, "\phi_E"']
\end{tikzcd}$$

Here ϕ_E and ϕ_F denote the canonical isometric embeddings, and $\|\overline{T}\| = \|T\|$ (simply consider the map $\phi_E T : F \to \mathrm{FBL}^{(p)}[E]$ and set $\overline{T} = \widehat{\phi_E T}$). It is easy to check that given operators $S : F \to G$ and $T : E \to F$, we have $\overline{S \circ T} = \overline{S} \circ \overline{T}$. In particular, if T is an isomorphism between Banach spaces E and F, then $\overline{T}$ is a lattice isomorphism between $\mathrm{FBL}^{(p)}[E]$ and $\mathrm{FBL}^{(p)}[F]$.

The goal of this section is to relate properties of T with properties of $\overline{T}$. More specifically, we will discover exactly when $\overline{T}$ is injective, surjective, a quotient map, etc. We will also study when $\overline{T}$ is an embedding, and when it is order continuous.

The following observation will be useful for our purposes:

Lemma 6.3.1. *Given $T : F \to E$, the extension $\overline{T} : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ is given, for $f \in \mathrm{FBL}^{(p)}[F]$, by*

$$T(f) = f \circ T^*.$$

Proof. Consider the composition operator induced by T^*: $C_{T^*}f(x^*) = f(T^*x^*)$ for $f \in \mathrm{FBL}^{(p)}[F]$ and $x^* \in E^*$. It is straightforward to check that $C_{T^*} : \mathrm{FBL}^{(p)}[F] \to H_p[E]$ is a well-defined lattice homomorphism. Moreover, $C_{T^*}\delta_x = \delta_{Tx}$ for $x \in F$, which implies that the range of C_{T^*} is actually contained in $\mathrm{FBL}^{(p)}[E]$. Because of uniqueness of extension, we must have $\overline{T} = C_{T^*}$. □

Characterizations of injectivity, surjectivity and density of the range

We begin with some simple observations on injectivity and surjectivity of the extended operator $\overline{T}$. We thank A. Avilés for sharing with us an argument leading to the characterization of surjectivity.

Proposition 6.3.2. *Let $T : F \to E$ be a bounded linear operator and let $\overline{T} : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ be its unique extension to a lattice homomorphism given above. Then*

(i) T is injective if and only if $\overline{T}$ is injective.

(ii) T has dense range if and only if $\overline{T}$ has dense range.

(iii) T is onto if and only if $\overline{T}$ is onto.

Proof. (i) Suppose T is injective. Let $T^* : E^* \to F^*$ be the adjoint operator; it is easy to check that its range $T^*(E^*)$ is weak* dense in F^* (cf. [111, Theorem 3.18]). Also, by Lemma 6.3.1, for $f \in \mathrm{FBL}^{(p)}[F]$, we can write $\overline{T}f = f \circ T^*$. Suppose $\overline{T}f = 0$ for some $f \in \mathrm{FBL}^{(p)}[F] \setminus \{0\}$. Since $\overline{T}$ is a lattice homomorphism, we can suppose without loss of generality that $f \in \mathrm{FBL}^{(p)}[F]_+$. By Theorem 6.2.9, we can find $g \in \mathrm{FVL}[F]$ such that $0 < g \leq f$, which by positivity also satisfies $\overline{T}g = 0$. It follows that $g(T^*y^*) = 0$ for every $y^* \in E^*$, and by weak* continuity of g we must have $g = 0$. Thus, $\overline{T}$ is injective. The converse is clear.

(ii) If T has dense range then for every $y \in E$ and $\varepsilon > 0$ there exists $x \in F$ with $\varepsilon > \|Tx - y\| = \left\|\overline{T}\delta_x - \delta_y\right\|$, hence $\overline{\mathrm{Range}\,\overline{T}}$ contains δ_y. Since $\overline{T}$ is a lattice homomorphism, $\overline{\mathrm{Range}\,\overline{T}}$ is a closed sublattice; it follows that $\overline{\mathrm{Range}\,\overline{T}} = \mathrm{FBL}^{(p)}[E]$. Suppose now that $\mathrm{Range}\,T$ is not dense. There exists $0 \neq y^* \in E^*$ which vanishes on it. Then the map $\widehat{y^*} \in \mathrm{FBL}^{(p)}[E]^*$ given by $\widehat{y^*}(g) = g(y^*)$ vanishes on $\overline{T}\delta_x$ for every $x \in F$. Since $\widehat{y^*}$ is a lattice homomorphism, it vanishes on Range $\overline{T}$; hence the range of $\overline{T}$ is not dense.

(iii) Suppose T is onto. Let $Z = \mathrm{FBL}^{(p)}[F]/\ker\overline{T}$ and let $Q\colon \mathrm{FBL}^{(p)}[F] \to Z$ be the canonical quotient map. Since $\overline{T}$ is a lattice homomorphism, $\ker\overline{T}$ is an ideal, hence Q is a lattice homomorphism, and, therefore, Z is a p-convex Banach lattice. There exists an injective operator $S\colon Z \to \mathrm{FBL}^{(p)}[E]$ such that $\overline{T} = SQ$. Since $\overline{T}$ and Q are lattice homomorphisms, so is S. Indeed, fix $z \in Z$. By the surjectivity of Q, we can find $x \in \mathrm{FBL}^{(p)}[F]$ such that $Qx = z$. Then

$$S|z| = S|Qx| = SQ|x| = \overline{T}|x| = |\overline{T}x| = |SQx| = |Sz|.$$

Since $\ker T \subseteq \ker Q\phi_F$, there exists an operator $R\colon E \to Z$ such that $Q\phi_F = RT$. Let $\widehat{R}\colon \mathrm{FBL}^{(p)}[E] \to Z$ be the canonical extension of R. Let $y \in E$. Pick $x \in F$ such that $y = Tx$. Then

$$S\widehat{R}\phi_E y = SRy = SRTx = SQ\phi_F x = \overline{T}\phi_F x = \phi_E y.$$

It follows that $S\widehat{R}$ is the identity on the range of ϕ_E and, therefore, on the sublattice generated by it. Since this sublattice is dense in $\mathrm{FBL}^{(p)}[E]$, $S\widehat{R}$ is the identity on $\mathrm{FBL}^{(p)}[E]$. It follows that S is surjective and, therefore, so is $\overline{T} = SQ$.

Conversely, suppose now that $\overline{T}$ is onto. Let $Q : F \to F/\ker T$ denote the canonical quotient map and let $S : F/\ker T \to E$ be the injective operator induced by T: $S(x + \ker T) = Tx$, for $x \in F$. Thus, we have $T = SQ$. Let us consider the corresponding lattice homomorphisms $\overline{Q} : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[F/\ker T]$, $\overline{S} : \mathrm{FBL}^{(p)}[F/\ker T] \to \mathrm{FBL}^{(p)}[E]$, which in particular satisfy $\overline{T} = \overline{S}\,\overline{Q}$. Note that since $\overline{T}$ is onto, so is $\overline{S}$. Moreover, as S is injective, by part (i) it follows that $\overline{S}$ is also injective. Hence, $\overline{S}$ is an isomorphism. In particular, it follows that S is bounded below and has closed range. But, by part (ii), it follows that S has dense range, thus S is onto. By construction of S, it follows that T must be onto as well. $\square$

In Section 6.3, we will study when $\overline{T}$ is an embedding. Unlike with injectivity, surjectivity, density of the range and being a quotient map, it is not true that T is an embedding if and only if $\overline{T}$ is an embedding. In fact, $\overline{T}$ is an embedding if and only if T is an embedding, and one can uniformly factor maps into ℓ_p^n through T. This will be made precise - and quantitative - in Theorem 6.3.7.

We also note the following form of "restricted projectivity" for $\mathrm{FBL}^{(p)}[E]$. The proof is essentially as in [185, Proposition 4.9], but on an operator-by-operator basis:

Proposition 6.3.3. *Let E be a Banach space, X a p-convex Banach lattice, J a closed ideal of X, $Q : X \to X/J$ the quotient map, and $T : E \to X/J$ an operator. Then*

(i) If $T : E \to X/J$ admits a lift to $\widetilde{T} : E \to X$ then $\widehat{\widetilde{T}} : \mathrm{FBL}^{(p)}[E] \to X$ is a lattice homomorphic lift of $\widehat{T} : \mathrm{FBL}^{(p)}[E] \to X/J$;

(ii) If $\widehat{T} : \mathrm{FBL}^{(p)}[E] \to X/J$ admits a linear lift $S := \widetilde{\widehat{T}} : \mathrm{FBL}^{(p)}[E] \to X$ then $S \circ \phi_E : E \to X$ is a lifting of T;

(iii) If the identity $I : X/J \to X/J$ admits a linear lift $\widetilde{I} : X/J \to X$ then for any lattice homomorphism $S : \mathrm{FBL}^{(p)}[E] \to X/J$, the canonical extension of $\widetilde{I} \circ S \circ \phi_E : E \to X$ to $\mathrm{FBL}^{(p)}[E]$ is a lattice homomorphic lifting of S.

Proof. Argue by diagram chasing. $\square$

Regularity of the inclusion

In this section, we prove that if F is a subspace of E, the canonical inclusion $\bar{\iota} : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ is order continuous (that is, if (f_α) is a decreasing net in $\mathrm{FBL}^{(p)}[F]$, whose infimum is 0, then the same is true for $(\bar{\iota} f_\alpha)$). This happens regardless of whether $\bar{\iota}$ is an embedding in its own right.

To set notation, throughout this subsection we equip B_{E^*} with its relative weak* topology. We let F be a closed subspace of E and $\iota\colon F \hookrightarrow E$ the canonical embedding. Then $\iota^*\colon E^* \to F^*$ is the restriction map: $\iota^* x^* = x^*_{|F}$, and $\bar{\iota}\colon \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ is injective.

Recall from the construction of $\mathrm{FBL}^{(p)}[E]$ that we defined $\mathrm{FVL}[E]$ to be the (non-closed) sublattice generated by $\{\delta_x\}_{x\in E}$ in $H[E]$. $\mathrm{FBL}^{(p)}[E]$ was then constructed as the closure of $\mathrm{FVL}[E]$ in $H_p[E]$.

Theorem 6.3.4. *Let F be a closed subspace of E; let $\iota\colon F \hookrightarrow E$ be the inclusion map. Then $\bar{\iota}\colon \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ is order continuous. That is, $\mathrm{FBL}^{(p)}[F]$ is a regular (not necessarily closed) sublattice of $\mathrm{FBL}^{(p)}[E]$.*

Proof. First we consider the case when F is complemented in E. In this case, the argument is an adaptation of [269, Proposition 5.9], which proves that $\mathrm{FBL}[\ell_1(B)]$ is regularly embedded in $\mathrm{FBL}[\ell_1(A)]$ whenever $B \subseteq A$. Let $P : E \to F$ denote a projection (so that $P\iota = id_F$).

To this end, let $f_\alpha \downarrow 0$ in $\mathrm{FBL}^{(p)}[F]$, and suppose $g \in \mathrm{FBL}^{(p)}[E]$ satisfies $0 < g \le \bar{\iota} f_\alpha$ for every α. Let $x_0^* \in E^*$ be such that $g(x_0^*) > 0$. It follows from $0 < g(x_0^*) \leqslant \bar{\iota} f_\alpha(x_0^*) = f_\alpha(\iota^* x_0^*)$ that $\iota^* x_0^* \neq 0$. We may assume without loss of generality that $\|\iota^* x_0^*\| = 1$. Pick any $y_0 \in F$ with $(\iota^* x_0^*)(y_0) = 1$. Put $z_0^* = x_0^* - P^* \iota^* x_0^*$. Note that $\iota^* P^* = id_{F^*}$, hence $\iota^* z_0^* = 0$.

Consider the operator $T : E \to \mathrm{FBL}^{(p)}[F]$ given by $Tx = \delta_{Px} + z_0^*(x)\,|\delta_{y_0}|$. Being a rank one perturbation of $\phi_F \circ P$, T is bounded, and, therefore, extends to a lattice homomorphism $\widehat{T} \colon \mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$. Put $h = \widehat{T} g$.

For every $y^* \in F^*$ and $x \in E$, we have

$$(\widehat{y^*} \circ \widehat{T})(\delta_x) = (\widehat{T}\delta_x)(y^*) = y^*(Px) + z_0^*(x)\,|y^*(y_0)| = \delta_x\big(\varphi(y^*)\big) = \widehat{\varphi(y^*)}(\delta_x),$$

where $\varphi(y^*) = P^* y^* + |y^*(y_0)|\, z_0^*$. The lattice homomorphisms $\widehat{y^*} \circ \widehat{T}$ and $\widehat{\varphi(y^*)}$ agree on every δ_x, hence they are equal. It follows that

$$h(y^*) = (\widehat{T} g)(y^*) = (\widehat{y^*} \circ \widehat{T})(g) = \widehat{\varphi(y^*)}(g) = g\big(\varphi(y^*)\big)$$

for every $y^* \in F^*$. This yields

$$h(y^*) \le \bar{\iota} f_\alpha\big(\varphi(y^*)\big) = f_\alpha\big(\iota^* \varphi(y^*)\big) = f_\alpha\big(\iota^* P^* y^* + |y^*(y_0)|\, \iota^* z_0^*\big) = f_\alpha(y^*)$$

for every $y^* \in F^*$ and every α because $\iota^* P^* = id_{F^*}$. Therefore, $0 \le h \le f_\alpha$ for every α, which yields $h = 0$. It follows from $\varphi(\iota^* x_0^*) = x_0^*$ that $g(x_0^*) = h(\iota^* x_0^*) = 0$. This contradiction proves the statement in the case when F is a complemented subspace of E.

We now proceed to the general case. For this, suppose that $f_\alpha \downarrow 0$ in $\mathrm{FBL}^{(p)}[F]$ and there exists $g \in \mathrm{FBL}^{(p)}[E]$ such that $0 < g \leqslant \bar{\iota} f_\alpha$ for every α. Since $\mathrm{FVL}[E]$ is order dense in $\mathrm{FBL}^{(p)}[E]$ by Theorem 6.2.9, we may assume without loss of generality that $g \in \mathrm{FVL}[E]$. Then g is a lattice-linear combination of $\delta_{y_1}, \dots, \delta_{y_n}$ for some $y_1, \dots, y_n$ in E.

Let G be the closed subspace of E spanned by F and $y_1, \dots, y_n$. Let j and k be the inclusion maps: $F \xrightarrow{j} G \xrightarrow{k} E$. Clearly, $\iota = k \circ j$. Let $h \in \mathrm{FVL}[G]$ be defined by the same lattice-linear combination of $\delta_{y_1}, \dots, \delta_{y_n}$ as g but viewed as an element of $\mathrm{FVL}[G]$. For every $z^* \in B_{G^*}$, we can extend it to some $y^* \in B_{E^*}$; it follows that

$$h(z^*) = g(y^*) \leqslant \bar{\iota} f_\alpha(y^*) = f_\alpha(\iota^* y^*) = f_\alpha(j^* z^*) = \bar{j} f_\alpha(z^*)$$

for every α, where $\bar{j}\colon \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[G]$ is the canonical inclusion induced by j. It follows that $0 \leqslant h \leqslant \bar{j} f_\alpha$ in $\mathrm{FBL}^{(p)}[G]$ for every α. Since F is complemented in G, the special case yields $h = 0$. For every $x^* \in B_{E^*}$, we have $g(x^*) = h(k^* x^*) = 0$, so $g = 0$. $\square$

Remark 6.3.5. In the above theorem we assumed that F is a subspace of E, so that ι is an embedding. However, since $T : F \to E$ is injective if and only if $\overline{T} : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ is injective, to identify $\mathrm{FBL}^{(p)}[F]$ as a vector sublattice of $\mathrm{FBL}^{(p)}[E]$ only requires the injectivity of ι. Since regularity is a pure vector lattice property, one may think that injectivity of ι would be enough to ensure regularity of the inclusion $\bar{\iota} : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$. However, the above proof fails under this weaker assumption, and it remains an open problem to characterize those $T : F \to E$ such that $\overline{T} : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ is order continuous.

The embedding problem, and its connection to extensions of operators

A direct consequence of Proposition 6.3.2 is that if E is a Banach space quotient of F, then $\mathrm{FBL}^{(p)}[E]$ is a Banach lattice quotient of $\mathrm{FBL}^{(p)}[F]$. This partly motivates the question of whether the dual version of this fact also holds. To properly formulate this, note first that Proposition 6.3.2 also yields that if F is a (closed non-zero) subspace of E, then the canonical embedding $\iota : F \hookrightarrow E$ induces an injective lattice homomorphism $\bar{\iota} : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ of norm 1. In this section, we consider the *embedding problem*: Suppose F is a subspace of E. Does the canonical embedding $\iota : F \hookrightarrow E$ induce a lattice embedding $\bar{\iota} : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$?

For context, recall that an inclusion of metric spaces always induces an isometric embedding of the associated Lipschitz free spaces, cf. [129, Lemma 2.3]. As we will see, however, the situation for free Banach lattices is more subtle. Our main result is Theorem 6.3.7 which shows that $\bar{\iota}$ being a lattice embedding is equivalent to every operator $T : F \to L_p(\mu)$ having an extension to E. In particular, this reduces a problem about Banach lattices to a purely Banach space one. In the next section, this criterion will be combined with various Banach space techniques to provide several examples where $\bar{\iota}$ is an embedding, as well as several examples where it is not.

To reiterate our goal, we aim to explore under which conditions the (injective) map $\bar{\iota}$ defines an isomorphic embedding, so that we can consider $\mathrm{FBL}^{(p)}[F]$ as a closed sublattice of $\mathrm{FBL}^{(p)}[E]$ in a natural way. Equivalently, we ask whether $\bar{\iota}$ is bounded below – that is, whether there exists $C > 0$ so that any $f \in \mathrm{FBL}^{(p)}[F]$ satisfies $\|\bar{\iota}f\| \geq \|f\|/C$. Since $\bar{\iota}$ is norm one, this is equivalent to asking $\bar{\iota}$ to be a lattice C-isomorphic embedding.

Remark 6.3.6. As alluded to above, we only consider isometric embeddings $\iota : F \hookrightarrow E$ in this subsection. Nevertheless, the results add to Proposition 6.3.2 a characterization of when an operator $T : F \to E$ induces a lattice isomorphic embedding $\overline{T} : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$. Indeed, the restriction to isometric embeddings presents little loss in generality, as given an operator $T : F \to E$, one can factor it as $T = j_2 j_1$, where $j_1 : F \to (T(F), \|\cdot\|_E)$, and $j_2 : (T(F), \|\cdot\|_E) \to E$ is an isometric inclusion. If $\overline{T}$ is an embedding, then it is easy to see that T is as well. On the other hand, if T is an embedding then $\overline{T} = \overline{j_2} \circ \overline{j_1}$ is an embedding if and only if $\overline{j_2}$ is. Thus, $\overline{T}$ is an embedding if and only if both T and $\overline{j_2}$ are. It therefore suffices to understand the map $\overline{j_2}$, which is the extension of the isometric mapping j_2.

We now reduce the problem of whether an embedding $\iota : F \hookrightarrow E$ induces a lattice embedding $\bar{\iota} : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ to a certain Banach space question involving extensions of operators:

Theorem 6.3.7. *Let $\iota : F \hookrightarrow E$ be an isometric embedding and $C > 0$. The following are equivalent:*

(i) $\bar{\iota} : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ is a lattice C-isomorphic embedding;

(ii) For every σ-finite measure μ, any $T\colon F \to L_p(\mu)$ extends to $\widetilde{T}\colon E \to L_p(\mu)$, with $\left\|\widetilde{T}\right\| \leq C\,\|T\|$;

(iii) For every $n \in \mathbb{N}$ and $\varepsilon > 0$, any $T\colon F \to \ell_p^n$ extends to $\widetilde{T}\colon E \to \ell_p^n$, with $\left\|\widetilde{T}\right\| \leq C(1+\varepsilon)\,\|T\|$.

Proof. (ii)⇒(iii) is trivial.

(iii)⇒(i): Fix $\varepsilon > 0$. Since $\bar{\iota}$ is a lattice homomorphism with $\|\bar{\iota}\| = \|\iota\| = 1$, we immediately get that

$$\|\bar{\iota}f\|_{\mathrm{FBL}^{(p)}[E]} \leq \|f\|_{\mathrm{FBL}^{(p)}[F]},$$

for all $f \in \mathrm{FBL}^{(p)}[F]$.

Now, take f in $\mathrm{FVL}[F]$. Given $x_1^*, \ldots, x_n^* \in F^*$, we define $T : F \to \ell_p^n$ by $T(x) = (x_k^*(x))_{k=1}^n$. Recall that

$$\|T\| = \sup_{x \in B_F} \left(\sum_{k=1}^n |x_k^*(x)|^p \right)^{\frac{1}{p}}.$$

By hypothesis, there is an extension $\widetilde{T} : E \to \ell_p^n$ with $\|\widetilde{T}\| \le C(1+\varepsilon)\|T\|$. Let $y_1^*, \ldots, y_n^* \in E^*$ be such that $\widetilde{T}(x) = (y_k^*(x))_{k=1}^n$ for each $x \in E$, so that $\iota^* y_k^* = x_k^*$. It follows that $f(x_k^*) = f(\iota^* y_k^*) = \bar{\iota} f(y_k^*)$ for $k = 1, \ldots, n$. Therefore, we have

$$\left(\sum_{k=1}^n |f(x_k^*)|^p \right)^{\frac{1}{p}} = \left(\sum_{k=1}^n |\bar{\iota} f(y_k^*)|^p \right)^{\frac{1}{p}} \le \|\bar{\iota} f\|_{\mathrm{FBL}^{(p)}[E]} \sup_{x \in B_E} \left(\sum_{k=1}^n |y_k^*(x)|^p \right)^{\frac{1}{p}}$$

$$\le C(1+\varepsilon) \|\bar{\iota} f\|_{\mathrm{FBL}^{(p)}[E]} \sup_{x \in B_F} \left(\sum_{k=1}^n |x_k^*(x)|^p \right)^{\frac{1}{p}}.$$

Taking supremum over $x_1^*, \ldots, x_n^* \in F^*$, it follows that

$$\|f\|_{\mathrm{FBL}^{(p)}[F]} \le C(1+\varepsilon) \|\bar{\iota} f\|_{\mathrm{FBL}^{(p)}[E]}.$$

By density, this inequality holds for all $f \in \mathrm{FBL}^{(p)}[F]$. Now let ε tend to zero.

(i)⇒(ii): The case of $p = \infty$ follows from the injectivity of L_∞-spaces, so we restrict ourselves to $1 \le p < \infty$. Let $T : F \to L_p(\mu)$. By the properties of a free Banach lattice T extends to a lattice homomorphism $\widehat{T} : \mathrm{FBL}^{(p)}[F] \to L_p(\mu)$, with $\|\widehat{T}\| = \|T\|$. Let S be the inverse of $\bar{\iota}$, taking $\bar{\iota}(\mathrm{FBL}^{(p)}[F])$ back to $\mathrm{FBL}^{(p)}[F]$; clearly S is a lattice isomorphism, with $\|S\| \le C$.

By [231, Theorem 1.c.4], there exists a band projection from $L_p(\mu)^{**}$ onto $L_p(\mu)$. By [278, Theorem 4] (for $p = 1$, see also [235]), $\widehat{T}S$ extends to a regular operator $U : \mathrm{FBL}^{(p)}[E] \to L_p(\mu)$, with $\|U\| \le \|\widehat{T}S\| \le C\|T\|$. Now let $\widetilde{T} = U\phi_E$ (here, as before, $\phi_E : E \to \mathrm{FBL}^{(p)}[E]$ is the canonical embedding). Clearly $\|\widetilde{T}\| \le \|U\| \le C\|T\|$. Moreover,

$$\widetilde{T}\iota = U\phi_E\iota = U\bar{\iota}\phi_F = \widehat{T}S\bar{\iota}\phi_F,$$

and, since S is the one-sided inverse of $\bar{\iota}$,

$$\widetilde{T}\iota = \widehat{T}\phi_F = T.$$

In other words, $\widetilde{T}$ extends T. $\square$

Theorem 6.3.7 motivates the following definition:

Definition 6.3.8. Fix $p \in [1,\infty]$. We say that a pair (F,E) with F a subspace of E has the *POE-p* with constant C, or C-POE-p, if for every $n \in \mathbb{N}$, every operator $T : F \to \ell_p^n$ extends to $\widetilde{T} : E \to \ell_p^n$ with $\|\widetilde{T}\| \le C\|T\|$. Here POE-$p$ stands for "Property of operator extension into L_p". A Banach space F is said to have *POE-p* with constant C (or C-POE-p) if, for any space E containing F, (F,E) has the POE-p with constant C. If (F,E) (or F) has the POE-p for some C, then we shall simply say that (F,E) (resp. F) has the POE-p.

In these terms, Theorem 6.3.7 yields the following equivalent characterizations of the POE-p:

Proposition 6.3.9. *For $C \ge 1$, $p \in [1,\infty]$, and a subspace F of a Banach space E, the following are equivalent:*

(i) (F,E) has the C-POE-p;

(ii) For any σ-finite measure μ, any $T : F \to L_p(\mu)$ has an extension $\widetilde{T} : E \to L_p(\mu)$ with $\|\widetilde{T}\| \le C\|T\|$;

(iii) For any $n \in \mathbb{N}$ and $\varepsilon > 0$, any $T : F \to \ell_p^n$ has an extension $\widetilde{T} : E \to \ell_p^n$ with $\|\widetilde{T}\| \le C(1+\varepsilon)\|T\|$.

We note that the 1-injectivity of ℓ_∞^n implies:

Proposition 6.3.10. *Any Banach space F has the POE-∞, with constant 1. Consequently, if $\iota : F \hookrightarrow E$ is an isometric embedding, then the map $\bar{\iota} : \mathrm{FBL}^{(\infty)}[F] \to \mathrm{FBL}^{(\infty)}[E]$ is a lattice isometric embedding.*

The case of $1 \le p < \infty$ is more interesting, and upcoming (sub)sections will discuss criteria for determining whether a pair (F,E), or a space F, has the POE-p.

Remark 6.3.11. As was noted in [26, Corollary 2.8], if F is a complemented subspace of E, then $\bar{\iota} : \mathrm{FBL}[F] \to \mathrm{FBL}[E]$ is a lattice isomorphic embedding. This, of course, also follows immediately from Theorem 6.3.7. However, [26, Corollary 2.8] (and slight modifications of its proof) show a lot more: If $\iota : F \hookrightarrow E$ is an embedding, and P is a projection from E onto $\iota(F)$, then $\overline{P}$ defines a lattice homomorphic projection from $\mathrm{FBL}^{(p)}[E]$ onto $\bar{\iota}(\mathrm{FBL}^{(p)}[F])$. A partial converse also holds.

Proposition 6.3.12. *Suppose F is isomorphic to a complemented subspace of a p-convex Banach lattice, and $\iota : F \hookrightarrow E$ is an embedding such that the induced map $\bar{\iota} : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ is an embedding, and there is a projection P (which is not assumed to be a lattice projection) from $\mathrm{FBL}^{(p)}[E]$ onto $\bar{\iota}(\mathrm{FBL}^{(p)}[F])$. Then F is complemented in E.*

Proof. As F is isomorphic to a complemented subspace of a p-convex Banach lattice, there is a projection $Q : \mathrm{FBL}^{(p)}[F] \to \phi_F(F)$ by [185, Proposition 4.2]. Diagram chasing shows that $V := \phi_F^{-1} \circ Q \circ \bar{\iota}^{-1} \circ P \circ \phi_E : E \to F$ satisfies $I_F = V \circ \iota$. In other words, F is complemented in E. $\square$

As an example, Theorem 6.3.7 (see Corollary 6.4.12 for additional details) shows that the inclusion $\iota : c_0 \hookrightarrow \ell_\infty$ induces a lattice embedding of $\mathrm{FBL}^{(p)}[c_0]$ into $\mathrm{FBL}^{(p)}[\ell_\infty]$. However, c_0 is not complemented in ℓ_∞, hence $\mathrm{FBL}^{(p)}[c_0]$ cannot be complemented in $\mathrm{FBL}^{(p)}[\ell_\infty]$.

Examples of lattice structures on a subspace spanned by Rademacher functions

In the previous subsection, we reduced the embedding problem for free Banach lattices to a pure Banach space problem involving extensions of operators into $L_p(\mu)$. This perspective on the embedding problem will be further expanded on in Section 6.4. However, before that, we examine the embedding problem from a lattice point of view. More specifically, here we consider an embedding $\iota : F \hookrightarrow E$, and explicitly calculate the norms of certain elements of $\bar{\iota}(\mathrm{FBL}^{(p)}[F]) \subseteq \mathrm{FBL}^{(p)}[E]$. By discovering that, for certain $f \in \mathrm{FBL}^{(p)}[F]$, $\|f\|_{\mathrm{FBL}^{(p)}[F]}$ may be very different from $\|\bar{\iota} f\|_{\mathrm{FBL}^{(p)}[E]}$, we conclude that $\bar{\iota}$ is not bounded below.

We denote by Rad_q ($1 \le q \le \infty$) the span of independent Rademacher random variables in L_q; R_q shall stand for the corresponding embedding. Khintchine's inequality shows that for finite q, Rad_q is isomorphic to ℓ_2, and it is easy to verify that Rad_∞ can be identified with ℓ_1.

It is well known that, for $1 < q < \infty$, Rad_q is complemented in L_q, hence the pair (Rad_q, L_q) has the POE-p for any p. Below we examine the edge cases $q = 1, \infty$. In the next section, we will revisit this question from an extension of operators point of view and prove in Proposition 6.4.25 that (Rad_1, L_1) fails the POE-p, for any $1 \le p < \infty$, and in Proposition 6.4.19 that $(\mathrm{Rad}_\infty, L_\infty)$ has the POE-p if and only if $2 \le p \le \infty$. However, this section presents a direct proof, in order to illustrate the structure of free Banach lattices:

Example 6.3.13. $\overline{R}_1 : \mathrm{FBL}^{(p)}[\mathrm{Rad}_1] \to \mathrm{FBL}^{(p)}[L_1]$ is not a lattice isomorphic embedding for any $p \in [1, \infty)$.

Proof. Let (e_k) denote the unit vector basis of ℓ_2 and (r_k) the sequence of Rademacher functions. Define

$$R : \ell_2 \to L_1[0,1] : \sum_k a_k e_k \mapsto \sum_k a_k r_k.$$

As Rad_1 is canonically isomorphic to a Hilbert space, it suffices to show that $\overline{R}$ is not bounded below.

Assume first that $p \in [1, 2]$. Then for each $m \in \mathbb{N}$ we have that

$$\Big\| \bigvee_{k=1}^m \delta_{e_k} \Big\|_{\mathrm{FBL}^{(p)}[\ell_2]} \geq \sqrt{m}.$$

Indeed, let $I : \ell_2 \to \ell_2$ be the identity map, and $\widehat{I} : \mathrm{FBL}^{(p)}[\ell_2] \to \ell_2$ the lattice homomorphism extending I, which exists because of the assumption that $p \leq 2$. It follows that

$$\sqrt{m} = \Big\| \bigvee_{k=1}^m e_k \Big\|_{\ell_2} = \Big\| \bigvee_{k=1}^m \widehat{I}\delta_{e_k} \Big\|_{\ell_2} \leq \Big\| \bigvee_{k=1}^m \delta_{e_k} \Big\|_{\mathrm{FBL}^{(p)}[\ell_2]}.$$

If instead $p \in (2, \infty)$, consider the inclusion $i : \ell_2 \hookrightarrow \ell_p$. Extend this to a contractive lattice homomorphism $\widehat{i} : \mathrm{FBL}^{(p)}[\ell_2] \to \ell_p$ to get

$$m^{\frac{1}{p}} = \Big\| \bigvee_{k=1}^m e_k \Big\|_{\ell_p} \leq \Big\| \bigvee_{k=1}^m \delta_{e_k} \Big\|_{\mathrm{FBL}^{(p)}[\ell_2]}.$$

On the other hand, for every $m \in \mathbb{N}$ we have that

$$\Big\| \bigvee_{k=1}^m \delta_{r_k} \Big\|_{\mathrm{FBL}[L_1]} = 1.$$

Indeed, note first that if K is a compact Hausdorff space and $(f_j)_{j=1}^n \subseteq C(K)$, then as a consequence of the fact that the extreme points of the dual unit ball $B_{C(K)^*}$ are point measures of the form $\pm\delta_k$ for $k \in K$, we have that

$$\sup_{x^* \in B_{C(K)^*}} \sum_{j=1}^n |x^*(f_j)| = \sup_{k \in K} \sum_{j=1}^n |f_j(k)| = \Big\| \sum_{j=1}^n |f_j| \Big\|_\infty.$$

Combining this observation with (6.1.2) yields that

$$\Big\|\bigvee_{k=1}^m \delta_{r_k}\Big\|_{\mathrm{FBL}[L_1]} = \sup\Big\{\sum_{j=1}^n\Big|\bigvee_{k=1}^m \int r_k f_j\Big| : n\in\mathbb{N},\ f_1,\dots,f_n\in L_\infty,\ \Big\|\sum_{j=1}^n |f_j|\Big\|_\infty \le 1\Big\}.$$

Since we have that

$$\sum_{j=1}^n\Big|\bigvee_{k=1}^m \int r_k f_j\Big| \le \sum_{j=1}^n \int |f_j| = \int \sum_{j=1}^n |f_j| \le \Big\|\sum_{j=1}^n |f_j|\Big\|_\infty,$$

it follows that

$$\Big\|\bigvee_{k=1}^m \delta_{r_k}\Big\|_{\mathrm{FBL}[L_1]} \le 1.$$

For the converse inequality,

$$\Big\|\bigvee_{k=1}^m \delta_{r_k}\Big\|_{\mathrm{FBL}[L_1]} \ge \Big|\bigvee_{k=1}^m \delta_{r_k}(r_1)\Big| = 1.$$

Now, since $\bigvee_{k=1}^m \delta_{r_k}$ lies in $\mathrm{FVL}[L_1]$, all $\|\cdot\|_{\mathrm{FBL}^{(p)}[L_1]}$-norms can be evaluated on this element, and we have

$$1 = \Big\|\bigvee_{k=1}^m \delta_{r_k}\Big\|_{\mathrm{FBL}[L_1]} \ge \Big\|\bigvee_{k=1}^m \delta_{r_k}\Big\|_{\mathrm{FBL}^{(p)}[L_1]} = \Big\|\overline{R}\bigvee_{k=1}^m \delta_{e_k}\Big\|_{\mathrm{FBL}^{(p)}[L_1]}.$$

Thus, $\overline{R}$ is not bounded below. □

Example 6.3.14. The lattice homomorphism $\overline{R}_\infty : \mathrm{FBL}^{(p)}[\mathrm{Rad}_\infty] \to \mathrm{FBL}^{(p)}[L_\infty[0,1]]$ is not an embedding for $p \in [1,2)$.

Here we provide a direct proof of this fact. Later, in Proposition 6.4.19, we will use a different technique to show that $\overline{R}_\infty : \mathrm{FBL}^{(p)}[\mathrm{Rad}_\infty] \to \mathrm{FBL}^{(p)}[L_\infty[0,1]]$ is an embedding if and only if $p \in [2,\infty]$.

Proof. Consider the Rademacher isometry $R := R_\infty : \ell_1 \to L_\infty : e_k \mapsto r_k$; here, (e_k) form the canonical basis in ℓ_1, while (r_k) are independent Rademacher random variables. As mentioned above, we shall show that $\overline{R}$ is not bounded below if $p \in [1,2)$.

To this end, first note that $\left\|\bigvee_{k=1}^m \delta_{e_k}\right\|_{\mathrm{FBL}^{(p)}[\ell_1]} = m^{1/p}$. Indeed, the upper estimate follows from the p-convexity of $\mathrm{FBL}^{(p)}[\ell_1]$:

$$\Big\|\bigvee_{k=1}^m \delta_{e_k}\Big\|^p_{\mathrm{FBL}^{(p)}[\ell_1]} \le \Big\|\Big(\sum_{k=1}^m |\delta_{e_k}|^p\Big)^{1/p}\Big\|^p_{\mathrm{FBL}^{(p)}[\ell_1]} \le \sum_{k=1}^m \|\delta_{e_k}\|^p = m.$$

For the opposite inequality, modify the arguments in Example 6.3.13 (using the formal identity from ℓ_1 to ℓ_p).

On the other hand, we shall show that

$$\Big\| \bigvee_{k=1}^{m} \delta_{r_k} \Big\|_{\mathrm{FBL}^{(p)}[L_\infty]} \sim \sqrt{m}.$$

By (6.1.1), $\Big\| \bigvee_{k=1}^{m} \delta_{r_k} \Big\|_{\mathrm{FBL}^{(p)}[L_\infty]}$ is the supremum of $\Big(\sum_{j=1}^{n} \Big| \bigvee_{k=1}^{m} \mu_j(r_k) \Big|^p \Big)^{1/p}$, with the supremum taken over all $\mu_1, \ldots, \mu_n \in L_\infty^*$ with

$$\sup_{x \in B_{L_\infty}} \sum_{j=1}^{n} |\mu_j(x)|^p \leq 1.$$

Now consider the contractive operator $u : L_\infty^* \to \ell_\infty^m : \mu \mapsto \big(\mu(r_k)\big)_k$. Note that $\Big| \bigvee_{k=1}^{m} \mu(r_k) \Big| \leq \|u\mu\|$, hence $\Big\| \bigvee_{k=1}^{m} \delta_{r_k} \Big\|_{\mathrm{FBL}^{(p)}[L_\infty]}$ is no greater than

$$\sup \Big\{ \Big(\sum_{j=1}^{n} \|u\mu_j\|^p \Big)^{1/p} : n \in \mathbb{N},\ \mu_1, \ldots, \mu_n \in L_\infty^*, \sup_{x \in B_{L_\infty}} \sum_{j=1}^{n} |\mu_j(x)|^p \leq 1 \Big\}.$$

Arguing as in (6.1.2), this last quantity equals $\pi_p(u)$, the p-summing norm of the operator u.

By [97, Theorem 2.8], $\pi_p(u) \leq \pi_1(u)$, so it suffices to bound $\pi_1(u)$. Denote by i the formal identity from ℓ_∞^m to ℓ_2^m. Note that $\|i^{-1}\| = 1$, and $\|i\| = \sqrt{m}$, hence $\|i \circ u\| \leq \sqrt{m}$. By [97, Theorem 3.1], $\pi_1(i \circ u) \leq K_G \|i \circ u\| \leq K_G \sqrt{m}$, where K_G is Grothendieck's constant. Thus, $\pi_1(u) = \pi_1\big(i^{-1} \circ (i \circ u)\big) \leq \|i^{-1}\| \pi_1(i \circ u) \leq K_G \sqrt{m}$ by [97, p. 37]. Consequently, $\Big\| \bigvee_{k=1}^{m} \delta_{r_k} \Big\|_{\mathrm{FBL}^{(p)}[L_\infty]} \lesssim \sqrt{m}$.

For the opposite inequality, recall that $\| \cdot \|_{\mathrm{FBL}^{(p)}[L_\infty]} \geq \| \cdot \|_{\mathrm{FBL}^{(2)}[L_\infty]}$. Therefore, it suffices to show that $\Big\| \bigvee_{k=1}^{m} \delta_{r_k} \Big\|_{\mathrm{FBL}^{(2)}[L_\infty]} \geq \sqrt{m}$. Let $\mu_j = r_j \in L_1 \subseteq L_\infty^*$. By Khintchine's inequality, the map $\ell_2 \to L_1 : e_j \mapsto \mu_j$ is contractive, where (e_j) now stands for the canonical basis in ℓ_2. Therefore, $\sup_{x \in B_{L_\infty}} \sum_{j=1}^{n} |\mu_j(x)|^2 \leq 1$. However, $\Big(\sum_{j=1}^{m} \Big| \bigvee_{k=1}^{m} \mu_j(r_k) \Big|^2 \Big)^{1/2} = \sqrt{m}$. □

6.4 Extensions of operators into Lebesgue spaces

In the previous section, we were able to reduce the embedding problem for $\mathrm{FBL}^{(p)}$ to the POE-p, or in other words, the study of extension properties of operators into L_p. We now embark on a detailed study of the POE-p. To begin, we provide several reformulations in terms of operator ideals and $\mathcal{L}_p$-spaces. We then study how the POE-p behaves under duality, which provides us with several examples of embeddings satisfying the POE-p; in particular, (F, F^{**}), $(F, F_{\mathcal{U}})$ and (F, E) whenever F is locally complemented or an ideal in E. We then show several stability properties of the POE-p, compare POE-p with POE-q, and provide numerous (non-)examples.

General facts about the POE-p

We begin with several basic facts about the POE-p; namely, its relation to operator ideals, extensions into $\mathcal{L}_p$-spaces, and previous literature. Firstly, the universality of $\ell_\infty(I)$ spaces allows us to reformulate the definition of the POE-p in terms of the ideal of ℓ_∞-factorable operators $(\Gamma_\infty, \gamma_\infty)$ (see [97] for information about this and other operator ideals).

Proposition 6.4.1. *Suppose F is a Banach space, and $1 \leq p \leq \infty$. The following statements are equivalent:*

(i) F has the C-POE-p;

(ii) For any operator $T : F \to \ell_p$, and any isometric embedding $F \hookrightarrow \ell_\infty(I)$, there exists an extension $\widetilde{T} : \ell_\infty(I) \to \ell_p$, with $\left\|\widetilde{T}\right\| \leqslant C\,\|T\|$;

(iii) For any operator $T : F \to \ell_p$, we have $\gamma_\infty(T) \leq C\|T\|$;

(iv) For any compact operator $T : F \to \ell_p$, we have $\gamma_\infty(T) \leq C\|T\|$.

In statements (2), (3), *and* (4), *ℓ_p can be replaced by any infinite dimensional L_p-space.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial.

(4) $\Rightarrow$ (1): We suppose $F \hookrightarrow E$ and show that, for any $\varepsilon > 0$, any $T : F \to \ell_p^n$ has an extension $\widetilde{T} : E \to \ell_p^n$ with $\|\widetilde{T}\| \leq (C+\varepsilon)\|T\|$, so the conclusion will follow by Proposition 6.3.9. Find a factorization $T = uv$, with $v : F \to \ell_\infty(I)$ and $u : \ell_\infty(I) \to \ell_p^n$, with $\|u\|\|v\| \leq (C+\varepsilon)\|T\|$. Extend v to $\widetilde{v} : E \to \ell_\infty(I)$, with $\|v\| = \|\widetilde{v}\|$. Then $\widetilde{T} = u\widetilde{v}$ has the desired properties. $\square$

In a similar fashion, we establish:

Proposition 6.4.2. *If (F, F_1) has the C_1-POE-p, and (F_1, F_2) has the C_2-POE-p, then (F, F_2) has the C_1C_2-POE-p. In particular, if (F, E) has the C_1-POE-p, and E is C_2-injective, then F has the C_1C_2-POE-p.*

For $1 \leq p \leq 2$, the POE-p can be characterized in terms of 2-summing operators as follows. First, recall that an operator $T : F \to E$ between Banach spaces is *p-summing* for $1 \leq p < \infty$ ($T \in \Pi_p(F, E)$) if there is a constant C such that for any finite collection $(x_k)_{k=1}^n \subseteq F$ we have

$$\left(\sum_{k=1}^n \|Tx_k\|^p\right)^{1/p} \leq C \sup\left\{\left(\sum_{k=1}^n |x^*(x_k)|^p\right)^{1/p} : x^* \in F^*, \|x^*\| \leq 1\right\}.$$

The smallest possible C appearing in this inequality is denoted $\pi_p(T)$ (cf. [97]).

Proposition 6.4.3. *Let F be a Banach space and $1 \leq p \leq 2$. The following are equivalent:*

(i) F has the POE-p;

(ii) There is a constant $C > 0$ such that, for all n and all $T : F \to \ell_p^n$, we have $\pi_2(T) \leq C\|T\|$;

(iii) $B(F, \ell_p) = \Pi_2(F, \ell_p)$;

(iv) $B(F, L_p) = \Pi_2(F, L_p)$.

Proof. (1)⇒(2): Consider an embedding $\iota : F \hookrightarrow C(K)$. By assumption any $T : F \to \ell_p^n$ has an extension $\widetilde{T} : C(K) \to \ell_p^n$ with $\|\widetilde{T}\| \leq C\|T\|$. By [97, Theorem 3.5],

$$\pi_2(T) \leq \pi_2(\widetilde{T}) \leq K_G\|\widetilde{T}\| \leq K_G C\|T\|,$$

where K_G is the Grothendieck constant.

(2)⇒(1): By the Π_2-extension theorem [97, Theorem 4.15], if E, F, Y are Banach spaces with F a subspace of E then any 2-summing operator $T : F \to Y$ has an extension $\widetilde{T} : E \to Y$ with $\pi_2(T) = \pi_2(\widetilde{T})$.

The equivalence of (2), (3), and (4) is a classical localization argument. □

Remark 6.4.4. Using Proposition 6.4.3 we see that POE-1 and POE-2 are actually well-studied Banach space properties. Indeed, by [184], F is POE-2 if and only if F is a Hilbert-Schmidt space. Moreover, by [279, Proposition 6.2], F is POE-1 if and only if F^* is a G.T. space.

Remark 6.4.5. The POE-p was also studied (under a different name) in [75]. Indeed, [75] investigates the spaces E so that (F, E) has the POE-p for every $F \subseteq E$. For instance, it is shown that, if E is a Banach lattice with such property for some $p \in (2, \infty)$, then E is weak Hilbert, and satisfies a lower 2-estimate. If E is a Köthe function space on $(0, 1)$, then it must be lattice isomorphic to $L_2(0, 1)$. If E is a space with a subsymmetric basis, then [282, Proposition 12.4] can be used to show that this basis is equivalent to the ℓ_2 basis. On the other hand, Maurey's Extension Theorem [97, p. 12.22] yields:

Proposition 6.4.6. *Suppose E has type 2, F is a subspace of E, and $1 \leq p \leq 2$. Then (F, E) has the POE-p.*

The definition of the POE-p involves extending operators into L_p-spaces. It turns out that we can extend operators into the wider class of $\mathcal{L}_p$-spaces.

Proposition 6.4.7. *Suppose $1 \leq p < \infty$, F is a Banach space, and X is an infinite dimensional $\mathcal{L}_p$-space. Consider the following statements:*

(i) (F, E) has the POE-p;

(ii) Any compact operator $T : F \to X$ has a bounded extension $\widetilde{T} : E \to X$;

(iii) Any compact operator $T : F \to X$ has a compact extension $\widetilde{T} : E \to X$;

(iv) Any bounded operator $T : F \to X$ has a bounded extension $\widetilde{T} : E \to X$.

Then $(1) \Leftrightarrow (2) \Leftrightarrow (3)$. Moreover, if X is complemented in X^{**}, then (4) is equivalent to the three preceding statements.

By [228], a $\mathcal{L}_p$-space X is complemented in X^{**} if and only if it embeds into L_p as a complemented subspace. It is well known (see e.g. [191]) that, for $1 < p < \infty$, any $\mathcal{L}_p$-space is reflexive, hence, in Proposition 6.4.7, $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$. For $p = 1$, [228, Section 5] provides an example of a $\mathcal{L}_1$-space which does not embed complementably into L_1.

Proof. Note that, if (4) holds, then there exists $C > 0$ so that any $T : F \to X$ has an extension $\widetilde{T} : E \to X$ with $\|\widetilde{T}\| \le C\|T\|$. Indeed, (4) implies that the map $\Phi : B(E, X) \to B(F, X) : S \mapsto S|_F$ is surjective; thus, there exists $C > 0$ so that for any T there exists $\widetilde{T}$ with $\|\widetilde{T}\| \le C\|T\|$, and $\Phi(\widetilde{T}) = T$. We can reach similar conclusions in cases (2) and (3).

By [228, Theorem 1], X contains a complemented copy of ℓ_p. Consequently, either (2), (3), or (4) implies (1). Clearly (3) $\Rightarrow$ (2). The implications (1) $\Rightarrow$ (3) and (modulo complementability of X in X^{**}) (1) $\Rightarrow$ (4) remain to be established. The proofs use [228, Theorem 3]: there exists a constant ρ so that for every finite dimensional $Z \subseteq X$, we can find Y so that $Z \subseteq Y \subseteq X$, Y is ρ-isomorphic to $\ell_p^{\dim Y}$, and ρ-complemented in X. Use this to find an increasing net of finite dimensional spaces $(Y_\alpha)_{\alpha \in \mathcal{I}}$, so that $X = \overline{\cup_\alpha Y_\alpha}$ and, for any α, denoting dim $Y_\alpha = n_\alpha$ we have $d(Y_\alpha, \ell_p^{n_\alpha}) \le \rho$, and there exists a projection $P_\alpha : X \to Y_\alpha$ so that $\|P_\alpha\| \le \rho$.

For the remainder of the proof, we assume that (F, E) has the POE-p with constant C, X is a $\mathcal{L}_p$-space, and $T : F \to X$ is a contraction.

(1) $\Rightarrow$ (3): Fix $\varepsilon > 0$. Assuming T is compact, we shall find a compact extension $\widetilde{T} : E \to X$, with $\|\widetilde{T}\| \le C\rho(1 + 3\varepsilon)$. We proceed recursively. Let $T_0 = 0$, $T_0' = T$. Our first goal is to find $\alpha_1 \prec \alpha_2 \prec \ldots$, and operators $T_k, T_k' \in B(F, X)$ so that, for any k, we have

$$T_{k-1}' = T_k + T_k', \ T_k = P_{\alpha_k} T_{k-1}', \ \|T_k'\| \le \varepsilon 2^{-k}.$$

Note that for any k we have $T = T_1 + \cdots + T_k + T_k'$. By the triangle inequality, $\|T_k\| \le \|T_{k-1}'\| + \|T_k'\| < \varepsilon 2^{2-k}$ for $k \ge 2$, and likewise, $\|T_1\| < 1 + \varepsilon$. Moreover, $\sum_{k=1}^\infty T_k$ converges to T.

Suppose we have already found $\alpha_1 \prec \ldots \prec \alpha_n$, and the operators $T_0, \ldots, T_n, T_n'$ with the desired properties (if $n = 0$, then we have taken $T_0 = 0$ and $T_0' = T$, and we ignore the condition about $\alpha_1, \ldots, \alpha_n$). Find $\alpha_{n+1} \succ \alpha_n$ so that

$$\sup_{f \in F, \|f\| \le 1} \inf_{y \in Y_{\alpha_{n+1}}} \|T_n' f - y\| < \varepsilon(\rho + 1)^{-1} 2^{-n-1}.$$

Let $T_{n+1} = P_{\alpha_{n+1}} T_n'$ and $T_{n+1}' = T_n' - T_{n+1}$, and note that $\|T_{n+1}'\| \le \varepsilon 2^{-n-1}$. Indeed, fix $f \in B_F$, and find $y \in Y_{\alpha_{n+1}}$ so that $\|T_n' f - y\| < \varepsilon 2^{-n-1}(\rho + 1)^{-1}$. Then

$$\|T_{n+1}' f\| = \big\|(I - P_{\alpha_{n+1}})(T_n' f - y)\big\| \le \big(1 + \|P_{\alpha_{n+1}}\|\big)\|T_n' f - y\| \le \varepsilon 2^{-n-1}.$$

So, T_{n+1} and T'_{n+1} have the desired properties.

Recall that $T_k(F) \subseteq Y_{\alpha_k}$, and the latter space is ρ-isomorphic to $\ell_p^{n_{\alpha_k}}$. Consequently, T_k has an extension $\widetilde{T_k} : E \to Y_{\alpha_k}$, with $\|\widetilde{T_k}\| \leq C\rho\|T_k\|$. Recalling the estimates on the norms $\|T_k\|$ obtained above, we conclude that $\|\widetilde{T_1}\| \leq C\rho(1+\varepsilon)$, and $\|\widetilde{T_k}\| \leq C\rho\varepsilon 2^{2-k}$ for $k \geq 2$. Then $\widetilde{T} = \sum_{k=1}^{\infty} \widetilde{T_k}$ extends T, and has norm not exceeding $C\rho(1+3\varepsilon)$. This proves (3).

Denote by Q a projection from X^{**} onto X; we will show that (1) $\Rightarrow$ (4). For each $\alpha \in \mathcal{I}$, we find $\widetilde{T_\alpha} : E \to Y_\alpha \subseteq X$, which extends $P_\alpha T$, and has norm at most $C\rho^2$. It is well-known (see e.g. [97, p. 120]) that $B(E, X^{**}) = \big(E\widehat{\otimes}X^*\big)^*$, where $\widehat{\otimes}$ denotes the projective tensor product. Hence, the net $(\widetilde{T_\alpha})$ has a subnet $\big(\widetilde{T_\beta}\big)_{\beta\in\mathcal{J}}$ which converges to some $S : E \to X^{**}$ in the $\sigma\big(B(E, X^{**}), E\widehat{\otimes}X^*\big)$ topology. Testing convergence on elementary tensor products $e \otimes x^*$ ($e \in E, x^* \in X^*$), we conclude that $\widetilde{T_\beta} \to S$ point-weak*, hence $\|S\| \leq \limsup_\beta \|\widetilde{T_\beta}\| \leq C\rho^2$.

Let $\widetilde{T} = QS$. Then $\|\widetilde{T}\| \leq \|Q\|C\rho^2$. We claim that $\widetilde{T}$ extends T – that is, $\widetilde{T}f = Tf$ for any $f \in F$. We shall show that, in fact, $Sf = Tf$. Indeed, fix $\varepsilon > 0$, and find $\beta_0 \in \mathcal{J}$ so large that, for any $\beta \succ \beta_0$, we have

$$\inf_{y\in Y_\beta} \big\|Tf - y\big\| < \varepsilon.$$

As in the proof of (1) $\Rightarrow$ (3), show that $\|Tf - P_\beta Tf\| = \|Tf - \widetilde{T_\beta}f\| < \varepsilon(\rho+1)$ when $\beta \succ \beta_0$. As Sf is the weak* limit of $(\widetilde{T_\beta}f)$, then $\|Tf - Sf\| \leq \varepsilon(\rho+1)$. To conclude that $Sf = Tf$, recall that ε can be arbitrarily small. $\square$

The POE-p: duality, local complementation and ultrapowers

We now explore the interplay between the POE-p and duality. To fix the terminology below, recall that an operator $Q : X \to Y$ between Banach spaces is said to be λ-surjective if for any $y \in Y$ with $\|y\| < 1$ there exists $x \in X$ with $Qx = y$, $\|x\| < \lambda$. A standard functional analysis result states that Q is λ-surjective if and only if Q^* is bounded below by $1/\lambda$ if and only if Q^{**} is λ-surjective.

For any Banach space Z, we can identify $B(Z, \ell_p^n)$ with $(Z^*)^n$, as a vector space. More precisely, any $T \in B(Z, \ell_p^n)$ can be written as $T = \sum_{k=1}^n z_k^* \otimes e_k$, where $e_1, \ldots, e_n$ form the canonical basis in ℓ_p^n, and $z_1^*, \ldots, z_n^* \in Z^*$. Then T, or $T^* \in B(\ell_{p'}^n, Z^*)$ $(1/p + 1/p' = 1)$, can be identified with $(z_1^*, \dots, z_n^*) \in (Z^*)^n$. By Local Reflexivity (as laid out in [95]), $B(Z, \ell_p^n)^{**} = B(Z^{**}, \ell_p^n)$.

Proposition 6.4.8. *A pair (F, E) has the C-POE-p if and only if the same is true for (F^{**}, E^{**}).*

Proof. Define, for any $n \in \mathbb{N}$, the operator $\Phi_{F,E}^{(n)} : B(E, \ell_p^n) \to B(F, \ell_p^n) : S \mapsto S|_F$. Fix n; we henceforth omit the upper index (n). By the preceding paragraphs, $\Phi_{F,E}^{**}$ can be identified with $\Phi_{F^{**},E^{**}}$, hence one is λ-surjective if and only if the other is. In light of Proposition 6.3.9, (F, E) has the C-POE-p if and only if $\Phi_{F,E}$ is C-surjective. By the above, (F, E) has the C-POE-p if and only if (F^{**}, E^{**}) does. □

For a Banach space E and $n \in \mathbb{N}$, let $E^{(n)}$ denote its n-th dual. The preceding result yields:

Proposition 6.4.9. *Let F be a closed subspace of E and suppose that $F^{(2k)}$ is C-complemented in $E^{(2k)}$ for some $k \in \mathbb{N}$. Then (F, E) has the C-POE-p.*

Similarly, since any operator $T : F \to \ell_p^n$ has an extension $T^{(2k)} : F^{(2k)} \to \ell_p^n$ $(k \in \mathbb{N})$ with the same norm, we see that:

Proposition 6.4.10. *For any Banach space F, $k \in \mathbb{N}$, and $p \in [1, \infty]$, $(F, F^{(2k)})$ has the 1-POE-p.*

Using $\ell_\infty(I)$ spaces we can convert Proposition 6.4.8 into a statement about Banach spaces with the POE-p (rather than pairs of Banach spaces with the POE-p):

Proposition 6.4.11. *F has the C-POE-p if and only if F^{**} does.*

Proof. Suppose F has the C-POE-p. Embed F isometrically into $\ell_\infty(I)$. By Proposition 6.4.1, F has the C-POE-p if and only if $(F, \ell_\infty(I))$ does. By Proposition 6.4.8, this, in turn, is equivalent to $(F^{**}, \ell_\infty(I)^{**})$ having the C-POE-p. By [333, Theorem 4.1], $\ell_\infty(I)^{**}$ is 1-injective. Thus, by Proposition 6.4.2, if $(F^{**}, \ell_\infty(I)^{**})$ has the C-POE-p, then so does F^{**}.

Conversely, suppose F^{**} has the C-POE-p. Embed F^{**} into $\ell_\infty(I)$, for a suitable index I. We have to show that $(F, \ell_\infty(I))$ has the C-POE-p. By Proposition 6.4.10 (F, F^{**}) has the 1-POE-p, hence Proposition 6.4.2 yields the desired result. □

We now give three examples where the above results apply. First, recall that by [333, Theorem 4.2], F^{**} is C-injective if and only if whenever F is a closed subspace of E and Y is finite dimensional, every operator $T : F \to Y$ extends to $\widetilde{T} : E \to Y$ with $\|\widetilde{T}\| \le C\|T\|$. Hence, $\mathcal{L}_\infty$-spaces have POE-p for all $p \in [1, \infty]$. To be more precise, by combining [229, Theorem 3.3] with [333, Theorem 4.2], we observe that, if F is a $\mathcal{L}_{\infty,\mu}$-space for all $\mu > \lambda$, then F^{**} is λ-injective. This implies:

Corollary 6.4.12. *If F is a $\mathcal{L}_{\infty,\mu}$-space for all $\mu > \lambda$, then it has the λ-POE-p for every $p \in [1, \infty]$. In particular, c_0 and $C(K)$ spaces have the 1-POE-p.*

In a similar fashion, we apply Proposition 6.4.8 and Proposition 6.4.9 to two well-studied classes of subspaces. Recall, following [197], that a closed subspace F of E is locally complemented in E if there is $\lambda > 0$ such that whenever G is a finite dimensional subspace of E and $\varepsilon > 0$, there is a linear operator $T : G \to F$ such that $\|T\| \le \lambda$ and $\|Tx - x\| \le \varepsilon\|x\|$ for $x \in F \cap G$. It follows from [197, Theorem 3.5] that F is locally complemented in E if and only if F^{**} is complemented in E^{**} under the natural embedding. Proposition 6.4.9 thus implies:

Corollary 6.4.13. *If F is locally complemented in E, then (F, E) has the POE-p for every $p \in [1, \infty]$.*

On the other hand, recall that a subspace F of a Banach space E is called an ideal (cf. [130]) if $F^\perp = \{x^* \in E^* : x^*(y) = 0 \text{ for } y \in F\}$ is the kernel of a contractive projection on E^*. In this case F^{**} is contractively complemented in E^{**}. Note that here, neither E nor F is assumed to have any order structure. One should distinguish between the "Hahn-Banach ideals" described above, and order ideals we are discussing in the context of Banach lattices.

For ideals (in the Banach space sense) Proposition 6.4.8 implies:

Corollary 6.4.14. *If F is an ideal in E, then (F, E) has the 1-POE-p for any $p \in [1, \infty]$.*

Embeddings of Banach spaces into their ultrapowers behave in a fashion similar to embeddings into second duals. Recall that given a Banach space E and a free ultrafilter $\mathcal{U}$ on an infinite set Γ, the ultrapower of E with respect to $\mathcal{U}$ is given by $E_{\mathcal{U}} = \ell_\infty(\Gamma, E)/N_{\mathcal{U}}$, where $N_{\mathcal{U}}$ is the subspace of elements in $\ell_\infty(\Gamma, E)$ which converge to zero along $\mathcal{U}$. A "natural embedding" of E into $E_{\mathcal{U}}$ is determined by mapping e to the equivalence class of $(e, e, \dots)$.

Proposition 6.4.15. *For any $p \in [1, \infty]$ and any Banach space F, the pair $(F, F_{\mathcal{U}})$ has the 1-POE-p.*

Proof. Given $T : F \to \ell_p^n$, let $T_{\mathcal{U}} : F_{\mathcal{U}} \to (\ell_p^n)_{\mathcal{U}}$ denote the natural extension (cf. [3, Theorem 1.64]) which satisfies $\|T_{\mathcal{U}}\| = \|T\|$. By compactness we have that $(\ell_p^n)_{\mathcal{U}} = \ell_p^n$. $\square$

Further characterizations of the POE-p

In the definition of POE-p there is a uniform constant C which is selected independently of the embedding $F \hookrightarrow E$. However, it is not necessary to require this:

Proposition 6.4.16. *For $1 \leq p \leq \infty$ and a Banach space F, the following are equivalent:*

(i) F has the POE-p;

(ii) For any Banach space E containing F there is a constant $C > 0$ such that every operator $T : F \to \ell_p^n$ extends to $\widetilde{T} : E \to \ell_p^n$ with $\|\widetilde{T}\| \leq C\|T\|$.

Proof. Clearly (1) $\Rightarrow$ (2). Now suppose (1) fails; we will show that (2) fails as well.

For each $k \in \mathbb{N}$, find an isometric embedding $j_k : F \hookrightarrow E_k$, and a contraction $T_k : F \to \ell_p^{n_k}$ so that any extension of T_k to E_k has norm at least k. We "amalgamate" the spaces E_k: let $E = (\sum_k E_k)_1/G$, where G consists of all elements $(a_k j_k y)_k \in (\sum_k E_k)_1$ ($y \in F$) with $\sum_k a_k = 0$ (this sum is well defined, since the membership in $(\sum_k E_k)_1$ implies $\sum_k |a_k| < \infty$). Define $u_k : E_k \to E$ by taking $x \in E_k$ to the equivalence class of $x^{(k)} := (0, \ldots, 0, x, 0, \ldots)$ (x is in the k-th position). Then u_k is an isometry. Indeed, clearly this map is contractive. On the other hand, for any $x \in E_k$,

$$\|u_k x\| = \inf_{g \in G} \left\|x^{(k)} + g\right\| = \inf_{y \in F, \sum_i a_i = 0} \left\{ \left\|a_k j_k y + x\right\| + \sum_{i \neq k} \left\|a_i j_i y\right\| \right\}$$
$$\geq \inf_{y \in F, \sum_i a_i = 0} \left\{ \|x\| - |a_k| \|y\| + \sum_{i \neq k} |a_i| \|y\| \right\} = \|x\|.$$

For $i, k \in \mathbb{N}$ and $y \in F$, $[j_i y]^{(i)} - [j_k y]^{(k)} \in G$, hence $u_i j_i = u_k j_k$. Denote $u_k j_k$ (no matter what k is – all these maps coincide) by j; then $j : F \to E$ is an isometric embedding. Consider $T_k : F \to \ell_p^{n_k}$ as described above (that is, T_k is an operator with no "small norm" extension to E_k). Suppose $S : E \to \ell_p^{n_k}$ extends T_k. Then $\|S\| \geq \left\|S|_{u_k(E_k)}\right\| \geq k$. As k is arbitrary, we conclude that (2) fails. $\square$

We can also restrict to superspaces of the same density. For a Banach space E, let us denote by $dens(E)$ the density character of E – that is, the least cardinality of a dense subset.

Proposition 6.4.17. *For $C \geq 1$, $1 \leq p \leq \infty$, and a Banach space F, the following are equivalent:*

(i) F has the C-POE-p;

(ii) Whenever F is a closed subspace of E with $dens(E) = dens(F)$, every operator $T : F \to \ell_p^n$ extends to $\widetilde{T} : E \to \ell_p^n$ with $\|\widetilde{T}\| \leq C\|T\|$.

Proof. Suppose (2) holds and let F be a closed subspace of an arbitrarily large E. By [165] (see also [302]), there exists a closed subspace G, such that $F \subseteq G \subseteq E$, $dens(G) = dens(F)$ and G is an ideal in E. If $T : F \to \ell_p^n$, then by hypothesis we can find an extension $\widetilde{T} : G \to \ell_p^n$ with $\|\widetilde{T}\| \leq C\|T\|$. Since G is an ideal in E, then we also have an extension $\widetilde{\widetilde{T}} : E \to \ell_p^n$ with $\|\widetilde{\widetilde{T}}\| \leq C\|T\|$ by Corollary 6.4.14. □

The relations between POE-p and POE-q, and several examples

In this section we give several examples of pairs (F, E) (or spaces F) which have the POE-p, and several which do not. We begin with $\mathcal{L}_1$-spaces:

Proposition 6.4.18. *Suppose $2 \leq p \leq \infty$, and F is a $\mathcal{L}_{1,\mu}$-space for all $\mu > \lambda$. Then F has the λ-POE-p.*

Proof. In light of Proposition 6.3.9, it suffices to show that, for any embedding $\iota : F \hookrightarrow E$, and any $C > \lambda$, any operator $T : F \to \ell_p^n$ has an extension $\widetilde{T} : E \to \ell_p^n$, with $\|\widetilde{T}\| \leq C\|T\|$. To this end, find $\mu > \lambda$ and $\varepsilon > 0$ so that $\mu(1 + \varepsilon) < C$. Let p' be the "conjugate" of p, so that $\frac{1}{p} + \frac{1}{p'} = 1$. As $1 \leq p' \leq 2$, there exists an isometric embedding $j : \ell_{p'}^n \to L_1$ (see e.g. [191, Section 4]). Then $j^* : L_\infty \to \ell_p^n$ is a quotient map. By [228, Theorem 4.2], T has a lifting $S : F \to L_\infty$, with $\|S\| \leq \mu(1+\varepsilon)\|T\|$, and $j^*S = T$. Find an extension $\widetilde{S} : E \to L_\infty$, with $\|\widetilde{S}\| = \|S\|$. Then $\widetilde{T} := j^*\widetilde{S}$ is the desired extension of T. □

We next discuss the relations between the POE-p, for different values of p. As a corollary, we deduce from Proposition 6.4.19 that Proposition 6.4.18 fails for $1 \leq p < 2$ and one cannot replace c_0 by ℓ_1 in Corollary 6.4.12 when $p \in [1, 2)$.

Proposition 6.4.19. *(i) If $1 \leq p \leq q \leq 2$, and F has the POE-p, then it has the POE-q;*

(ii) If $2 \leq p < \infty$, and F has the POE-p, then it has the POE-2;

(iii) The space ℓ_1 has the POE-p if and only if $2 \leq p \leq \infty$.

Proof. (1) By [191, Section 4], ℓ_q (or even L_q) embeds isometrically into L_p. If F has the POE-p, then, by Proposition 6.4.3, there exists a constant C so that the inequality $\pi_2(T) \leq C\|T\|$ holds for any $T : F \to L_p$. The ideal of 2-summing operators is injective, hence we have $\pi_2(T) \leq C\|T\|$ holds for any $T : F \to \ell_q$. Thus, F has the POE-q.

(2) Suppose F has the POE-p with constant c. We need to show that, for any $E \supseteq F$, any operator $T : F \to \ell_2^n$ has an extension $\widetilde{T} : E \to \ell_2^n$, with $\|\widetilde{T}\| \leq C\|T\|$ (C is a universal constant). Denote the canonical basis in ℓ_2^n by (e_k), and let $j : \ell_2^n \to \ell_p^{2^n}$ be the "Khintchine" embedding – that is, $je_k = r_k$ for $1 \leq k \leq n$, with $r_1, \ldots, r_n$ being Rademacher random variables realized in $\ell_p^{2^n}$. Then $\|jx\| \geq \|x\|$ for any x. Further, there exists $\lambda = \lambda_p$ so that $\|j\| \leq \lambda$, and there exists a projection $P : \ell_p^{2^n} \to j(\ell_2^n)$ with $\|P\| \leq \lambda$. Consider $T : F \to \ell_2^n$. As F has the POE-p, $jT : F \to \ell_p^{2^n}$ has an extension $S : E \to \ell_p^{2^n}$ with $\|S\| \leq c\|j\|\|T\| \leq c\lambda\|T\|$. Then $\widetilde{T} = j^{-1}PS$ extends T, and $\|\widetilde{T}\| \leq c\lambda^2\|T\|$.

(3) The fact that ℓ_1 has the POE-p for $p \geq 2$ follows directly from Proposition 6.4.18. Now suppose $1 \leq p < 2$. By Proposition 6.4.3, it suffices to show that for every $C > 0$ there exists a contractive $T : \ell_1 \to \ell_p^n$ so that $\pi_2(T) \geq C$. Denote by (e_k) the canonical bases in both ℓ_1 and ℓ_p^n, and set $Te_k = e_k$ if $1 \leq k \leq n$, $Te_k = 0$ otherwise. By [125, Theorem 9(v)], $\pi_2(T) \sim n^{1/p-1/2}$. □

The class of POE-1 spaces, albeit more restrictive than that of POE-2 spaces (by Proposition 6.4.19), is still fairly large. For instance, by Proposition 6.4.3 and [329, Corollary III.I.13], the disk algebra A has the POE-1. However, for spaces with an unconditional basis, the POE-1 condition is very restrictive, as we will next see.

Proposition 6.4.20. *A space F with a normalized unconditional basis (x_k) has the POE-1 if and only if (x_k) is equivalent to the c_0 basis.*

Proof. Due to Corollary 6.4.12, c_0 has the POE-1. Now suppose F possesses a normalized unconditional basis (x_k), and has the POE-1. It is easy to see that the (semi-normalized) biorthogonal functionals (x_k^*) form an unconditional sequence in F^*. By Remark 6.4.4, F^* is a G.T. space. For $m \in \mathbb{N}$, denote by P_m the canonical basis projection from F onto $\operatorname{span}[x_k : 1 \leq k \leq m]$. Then P_m^* is a projection from F^* onto $\operatorname{span}[x_k^* : 1 \leq k \leq m]$. As

$\sup_m \|P_m\| < \infty$, we conclude that there exists a constant C so that, for any $m, n \in \mathbb{N}$, any operator $T : \mathrm{span}[x_k^* : 1 \leq k \leq m] \to \ell_2^n$ satisfies $\pi_1(T) \leq C\|T\|$. The proof of [279, Theorem 8.21] shows the existence of a constant C' so that $(x_k^*)_{k=1}^m$ is C'-equivalent to the canonical basis of ℓ_1^m. In other words, the inequality

$$\frac{1}{C'} \sum_k |a_k| \leq \Big\| \sum_k a_k x_k^* \Big\| \leq \sum_k |a_k|$$

holds for any finite sequence (a_k). Thus, (x_k) is equivalent to the c_0 basis. $\square$

Remark 6.4.21. An alternative proof for Proposition 6.4.20 can also be deduced from [291], where it is shown that a space with an unconditional basis has the POE-2 if and only if it is isomorphic to ℓ_1, c_0, or $c_0 \oplus \ell_1$. Indeed, if F has the POE-1, then by Proposition 6.4.19, it also has the POE-2, and since ℓ_1 fails POE-1, the previous characterization yields that F can only be isomorphic to c_0.

Remark 6.4.22. The relations between the POE-p for different values of p remain unclear. For instance, we do not know whether POE-p implies POE-q in the following situations:

(i) $p \in [1, \infty)$, $q \in (2, \infty)$.

(ii) $1 \leq q < p < 2$.

Also, we do not have a characterization of POE-p ($2 < p < \infty$) in terms of operator ideals, along the lines of Proposition 6.4.3. Using [97, Corollary 10.10], one can observe that, if F has the POE-p for $2 < p < \infty$, then $B(F, \ell_p) = \Pi_{p,s}(F, \ell_p) = \Pi_r(F, \ell_p)$ whenever $s < p < r$. However, this condition does not seem to be sufficient.

Above, we have observed that the disk algebra A has the POE-p for $1 \leq p \leq 2$. We do not know whether A has the POE-p for $2 < p < \infty$. We know that, by [97, Corollary 10.10] and [329, Corollary III.I.13], $B(A, \ell_p) = \Pi_r(A, \ell_p)$ when $2 < p < r$. Also, by F. and M. Riesz Theorem, $A^{**} = H_\infty \oplus_\infty M_s^*$, where M_s is the set of measures singular with respect to the Lebesgue measure (cf. [329, p. 181]). As the POE-p passes to the double dual, we conclude that H_∞ has the POE-p for $1 \leq p \leq 2$. As with A, we do not know what the situation is for $2 < p < \infty$.

Finally, we examine the POE-p for some "natural" pairs (F, E), where F is a subspace of $L_p(\mu)$.

Proposition 6.4.23. *Suppose $1 \leq p < \infty$, and F is isomorphic to a complemented subspace of $L_p(\mu)$, for some measure μ. If a Banach space E contains F, then the following are equivalent:*

(i) F is complemented in E.

(ii) F is locally complemented in E.

(iii) (F, E) has the POE-p.

Remark 6.4.24. Proposition 6.4.23 is applicable in the following situations:

(i) $1 < p < \infty$, and F is isomorphic to a Hilbert space. Indeed, suppose I is an index set, and μ is the uniform probability measure on $\{0, 1\}$. Then $\ell_2(I)$ is isomorphic to the span of independent Rademacher functions in $L_p(\mu^{\otimes I})$; the latter space is complemented in $L_p(\mu^{\otimes I})$.

(ii) $1 < p < \infty$, and F is a $\mathcal{L}_p$-space. Indeed, by [228], such an F embeds complementably into an L_p-space.

Proof of Proposition 6.4.23. (1) $\Rightarrow$ (2) is easy, and (2) $\Rightarrow$ (3) follows from Corollary 6.4.13.

(3) $\Rightarrow$ (1): Let F' be a complemented subspace of $L_p(\mu)$ isomorphic to F, $P : L_p(\mu) \to F'$ a projection, and $T : F \to F'$ an isomorphism. By Proposition 6.3.9, T has an extension $\widetilde{T} \colon E \to L_p(\mu)$. Then $Q := T^{-1}|_{F'} P \widetilde{T}$ is a projection from E onto F. □

Specializing to Hilbertian subspaces of L_1, we obtain:

Proposition 6.4.25. *If F is an infinite dimensional subspace of L_1, isomorphic to a Hilbert space, then (F, L_1) fails the POE-p for $1 \leq p < \infty$.*

Proof. If $1 < p < \infty$, the result follows from Remark 6.4.24. It remains to examine the case of $p = 1$. Note that, if F is Hilbertian, and (F, E) has the POE-p, then (F', E) has the POE-p whenever F' is a subspace of F. Indeed, there exists a projection P from F onto F'. For any $T \in B(F', L_p)$, the operator TP has an extension $S : E \to L_p$, which clearly also extends T. Therefore, it suffices to establish the failure of POE-1 for separable F. Also, we can restrict ourselves to $F \subseteq L_1(\mu)$, where μ is a probability measure.

Suppose, for the sake of contradiction, that $(F, L_1(\mu))$ has the POE-1 with constant C. In particular, any operator $T : F \to F$ extends to $\widetilde{T} : L_1(\mu) \to L_1(\mu)$, with $\|\widetilde{T}\| \leq C\|T\|$.

Recall that a normalized (in L_1) Gaussian random variable g can be realized on a measure space (Ω_0, ν_0). Then independent normalized Gaussian variables $(g_i)_{i\in\mathbb{N}}$ can be realized in $L_1(\nu)$, with $\nu = \otimes_{i\in\mathbb{N}}\nu_0$; denote by G the closure of their linear span. It is well-known that G is Hilbertian. As operators on F have an extension property described above, [286] shows that, for any isomorphism $J : F \to G$ there exists operators $u : L_1(\mu) \to L_1(\nu)$ and $v : L_1(\nu) \to L_1(\mu)$, extending J and J^{-1}, respectively. From this, we conclude that $(G, L_1(\nu))$ has the POE-1. Indeed, fix J, u, v as in the preceding paragraph. For any $T \in B(G, L_1)$, the operator $S = TJ$ has an extension $\widetilde{S} : L_1(\mu) \to L_1$, with $\|\widetilde{S}\| \leq C\|J\|\|T\|$. Then $\widetilde{S}v : L_1(\nu) \to L_1$ extends T.

Therefore, by [279, Theorem 6.6, and the remark following it], $L_1(\nu)^* = L_\infty(\nu)$ has cotype 2 (and satisfies Grothendieck's Theorem), which is clearly false. This is the desired contradiction. □

Remark 6.4.26. Fix p, and suppose F is a closed subspace of a Banach space E. We have found a characterization of when the canonical embedding extends to a lattice embedding of $\mathrm{FBL}^{(p)}[F]$ inside $\mathrm{FBL}^{(p)}[E]$. However, one might still wonder when $\mathrm{FBL}^{(p)}[E]$ at least contains some lattice isomorphic (or isometric) copy of $\mathrm{FBL}^{(p)}[F]$. Similarly, if F is, moreover, a p-convex Banach lattice, when does $\mathrm{FBL}^{(p)}[E]$ contain a (nicely complemented) lattice copy of F? In general, these questions will have a negative answer: take for instance $E = C[0, 1]$, and let F be a subspace which is isomorphic to ℓ_1. It will follow from Theorem 6.9.20 that $\mathrm{FBL}[C[0, 1]]$ never contains a sublattice isomorphic to ℓ_1, so it fails to contain $\mathrm{FBL}[\ell_1]$ as a sublattice as well (this is due to Theorem 6.8.3).

6.5 Basic sequences in free spaces

In this section we study the structure of basic sequences in free Banach lattices. More specifically, we begin with a basic sequence (x_k) in a Banach space E, and try to understand the sequence of moduli $(|\delta_{x_k}|)$ it generates in $\mathrm{FBL}^{(p)}[E]$. This is important, since, due to the universal nature of free Banach lattices, the behaviour of the sequence $(|\delta_{x_k}|)$ reflects all possible embeddings of E into arbitrary p-convex lattices. As an illustration of this, we note the following:

Proposition 6.5.1. *Suppose (x_k) is a sequence in E, and $(|\delta_{x_k}|)$ is weakly null in $\mathrm{FBL}^{(p)}[E]$. Then, for any p-convex Banach lattice X, and any bounded map $T : E \to X$, the sequence $(|Tx_k|)$ is weakly null.*

Proof. For T as above, $\widehat{T} : \mathrm{FBL}^{(p)}[E] \to X$ is bounded, hence weak-to-weak continuous, hence if $(|\delta_{x_k}|)$ is weakly null then so is $(\widehat{T}|\delta_{x_k}|) = (|Tx_k|)$. □

Taking into account the description of $(|\delta_{e_k}|) \subseteq \mathrm{FBL}[\ell_r]$ obtained in [30], we obtain:

Corollary 6.5.2. *Suppose $2 < r \leq \infty$, and (e_k) is the canonical basis in ℓ_r (if $r = \infty$, we take c_0 instead of ℓ_∞). If X is a Banach lattice, and $T : \ell_r \to X$ is a bounded operator, then $(|Te_k|)$ is weakly null in X.*

The preceding result fails for $1 \leq r \leq 2$. Indeed, [26, Theorem 5.4] (see also Proposition 6.5.14 below) shows that $(|\delta_{e_k}|)$ is equivalent to the ℓ_1 basis for $1 \leq r \leq 2$. However, by Proposition 6.6.4 $(|\delta_{e_k}|)$ is equivalent to the ℓ_r basis (hence weakly null, if $r > 1$) in $\mathrm{FBL}^{(\infty)}[\ell_r]$. These observations are consistent with the fact that the standard Rademacher random variables give a copy of ℓ_2 in L_r $(1 \leq r < \infty)$ but their moduli are not weakly null. Actually, the unit vector basis of ℓ_r $(1 < r \leq 2)$ has weakly null moduli in $\mathrm{FBL}^{(p)}[\ell_r]$ if and only if $p = \infty$; the Rademacher functions in L_∞ shows that the moduli of ℓ_1 in $\mathrm{FBL}^{(p)}[\ell_1]$ can never be weakly null. We will expand on these observations significantly in the results below.

Next we generalize [30, Proposition 1]. Recall that a basic sequence (x_k) is called C-suppression unconditional if for every choice of scalars (a_k) and any set $A \subseteq \mathbb{N}$ we have $\|\sum_{k\in A} a_k x_k\| \leq C\|\sum_{k\in\mathbb{N}} a_k x_k\|$. It is standard to check that every C-suppression unconditional sequence is $2C$-unconditional.

Proposition 6.5.3. *Let (x_k) be a sequence in a Banach space E. Then, for the sequence $(|\delta_{x_k}|)$ in $\mathrm{FBL}^{(p)}[E]$, we have:*

(i) If (x_k) is minimal then $(|\delta_{x_k}|)$ is minimal;

(ii) If (x_k) is a basis then $(|\delta_{x_k}|)$ is basic;

(iii) If (x_k) is a C-suppression unconditional basis then $(|\delta_{x_k}|)$ is C-suppression unconditional, hence $2C$-unconditional;

(iv) If (x_k) is a symmetric basis then $(|\delta_{x_k}|)$ is symmetric.

Recall (see [303, p. 54]) that (x_k) is called minimal if it admits a system of biorthogonal functionals.

Proof. (1): Let (x_k^*) be biorthogonal functionals for (x_k), and extend them so that $x_k^* \in E^*$. Then $\widehat{x_k^*}\colon \mathrm{FBL}^{(p)}[E] \to \mathbb{R}$ is a lattice homomorphism for every k, so that $\widehat{x_k^*}(|\delta_{x_l}|) = \big|x_k^*(x_l)\big| = \delta_{k,l}$, showing that $\big(\widehat{x_k^*}\big)$ are biorthogonal functionals for $\big(|\delta_{x_k}|\big)$. The proofs of statements (2)-(4) are similar to [30, Proposition 1]. $\square$

Remark 6.5.4. In (3), $2C$-unconditionality cannot be replaced by C-unconditionality, even if $C = 1$. Indeed, suppose (e_k) is the canonical basis in ℓ_2, and $p = 1$. We first show that $\big\|\big|\delta_{e_1}\big| + \big|\delta_{e_2}\big|\big\|_{\mathrm{FBL}[\ell_2]} = 2$. Let $R : \ell_2 \to L_2$ be the Rademacher mapping, which is known to be isometric. This lifts to a lattice homomorphism $\widehat{R} : \mathrm{FBL}[\ell_2] \to L_2$ of norm one. Hence,

$$2 \geq \left\|\sum_{k=1}^{2} |\delta_{e_k}|\right\| = \|R\|\left\|\sum_{k=1}^{2} |\delta_{e_k}|\right\| \geq \left\|\widehat{R}\sum_{k=1}^{2} |\delta_{e_k}|\right\| = \left\|\sum_{k=1}^{2} |r_k|\right\| = 2.$$

To contrast this we shall show that $\big\||\delta_{e_1}| - |\delta_{e_2}|\big\| = \sqrt{2}$. It follows from $\big||\delta_{e_1}| - |\delta_{e_2}|\big| \leqslant |\delta_{e_1} - \delta_{e_2}|$ that

$$\big\||\delta_{e_1}| - |\delta_{e_2}|\big\| \leqslant \|\delta_{e_1} - \delta_{e_2}\| = \|e_1 - e_2\| = \sqrt{2}.$$

For the converse inequality, let $T\colon \ell_2^2 \to \ell_1^2$ be the formal identity. Then

$$\big\||\delta_{e_1}| - |\delta_{e_2}|\big\| \geqslant \frac{1}{\|T\|}\big\||Te_1| - |Te_2|\big\| = \frac{1}{\sqrt{2}} \cdot 2 = \sqrt{2}.$$

Note that in Proposition 6.5.3 it was shown that if (x_k) is a *basis* of E, then $(|\delta_{x_k}|)$ is basic in $\mathrm{FBL}^{(p)}[E]$. However, the following question is open.

Question 6.5.5. Suppose (x_k) is a basic sequence in a Banach space E. Is the sequence $(|\delta_{x_k}|)$ basic in $\mathrm{FBL}^{(p)}[E]$? If (x_k) is, further, unconditional, is $(|\delta_{x_k}|)$ unconditional as well?

We now present some partial progress on this question:

Proposition 6.5.6. *If E has a basis (u_k), then for every block basic sequence (x_k) of (u_k) the sequence $(|\delta_{x_k}|)$ is basic in* $\mathrm{FBL}^{(p)}[E]$.

Proof. By a well-known result of Zippin (cf. [9, Lemma 9.5.5]), a basis (f_n) of E can be constructed such that (x_k) is a subbasis of (f_n), say $f_{n_k} = x_k$. By Proposition 6.5.3, we have that $(|\delta_{f_n}|)$ is a basic sequence in $\mathrm{FBL}^{(p)}[E]$. Hence, being a subsequence, so is $(|\delta_{x_k}|)$. $\square$

Remark 6.5.7. Using that $\||\delta_{x_k}| - |\delta_{y_k}|\| \leq \|x_k - y_k\|$, it follows from Proposition 6.5.6 and the principle of small perturbations that if E has a basis (u_k) then small perturbations of blocks of (u_k) have moduli that are basic in $\mathrm{FBL}^{(p)}[E]$.

A well-known result due to Bessaga and Pełczyński allows one to extract a basic sequence from every semi-normalized weakly null sequence in a Banach space E. We will see next that this extraction can be made so that the corresponding sequence of moduli in $\mathrm{FBL}^{(p)}[E]$ is also basic:

Proposition 6.5.8. *Let E be a Banach space and (x_n) a weakly null semi-normalized sequence in E. There is a subsequence such that $(|\delta_{x_{n_k}}|)$ is basic in* $\mathrm{FBL}^{(p)}[E]$.

Proof. Due to [302] there is a separable subspace $F \subseteq E$ which is an ideal in E and such that $(x_n) \subseteq F$. Hence, by Corollary 6.4.14, $\mathrm{FBL}^{(p)}[F]$ is an isometric sublattice of $\mathrm{FBL}^{(p)}[E]$. Therefore, for our purposes, we can assume without loss of generality that E is actually separable.

Let us suppose that the set $S = \{|\delta_{x_n}|\} \subseteq \mathrm{FBL}^{(p)}[E]$ does not contain any basic sequence. Since (x_n) is semi-normalized, $0 \notin \overline{S}^{\|\cdot\|}$, hence, by [9, Theorem 1.5.6], the weak-closure of S, $K = \overline{S}^{w}$ is a weakly compact set with $0 \notin K$.

Now, let $\iota : E \to C[0,1]$ be an isometric embedding and $\bar{\iota} : \mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[C[0,1]]$ the induced lattice homomorphism. Since $\bar{\iota}$ is weak-weak continuous and injective, it follows that $\bar{\iota}(K)$ is a weakly-compact set and $0 \notin \bar{\iota}(K)$. Thus, again by [9, Theorem 1.5.6], $(|\delta_{\iota x_n}|) = (\bar{\iota}|\delta_{x_n}|) \subseteq \mathrm{FBL}^{(p)}[C[0,1]]$ does not contain a basic sequence.

However, $\iota(x_n)$ is a semi-normalized weakly null sequence in $C[0,1]$. Therefore, we can extract a subsequence $\iota(x_{n_k})$ which is a small perturbation of a block basic sequence of the monotone basis of $C[0,1]$. Hence, Remark 6.5.7 yields that for some subsequence $(|\delta_{\iota(x_{n_m})}|)$ is a basic sequence in $\mathrm{FBL}^{(p)}[C[0,1]]$. This is a contradiction. We can thus assume that S contains a basic sequence, which implies that we can extract an increasing sequence $(n_k) \subseteq \mathbb{N}$ such that $(|\delta_{x_{n_k}}|)$ is basic in $\mathrm{FBL}^{(p)}[E]$, as claimed. ∎

Remark 6.5.9. Of course, building on Proposition 6.5.3, it is natural to study how other properties pass between the sequences (x_k) and $(|\delta_{x_k}|)$ (e.g. shrinking, boundedly complete, etc.) Although this will not be our focus, our results will indirectly shed partial light on such questions. In particular, we will discover some "rigidity" results, i.e., properties of $(|\delta_{x_k}|)$ that force (x_k) to take a particular form.

Lower 2-estimates, ℓ_1, and c_0

In this subsection we explicitly compute the moduli of certain bases, refining some results from [30]. We begin by characterizing the behaviour of c_0 in $\mathrm{FBL}^{(p)}[c_0]$:

Proposition 6.5.10. *If (e_k) is the canonical basis of c_0 then the sequence $(|\delta_{e_k}|)$ in $\mathrm{FBL}^{(p)}[c_0]$ is equivalent to the canonical basis of ℓ_2 for all $1 \le p < \infty$.*

Proof. By Proposition 6.5.3, the sequence $\big(|\delta_{e_k}|\big)_k$ is an unconditional basic sequence. We shall show that, for finitely supported sequences (a_k),

$$\Big\| \sum_{k=1}^{\infty} a_k |\delta_{e_k}| \Big\|_{\mathrm{FBL}^{(p)}[c_0]} \sim \left(\sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2}$$

holds (with an equivalence constant depending on $p \in [1, \infty)$). Fix such (a_k). Let

$$A_+ = \{k : a_k \ge 0\} \quad \text{and} \quad A_- = \{k : a_k < 0\}.$$

Define an operator $T \colon c_0 \to L_p[0,1]$ via

$$Tx = \sum_{k \in A_+} a_k e_k^*(x) r_k.$$

If $\|x\| \leqslant 1$ then Khintchine's inequality yields

$$\|Tx\| \leqslant B_p \Big(\sum_{k \in A_+} \big|a_k e_k^*(x)\big|^2 \Big)^{\frac{1}{2}} \leqslant B_p \Big(\sum_{k \in A_+} a_k^2 \Big)^{\frac{1}{2}}.$$

It follows that $\|T\| \leqslant B_p \Big(\sum_{k \in A_+} a_k^2 \Big)^{\frac{1}{2}}$. Note that Te_k equals $a_k r_k$ if $k \in A_+$ and zero otherwise.

Let $\widehat{T} \colon \mathrm{FBL}^{(p)}[c_0] \to L_p[0,1]$ be the canonical extension of T. Then $\|\widehat{T}\| \leqslant B_p \Big(\sum_{k \in A_+} a_k^2 \Big)^{\frac{1}{2}}$ and

$$\widehat{T}\Big(\sum_{k=1}^{\infty} a_k \,|\delta_{e_k}| \Big) = \sum_{k=1}^{\infty} a_k \,|Te_k| = \sum_{k \in A_+} a_k \,|a_k r_k| = \Big(\sum_{k \in A_+} a_k^2 \Big) \mathbf{1}.$$

It follows that

$$\sum_{k\in A_+} a_k^2 = \Big\|\widehat{T}\Big(\sum_{k=1}^{\infty} a_k\,|\delta_{e_k}|\Big)\Big\| \leqslant B_p\Big(\sum_{k\in A_+} a_k^2\Big)^{\frac{1}{2}}\Big\|\sum_{k=1}^{\infty} a_k\,|\delta_{e_k}|\Big\|,$$

so that

$$B_p\Big\|\sum_{k=1}^{\infty} a_k\,|\delta_{e_k}|\Big\| \geqslant \Big(\sum_{k\in A_+} a_k^2\Big)^{\frac{1}{2}}.$$

Similarly, we get

$$B_p\Big\|\sum_{k=1}^{\infty} a_k\,|\delta_{e_k}|\Big\| \geqslant \Big(\sum_{k\in A_-} a_k^2\Big)^{\frac{1}{2}}.$$

Combining these estimates, we get

$$\sqrt{2}B_p\Big\|\sum_{k=1}^{\infty} a_k\,|\delta_{e_k}|\Big\|_{\mathrm{FBL}^{(p)}[c_0]} \geqslant \Big(\sum_{k=1}^{\infty} |a_k|^2\Big)^{\frac{1}{2}}.$$

Conversely, it was shown in [30] that

$$\Big\|\sum_{k=1}^{\infty} a_k|\delta_{e_k}|\Big\|_{\mathrm{FBL}[c_0]} \sim \left(\sum_{k=1}^{\infty} |a_k|^2\right)^{1/2}.$$

Hence, since the FBL norm is the largest of the $\mathrm{FBL}^{(p)}$ norms,

$$\Big\|\sum_{k=1}^{\infty} a_k\,|\delta_{e_k}|\Big\|_{\mathrm{FBL}^{(p)}[c_0]} \leq \Big\|\sum_{k=1}^{\infty} a_k\,|\delta_{e_k}|\Big\|_{\mathrm{FBL}[c_0]} \sim \left(\sum_{k=1}^{\infty} |a_k|^2\right)^{1/2}.$$

□

Corollary 6.5.11. *Suppose a sequence (e_k) in E is equivalent to the canonical basis of c_0. Then the sequence $(|\delta_{e_k}|)$ in $\mathrm{FBL}^{(p)}[E]$ is equivalent to the canonical basis of ℓ_2 for all $1 \leq p < \infty$.*

Proof. Combine Proposition 6.5.10 with the fact that c_0 has the POE-p (Corollary 6.4.12).

□

Combined with Proposition 6.5.10 the next result establishes a lower 2-estimate for the moduli of an arbitrary basic sequence:

Proposition 6.5.12. *Let (x_k) be a sequence in E, and assume that there are biorthogonal functionals (x_k^*) to (x_k) such that $K := \sup_k \|x_k^*\| < \infty$. Then for any finitely supported sequence of scalars (a_k) we have*

$$\Big\|\sum_{k=1}^{\infty} a_k|\delta_{e_k}|\Big\|_{\mathrm{FBL}^{(p)}[c_0]} \leq K\Big\|\sum_{k=1}^{\infty} a_k|\delta_{x_k}|\Big\|_{\mathrm{FBL}^{(p)}[E]},$$

where (e_k) denotes the unit vector basis of c_0. Consequently, for $1 \leq p < \infty$,

$$\left(\sum_{k=1}^{\infty} a_k^2\right)^{1/2} \lesssim \Big\|\sum_{k=1}^{\infty} a_k|\delta_{x_k}|\Big\|_{\mathrm{FBL}^{(p)}[E]}.$$

Proof. The assumptions tell us that the operator $T : [x_k] \to \ell_\infty$ given by $Tx = (x_k^*(x))$ has norm at most K and $Tx_k = e_k$. By injectivity of ℓ_∞ (or Hahn-Banach), we have an extension $\widetilde{T} : E \to \ell_\infty$ with $\|\widetilde{T}\| \leq K$. Let $\phi_{\ell_\infty} : \ell_\infty \to \mathrm{FBL}^{(p)}[\ell_\infty]$ denote the canonical isometric embedding and let $S = \phi_{\ell_\infty}\widetilde{T} : E \to \mathrm{FBL}^{(p)}[\ell_\infty]$. Let now $\widehat{S} : \mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[\ell_\infty]$ be the lattice homomorphism extending S, and note that $\|\widehat{S}\| \leq K$. It follows that for any finitely supported sequence of scalars (a_k) we have

$$\Big\|\sum_{k=1}^{\infty} a_k|\delta_{e_k}|\Big\|_{\mathrm{FBL}^{(p)}[\ell_\infty]} = \Big\|\sum_{k=1}^{\infty} a_k|\widehat{S}\delta_{x_k}|\Big\|_{\mathrm{FBL}^{(p)}[\ell_\infty]} \leq K\Big\|\sum_{k=1}^{\infty} a_k|\delta_{x_k}|\Big\|_{\mathrm{FBL}^{(p)}[E]}.$$

Using Proposition 6.4.10 and the above estimate we get that

$$\Big\|\sum_{k=1}^{\infty} a_k|\delta_{e_k}|\Big\|_{\mathrm{FBL}^{(p)}[c_0]} = \Big\|\sum_{k=1}^{\infty} a_k|\delta_{e_k}|\Big\|_{\mathrm{FBL}^{(p)}[\ell_\infty]} \leq K\Big\|\sum_{k=1}^{\infty} a_k|\delta_{x_k}|\Big\|_{\mathrm{FBL}^{(p)}[E]}.$$

Finally, the "consequently" statement follows from Proposition 6.5.10. □

Statement (3) of Proposition 6.4.19 suggests that ℓ_1 could be a counterexample to Question 6.5.5. However, basic sequences equivalent to ℓ_1 will always have moduli equivalent to ℓ_1 in free spaces. More generally, we have the following proposition:

Proposition 6.5.13. *Suppose (x_k) is a C-unconditional basic sequence in a Banach space E. Then, for $1 \leq p \leq \infty$, and for any $a_1, \dots, a_n \in \mathbb{R}$,*

$$\|\sum_{k=1}^{n} a_k|\delta_{x_k}|\|_{\mathrm{FBL}^{(p)}[E]} \geq \frac{1}{2C}\|\sum_{k=1}^{n} a_k x_k\|.$$

Proof. It suffices to consider $p = \infty$, as this is the weakest of the $\mathrm{FBL}^{(p)}$-norms. By Proposition 6.3.10, we may assume that (x_k) is a basis. The result then follows from statement (3) of Proposition 6.5.3:

$$\begin{aligned} 2C\Big\|\sum_{k=1}^n a_k|\delta_{x_k}|\Big\|_{\mathrm{FBL}^{(\infty)}[E]} &\geq \Big\|\sum_{k=1}^n |a_k||\delta_{x_k}|\Big\|_{\mathrm{FBL}^{(\infty)}[E]} \\ &\geq \Big\|\sum_{k=1}^n a_k\delta_{x_k}\Big\|_{\mathrm{FBL}^{(\infty)}[E]} = \Big\|\sum_{k=1}^n a_k x_k\Big\|. \end{aligned}$$

□

Remark 6.5.19 below will show that the unconditionality assumption in the preceding proposition is essential, in general, for the sequence of moduli to dominate the original sequence.

We now look to characterize those bases (x_k) of E such that the sequence $(|\delta_{x_k}|)$ in $\mathrm{FBL}^{(p)}[E]$ is equivalent to the unit vector basis of ℓ_1. We begin with a sufficient condition:

Proposition 6.5.14. *Let E be a Banach space with a normalized basis (x_k) satisfying a lower 2-estimate. Then for all $p \in [1,\infty)$ the sequence $(|\delta_{x_k}|)$ in $\mathrm{FBL}^{(p)}[E]$ is equivalent to the unit vector basis of ℓ_1.*

Proof. We first prove this for the unit vector basis (e_k) of ℓ_2. Let $R : \ell_2 \to L_p$ be the Rademacher mapping. This extends to a lattice homomorphism $\widehat{R} : \mathrm{FBL}^{(p)}[\ell_2] \to L_p$. Now,

$$\|R\|\left\|\sum_{k=1}^n a_k|\delta_{e_k}|\right\| \gtrsim \|R\|\left\|\sum_{k=1}^n |a_k||\delta_{e_k}|\right\| \geq \left\|\widehat{R}\sum_{k=1}^n |a_k||\delta_{e_k}|\right\| = \left\|\sum_{k=1}^n |a_k||r_k|\right\| = \sum_{k=1}^n |a_k|,$$

where the first domination is by the unconditionality statement in Proposition 6.5.3.

Now suppose that (x_k) is normalized with a lower 2-estimate. Then the basis to basis map $T : E \to \ell_2$ is bounded, so we can extend it to a lattice homomorphism $\overline{T} : \mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[\ell_2]$. Note $\overline{T}(|\delta_{x_k}|) = |\delta_{e_k}|$. From this we get that

$$\|T\|\left\|\sum_{k=1}^n a_k|\delta_{x_k}|\right\| \geq \left\|\sum_{k=1}^n a_k|\delta_{e_k}|\right\| \sim \sum_{k=1}^n |a_k|.$$

□

Example 6.5.15. Let $1 \le r \le 2$ and let (e_k) be the canonical basis of ℓ_r. Then the sequence $(|\delta_{e_k}|)$ in $\mathrm{FBL}^{(p)}[\ell_r]$ is equivalent to the canonical basis of ℓ_1, when $p \in [1, \infty)$. Similarly, the Walsh basis in $L_r[0,1]$ is normalized and satisfies a lower 2-estimate if $1 \le r \le 2$.

One cannot replace "basis" with "basic sequence" in Proposition 6.5.14, see Proposition 6.6.5. The dual to the summing basis in c_0 (see below) satisfies the conclusion but not the hypothesis of Proposition 6.5.14:

Example 6.5.16. Let (x_k) be the basis for ℓ_1 such that $x_1 = e_1$ and $x_k = e_k - e_{k-1}$ for $k \ge 2$. Then for all $p \in [1, \infty]$ the sequence $(|\delta_{x_k}|)$ in $\mathrm{FBL}^{(p)}[\ell_1]$ is equivalent to the unit vector basis of ℓ_1.

Proof. By Remark 6.2.8 it suffices to work with $\mathrm{FBL}^{(\infty)}[\ell_1]$. Recall that the norm

$$\left\| \sum_{k=1}^{m} a_k |\delta_{x_k}| \right\|$$

is computed by taking $\sup_{f \in B_{\ell_\infty}} \left| \sum_{k=1}^m a_k |f(x_k)| \right|$.

Choosing $f = (1, -1, 1, -1, 1, -1, 1, -1, \dots)$ we get

$$\left| \sum_{k=1}^{m} a_k |f(x_k)| \right| = |a_1 + 2a_2 + 2a_3 + \cdots + 2a_m|.$$

This tells us $\left(|\delta_{x_k}|\right)_k$ is either conditional, or equivalent to the ℓ_1 basis.

Now, in general, to take care of signs, one picks f to be a sequence of ones and negative ones, but now aligns them with the signs of the a_k (we can safely ignore x_1). This easily gives

$$\left\| \sum_{k=2}^{m} a_k |\delta_{x_k}| \right\|_{\mathrm{FBL}^{(\infty)}[\ell_1]} \ge \max \left(\left| \sum_{k: a_k \ge 0} 2a_k \right|, \left| \sum_{k: a_k \le 0} 2a_k \right| \right) \ge \sum_{k=2}^{m} |a_k|,$$

and proves the claim. $\square$

In contrast to Example 6.5.16, Proposition 6.5.14 is sharp for unconditional bases:

Proposition 6.5.17. *Suppose that (x_k) is a 1-unconditional basic sequence in E, $a_1, \dots, a_n \geqslant 0$, and $1 \leqslant p \leqslant \infty$. Then*

$$\left\| \sum_{k=1}^{n} a_k \, |\delta_{x_k}| \right\|_{\mathrm{FBL}^{(p)}[E]} \leqslant K_G \Big(\sum_{k=1}^{n} a_k \Big)^{\frac{1}{2}} \left\| \sum_{k=1}^{n} \sqrt{a_k} x_k \right\|_E. \tag{6.5.1}$$

Here, K_G denotes the universal Grothendieck constant. In particular, suppose (x_k) fails to have a lower 2-estimate – that is, there exist $t_1, \dots, t_n$ so that $\| \sum_{k=1}^n t_k x_k \| \ll (\sum_{k=1}^n t_k^2)^{1/2}$. Then by (6.5.1),

$$\Big\| \sum_{k=1}^n t_k^2 |\delta_{x_k}| \Big\|_{\mathrm{FBL}^{(p)}[E]} \ll \sum_{k=1}^n t_k^2,$$

implying that $(|\delta_{x_k}|)$ is not equivalent to the ℓ_1 basis.

Proof. Let $F = \overline{\mathrm{span}}[x_k : k \in \mathbb{N}]$. Let $T = \phi_E|_F \colon F \to \mathrm{FBL}^{(p)}[E]$ be the natural inclusion. In $\mathrm{FBL}^{(p)}[E]$, using Cauchy-Schwarz inequality we have

$$\sum_{k=1}^n a_k \, |\delta_{x_k}| \leqslant \Big(\sum_{k=1}^n a_k \Big)^{\frac{1}{2}} \Big(\sum_{k=1}^n a_k \, |\delta_{x_k}|^2 \Big)^{\frac{1}{2}} = \Big(\sum_{k=1}^n a_k \Big)^{\frac{1}{2}} \Big(\sum_{k=1}^n \big| T(\sqrt{a_k} x_k) \big|^2 \Big)^{\frac{1}{2}}.$$

View F as a Banach lattice under the order induced by (x_k). Using Krivine's inequality [231, Theorem 1.f.14.], and the fact that (x_k) are disjoint in F, we get

$$\begin{aligned}\Big\| \sum_{k=1}^n a_k \, |\delta_{x_k}| \Big\|_{\mathrm{FBL}^{(p)}[E]} &\leqslant \Big(\sum_{k=1}^n a_k \Big)^{\frac{1}{2}} \cdot K_G \, \|T\| \, \Big\| \Big(\sum_{k=1}^n \big| (\sqrt{a_k} x_k) \big|^2 \Big)^{\frac{1}{2}} \Big\|_F \\ &= K_G \Big(\sum_{k=1}^n a_k \Big)^{\frac{1}{2}} \Big\| \sum_{k=1}^n \sqrt{a_k} x_k \Big\|_E .\end{aligned}$$

□

As already mentioned, if (x_k) is an unconditional basis of E, then the sequence $(|\delta_{x_k}|)$ is unconditional in $\mathrm{FBL}^{(p)}[E]$. As we saw in Example 6.5.16, the modulus of a conditional basis need not be conditional - it also need not be unconditional. For the sake of an example, consider the summing basis in c_0, consisting of vectors $s_k = (1, \dots, 1, 0, \dots)$ (k 1's in a row).

Proposition 6.5.18. *For all $p \in [1, \infty]$ the basic sequence $\big(|\delta_{s_k}| \big)$ is conditional (even for constant coefficients) in $\mathrm{FBL}^{(p)}[c_0]$. In fact, if n is an even integer, then*

$$\Big\| \sum_{k=1}^n |\delta_{s_k}| \Big\|_{\mathrm{FBL}^{(\infty)}[c_0]} = n,$$

but:

(i) $\big\| \sum_{k=1}^n (-1)^k |\delta_{s_k}| \big\|_{\mathrm{FBL}^{(\infty)}[c_0]} = 1.$

(ii) $\kappa\sqrt{n} \leq \big\|\sum_{k=1}^n (-1)^k \big|\delta_{s_k}\big|\big\|_{\mathrm{FBL}[c_0]} \leq K_G\sqrt{n}$*, where* K_G *is Grothendieck's constant, and* $\kappa > 0$ *is a universal constant.*

Proof. (1) The case of $\mathrm{FBL}^{(\infty)}[c_0]$:

The norm on $\mathrm{FBL}^{(\infty)}[c_0]$ arises from $\|f\| = \sup\big\{|f(x^*)| : \|x^*\|_{\ell_1} \leq 1\big\}$. By the triangle inequality, $\big\|\sum_{k=1}^n \big|\delta_{s_k}\big|\big\| \leq \sum_{k=1}^n \|s_k\| = n$. For a lower estimate take $x^* = (1, 0, 0, \ldots) \in \ell_1$. Then

$$\big\|\sum_{k=1}^n \big|\delta_{s_k}\big|\big\| \geq \sum_{k=1}^n \big|x^*(s_k)\big| = n.$$

Now recall that

$$\big\|\sum_{k=1}^n (-1)^k \big|\delta_{s_k}\big|\big\| = \sup_{\|x^*\|\leq 1} \Big|\sum_{k=1}^n (-1)^k |x^*(s_k)|\Big|.$$

Write $x^* = (a_1, a_2, \ldots)$. Then $x^*(s_k) = a_1 + \cdots + a_k$; hence

$$\big|\sum_{k=1}^n (-1)^k |x^*(s_k)|\big| = \Big|\sum_{j=1}^{n/2} \Big(\big|\sum_{k=1}^{2j} a_k\big| - \big|\sum_{k=1}^{2j-1} a_k\big|\Big)\Big| \leq \sum_{j=1}^{n/2} \Big|\big|\sum_{k=1}^{2j} a_k\big| - \big|\sum_{k=1}^{2j-1} a_k\big|\Big|.$$

By the triangle inequality, $\Big|\big|\sum_{k=1}^{2j} a_k\big| - \big|\sum_{k=1}^{2j-1} a_k\big|\Big| \leq |a_{2j}|$, hence

$$\big|\sum_{k=1}^n (-1)^k |x^*(s_k)|\big| \leq \sum_{j=1}^{n/2} \big|a_{2j}\big| \leq \|x^*\| \leq 1,$$

which leads us to conclude that

$$\big\|\sum_{k=1}^n (-1)^k \big|\delta_{s_k}\big|\big\| = \sup_{\|x^*\|\leq 1} \big|\sum_{k=1}^n (-1)^k |x^*(s_k)|\big| \leq 1.$$

On the other hand, testing on $x^* = (0, 1, 0, \ldots)$, we obtain $\big\|\sum_{k=1}^n (-1)^k \big|\delta_{s_k}\big|\big\| \geq 1$.

(2) The case of $\mathrm{FBL}[c_0]$:

The lower estimate for $\big\|\sum_{k=1}^n (-1)^k \big|\delta_{s_k}\big|\big\|$ follows from Proposition 6.5.12 (the norms of the biorthogonal functionals $(0, \ldots, 0, 1, -1, 0, \ldots) \in \ell_1$ do not exceed 2).

For an upper estimate, we view $s_1, \dots, s_n$ as living in ℓ_∞^n (spanned by the first n coordinates of c_0). We need to prove that, if $x_1^*, \dots, x_m^* \in \ell_1^n$ are such that $\max_\pm \big\|\sum_{j=1}^m \pm x_j^*\big\| \le 1$, then

$$\sum_{j=1}^m \Big| \sum_{k=1}^n (-1)^k |x_j^*(s_k)| \Big| \le K_G \sqrt{n}.$$

As noted in part (1),

$$\Big| \sum_{k=1}^n (-1)^k |x_j^*(s_k)| \Big| \le \|x_j^*\|.$$

We have to therefore show that, for our sequence (x_j^*), $\sum_j \|x_j^*\| \le K_G\sqrt{n}$. To this end, consider the operator $u : \ell_\infty^m \to \ell_1^n : e_j \mapsto x_j^*$ (here $(e_j)_{j=1}^m$ is the canonical basis of ℓ_∞^m). Note that $\sum_j \|x_j^*\| = \sum_j \|ue_j\|$, and $\|\sum_j \delta_j e_j\| = 1$ whenever $\delta_j = \pm 1$; thus, $\sum_j \|ue_j\| \le \pi_1(u)$. Therefore, it suffices to show that $\pi_1(u) \le K_G\sqrt{n}$.

Note that $\|u\| = \sup_{\delta_j=\pm 1}\big\|\sum_{j=1}^m \delta_j x_j^*\big\| \le 1$, hence, by Grothendieck's Theorem (see e.g. [97, Theorem 3.5]), $\pi_2(u) \le K_G$. Write $u = id \circ u$, where id is the identity operator on ℓ_1^n. Then by [97, Theorem 4.17], $\pi_2(id) = \sqrt{n}$, and therefore [97, Theorem 2.22] implies that $\pi_1(u) = \pi_1(id \circ u) \le \pi_2(id)\pi_2(u) \le K_G\sqrt{n}$.

We can now interpolate these results to general p: We know that, for even n,

$$\Big\|\sum_{k=1}^n |\delta_{s_k}|\Big\|_{\mathrm{FBL}^{(\infty)}[c_0]} = n \quad \text{ and } \quad \Big\|\sum_{k=1}^n (-1)^k |\delta_{s_k}|\Big\|_{\mathrm{FBL}[c_0]} \lesssim n^{1/2}.$$

If $(|\delta_{s_k}|)$ were unconditional in $\mathrm{FBL}^{(p)}[c_0]$ then there would exist a C such that

$$\Big\|\sum_{k=1}^n |\delta_{s_k}|\Big\|_{\mathrm{FBL}^{(p)}[c_0]} \le C\Big\|\sum_{k=1}^n (-1)^k |\delta_{s_k}|\Big\|_{\mathrm{FBL}^{(p)}[c_0]}.$$

But now using that the $\mathrm{FBL}^{(\infty)}$-norm is minimal and the FBL-norm is maximal, we get

$$\begin{aligned} n = \Big\|\sum_{k=1}^n |\delta_{s_k}|\Big\|_{\mathrm{FBL}^{(\infty)}[c_0]} &\le \Big\|\sum_{k=1}^n |\delta_{s_k}|\Big\|_{\mathrm{FBL}^{(p)}[c_0]} \\ &\le C\Big\|\sum_{k=1}^n (-1)^k |\delta_{s_k}|\Big\|_{\mathrm{FBL}^{(p)}[c_0]} \le C\Big\|\sum_{k=1}^n (-1)^k |\delta_{s_k}|\Big\|_{\mathrm{FBL}[c_0]} \lesssim n^{1/2}, \end{aligned}$$

a contradiction. $\square$

Remark 6.5.19. Proposition 6.5.13 states that, if (x_k) is an unconditional basis in a Banach space E, then the inequality $\big\| \sum_k a_k \big|\delta_{x_k}\big|\big\| \geq c \big\| \sum_k a_k x_k \big\|$ holds, with a constant c independent of (a_k). However, this is false for conditional bases. Indeed, consider the "alternating summing" basis $s'_k = (-1)^k s_k$ in c_0. It is easy to see that, for any n, $\big\| \sum_{k=1}^n (-1)^k s'_k \big\| = \big\| \sum_{k=1}^n s_k \big\| = n$. However, $\big\|\sum_{k=1}^n (-1)^k \big|\delta_{s'_k}\big|\big\|_{\mathrm{FBL}[c_0]} = \big\|\sum_{k=1}^n (-1)^k \big|\delta_{s_k}\big|\big\|_{\mathrm{FBL}[c_0]} \sim \sqrt{n}$, due to Proposition 6.5.18.

Example: moduli of the Haar system in $L_1[0,1]$

In what follows, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $(h_{n,k})_{1\leq k\leq 2^n, n\in\mathbb{N}_0}$ denote the normalized Haar basis in L_1. That is,

$$h_{n,k} = 2^n \chi_{[\frac{k-1}{2^n}, \frac{2k-1}{2^{n+1}}]} - 2^n \chi_{[\frac{2k-1}{2^{n+1}}, \frac{k}{2^n}]}.$$

For more information about the Haar system the reader is referred to e.g. [231, Section 2.c]. Clearly, for a fixed $n \in \mathbb{N}_0$, $(h_{n,k})_{k=1}^{2^n}$ is 1-equivalent to the basis of $\ell_1^{2^n}$, and so is $(|\delta_{h_{n,k}}|)_{k=1}^{2^n}$ in $\mathrm{FBL}[L_1]$ as the following shows:

Lemma 6.5.20. *For every $n \in \mathbb{N}$, and scalars $(a_k)_{k=1}^{2^n}$ we have*

$$\Big\| \sum_{k=1}^{2^n} a_k |\delta_{h_{n,k}}| \Big\|_{\mathrm{FBL}[L_1]} = \sum_{k=1}^{2^n} |a_k|.$$

Proof. Since $\|h_{n,k}\| = 1$, the triangle inequality trivially implies that $\| \sum_{k=1}^{2^n} a_k |\delta_{h_{n,k}}| \| \leq \sum_{k=1}^{2^n} |a_k|$.

For the converse, let $T : L_1 \to \ell_1^{2^n}$ be the operator given by

$$Tf = \left(\int_{\frac{k-1}{2^n}}^{\frac{2k-1}{2^{n+1}}} f d\mu - \int_{\frac{2k-1}{2^{n+1}}}^{\frac{k}{2^n}} f d\mu \right)_{k=1}^{2^n}.$$

Clearly, $\|T\| = 1$ and $Th_{n,k} = e_k$ for $1 \leq k \leq 2^n$. Let $\widehat{T} : \mathrm{FBL}[L_1] \to \ell_1^{2^n}$ denote the lattice homomorphism extending T. Note $\widehat{T}|\delta_{h_{n,k}}| = |Th_{n,k}| = e_k$, which implies that

$$\sum_{k=1}^{2^n} |a_k| = \Big\| \sum_{k=1}^{2^n} a_k e_k \Big\| = \Big\| \widehat{T}\Big(\sum_{k=1}^{2^n} a_k |\delta_{h_{n,k}}| \Big) \Big\| \leq \Big\| \sum_{k=1}^{2^n} a_k |\delta_{h_{n,k}}| \Big\|.$$

$\square$

By a branch of the Haar basis, we mean any sequence $(h_{n_j,k_j})_{j\in\mathbb{N}}$ such that, for each $j \in \mathbb{N}$, the support of $h_{n_{j+1},k_{j+1}}$ is contained in that of h_{n_j,k_j}.

Lemma 6.5.21. *For every branch $(h_{n_j,k_j})_{j\in\mathbb{N}}$ of the Haar basis, we have that the sequence $(|\delta_{h_{n_j,k_j}}|)_{j\in\mathbb{N}}$ in $\mathrm{FBL}[L_1]$ is equivalent to the ℓ_1 basis.*

Proof. We will do the computations for $(h_{n,1})_{n\in\mathbb{N}_0}$, since this can be translated to any other branch. Let $T : L_1 \to \ell_1(\mathbb{N}_0)$ be the norm 1 operator defined by $Tf = (\int_{2^{-k-1}}^{2^{-k}} f d\mu)_{k\in\mathbb{N}_0}$, and let $\widehat{T} : \mathrm{FBL}[L_1] \to \ell_1(\mathbb{N}_0)$ denote the lattice homomorphism extending T. Note that

$$(Th_{n,1})_k = \begin{cases} 0 & \text{if } k < n, \\ -\frac{1}{2} & \text{if } k = n, \\ \frac{1}{2^{k-n+1}} & \text{if } k > n. \end{cases}$$

We claim that $(\widehat{T}|\delta_{h_{n,1}}|)_{n\in\mathbb{N}_0}$ is equivalent to the ℓ_1 basis. Indeed, $\widehat{T}|\delta_{h_{n,1}}| = |Th_{n,1}|$, which coincides with the sequence $(0, \ldots, 0, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$, starting with n zeros. Note that if $S : \ell_1 \to \ell_1$ denotes the right shift, and we set $R = \sum_{n\in\mathbb{N}_0} \frac{1}{2^{n+1}} S^n$, then this defines an invertible operator with $R^{-1} = 2I - S$. Now, notice that for the unit vector basis (e_n) in ℓ_1 we have $Re_n = \widehat{T}|\delta_{h_{n,1}}|$ and this proves the claim. Therefore, there is $C > 0$ such that for every sequence of scalars $(a_n)_{n\in\mathbb{N}}$ we have

$$C\sum_{n\in\mathbb{N}} |a_n| \le \Big\| \sum_{n\in\mathbb{N}} a_n \widehat{T}|\delta_{h_{n,1}}| \Big\|_{\ell_1} \le \|T\| \Big\| \sum_{n\in\mathbb{N}} a_n |\delta_{h_{n,1}}| \Big\|_{\mathrm{FBL}[L_1]} \le \sum_{n\in\mathbb{N}} |a_n|.$$

This finishes the proof. $\square$

Remark 6.5.22. We do not know whether the double indexed sequence $(|\delta_{h_{n,k}}|)$ is equivalent to the ℓ_1 basis. However, $(h_{n,k})$ is a monotone basis in $L_1(0,1)$ (see e.g. [231, Section 2.c]), hence, by Proposition 6.6.13 below,

$$\| \sum_{n,k} a_{n,k} |\delta_{h_{n,k}}| \| \ge \frac{1}{2} \sum_{n,k} a_{n,k}$$

whenever $(a_{n,k})$ are positive scalars.

6.6 Sequences with prescribed moduli

In this section, we continue our investigation of connections between properties of the sequence $(x_k) \subseteq E$, and those of $\big(|\delta_{x_k}|\big) \subseteq \mathrm{FBL}^{(p)}[E]$. In Section 6.6, we show how p-summing norms can be used to compute the norm of certain expressions on $\mathrm{FBL}^{(p)}[E]$, which will be a helpful tool in the sequel. In Section 6.6, we show that, under fairly general conditions, if (x_k) and $\left(\left|\delta_{x_k}\right|\right)$ are equivalent, then they both are equivalent to the ℓ_1 basis. We also give examples showing that our conditions are necessary, and characterize when the span of $(|\delta_{x_k}|)$ is complemented in $\mathrm{FBL}^{(p)}[E]$. Section 6.6 is devoted to analysing $\left(\left|\delta_{x_k}\right|\right)$ for sequences $(x_k) \subseteq L_1$. Finally, in Section 6.6 we investigate (x_k)'s for which $\left(\left|\delta_{x_k}\right|\right)$ is equivalent to the ℓ_2 basis.

Using linear operators to compute non-linear expressions

Throughout this subsection, we fix $p \in [1, \infty]$. For $\overline{x} = (x_1, \dots, x_n) \in E^n$, we define the operator

$$T_{\overline{x}} : E^* \to \ell_p^n : x^* \mapsto (x^*(x_k))_{k=1}^n.$$

Note $T_{\overline{x}} = S_{\overline{x}}^*$ for $S_{\overline{x}}$ the operator given by

$$S_{\overline{x}} : \ell_q^n \to E : e_k \mapsto x_k,$$

where q is conjugate to p ($\frac{1}{p} + \frac{1}{q} = 1$) and $(e_k)_{k=1}^n$ is the canonical basis of ℓ_q^n.

Proposition 6.6.1. *In the above notation,*

$$\Big\| \Big(\sum_{k=1}^n |\delta_{x_k}|^p \Big)^{1/p} \Big\|_{\mathrm{FBL}^{(p)}[E]} = \pi_p(T_{\overline{x}}). \tag{6.6.1}$$

Proof. Put $T = T_{\overline{x}}$ and $f = \left(\sum_{k=1}^n |\delta_{x_k}|^p \right)^{\frac{1}{p}}$. By the definition of π_p, we have

$$\begin{aligned} \pi_p(T) &= \sup\Bigg\{ \Big(\sum_{i=1}^m \|Tx_i^*\|^p \Big)^{\frac{1}{p}} : x_1^*, \dots, x_m^* \in E^*, \sup_{x^{**} \in B_{E^{**}}} \Big(\sum_{i=1}^m \big|x^{**}(x_i^*)\big|^p \Big)^{\frac{1}{p}} \le 1 \Bigg\} \\ &= \sup\Bigg\{ \Big(\sum_{i=1}^m \|Tx_i^*\|^p \Big)^{\frac{1}{p}} : x_1^*, \dots, x_m^* \in E^*, \sup_{x \in B_E} \Big(\sum_{i=1}^m \big|x_i^*(x)\big|^p \Big)^{\frac{1}{p}} \le 1 \Bigg\}, \end{aligned}$$

by an argument similar to (6.1.2). For each i, we have

$$\|Tx_i^*\| = \Big(\sum_{k=1}^n \big|x_i^*(x_k)\big|^p \Big)^{\frac{1}{p}} = f(x_i^*),$$

hence

$$\pi_p(T) = \sup\Bigg\{ \Big(\sum_{i=1}^m |f(x_i^*)|^p \Big)^{\frac{1}{p}} : \sup_{x \in B_E} \Big(\sum_{i=1}^m \big|x_i^*(x)\big|^p \Big)^{\frac{1}{p}} \le 1 \Bigg\} = \|f\|_{\mathrm{FBL}^{(p)}[E]}.$$

□

We now mention some corollaries. Suppose (x_k) is a 1-unconditional basis of E and $\alpha = (a_1, \dots, a_n) \in \mathbb{R}^n$. We use the notation $\overline{x}[\alpha] = (a_1x_1, \dots, a_nx_n) \in E^n$. The corresponding operator $T_{\overline{x}[\alpha]} : E^* \to \ell_p^n : x^* \mapsto (a_k x^*(x_k))_{k=1}^n$ is *diagonal* with respect to the 1-unconditional bases given by the biorthogonal functionals in E^* and ℓ_p^n respectively. The next result shows equivalence between the problem of computing the moduli of an unconditional basis in $\mathrm{FBL}[E]$, and the problem of computing a certain 1-summing norm:

Corollary 6.6.2. *Assume that (x_k) is a 1-unconditional basis of E, and let the notation be as above. For $\alpha = (a_1, \dots, a_n) \in \mathbb{R}^n$,*

$$\Big\|\sum_{k=1}^n a_k|\delta_{x_k}|\Big\|_{\mathrm{FBL}[E]} \le \pi_1(T_{\overline{x}[\alpha]}) \le 2\Big\|\sum_{k=1}^n a_k|\delta_{x_k}|\Big\|_{\mathrm{FBL}[E]}.$$

Proof. First apply Proposition 6.6.1 with x_k replaced by a_kx_k to obtain

$$\Big\|\sum_{k=1}^n |a_k||\delta_{x_k}|\Big\|_{\mathrm{FBL}[E]} = \pi_1\big(T_{\overline{x}[\alpha]}\big).$$

Now invoke Proposition 6.5.3 to get that $(|\delta_{x_k}|)$ is 2-unconditional. $\square$

In particular, for the canonical basis of ℓ_r, setting $\frac{1}{r} + \frac{1}{r'} = 1$ we have

$$\Big\|\sum_{k=1}^n a_k|\delta_{e_k}|\Big\|_{\mathrm{FBL}[\ell_r]} \sim \pi_1(\overline{\alpha}), \text{ with } \overline{\alpha} = \mathrm{diag}\big((a_k)_{k\in\mathbb{N}}\big) : \ell_{r'} \to \ell_1. \tag{6.6.2}$$

Summing norms of diagonal operators between ℓ_p-spaces have been investigated in [125] (specifically, Theorems 4 and 9 – although the latter contains some typos). Combining these results with (6.6.2) gives an alternative proof of some of the results from [30]. Generally, Corollary 6.6.2 is a useful tool for computing the moduli of unconditional bases in $\mathrm{FBL}[E]$, as there is a large theory concerning 1-summing norms. However, due to the p-sum inside the norm of (6.6.1), the case when $p \in (1, \infty)$ is more difficult, and, in particular, we don't know the behaviour in $\mathrm{FBL}^{(p)}[\ell_r]$ of the moduli of the basis vectors from ℓ_r when both $p, r \in (2, \infty)$. The case when the basis is conditional is also more difficult, as Corollary 6.6.2 only allows us to control positive scalars, but the moduli of a conditional basis need not be unconditional (see Proposition 6.5.18).

When are (x_k) and $\big(|\delta_{x_k}|\big)$ equivalent?

Our next result can be considered as a converse of Proposition 6.5.13 when $p \in [1, \infty)$.

Proposition 6.6.3. *Fix $p \in [1, \infty)$ and let (x_k) be a normalized basis of E such that $(|\delta_{x_k}|)$ in $\mathrm{FBL}^{(p)}[E]$ is equivalent to (x_k). Then (x_k) must be equivalent to the unit vector basis of ℓ_1.*

Proof. By Proposition 6.5.10, Proposition 6.5.12 and the hypothesis, we have that

$$\Big(\sum_k a_k^2\Big)^{\frac{1}{2}} \lesssim \Big\| \sum_k a_k |\delta_{x_k}| \Big\|_{\mathrm{FBL}^{(p)}[E]} \lesssim \Big\| \sum_k a_k x_k \Big\|_E .$$

Therefore, we have a bounded map $T : E \to \ell_2$ with $T(x_k) = e_k$, where (e_k) is the unit vector basis of ℓ_2. Let $\overline{T} : \mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[\ell_2]$ be the lattice homomorphism extending T. By Proposition 6.5.14 it follows that

$$\begin{aligned} \sum_k |a_k| \lesssim \Big\| \sum_k a_k |\delta_{e_k}| \Big\|_{\mathrm{FBL}^{(p)}[\ell_2]} &\leq \|\overline{T}\| \Big\| \sum_k a_k |\delta_{x_k}| \Big\|_{\mathrm{FBL}^{(p)}[E]} \\ &\lesssim \Big\| \sum_k a_k x_k \Big\|_E \leq \sum_k |a_k|. \end{aligned}$$

□

The case $p = \infty$ is completely different.

Proposition 6.6.4. *Let (x_k) be an unconditional basic sequence in a Banach space E. Then $(\delta_{x_k}) \sim (|\delta_{x_k}|)$ in $\mathrm{FBL}^{(\infty)}[E]$.*

Proof. By Propositions 6.3.10 and 6.5.3, we may assume that (x_k) is a basis and that $\big(|\delta_{x_k}|\big)$ is unconditional. For any $a_1, \ldots, a_n$, we have

$$\Big\|\sum_{k=1}^n a_k x_k\Big\| = \Big\|\sum_{k=1}^n a_k \delta_{x_k}\Big\| \lesssim \Big\|\sum_{k=1}^n |a_k|\, |\delta_{x_k}|\Big\| \sim \Big\|\sum_{k=1}^n a_k\, |\delta_{x_k}|\Big\|.$$

Fix $x^* \in B_{E^*}$ and put $\varepsilon_k = \operatorname{sign} x^*(x_k)$; we then have

$$\Big|\sum_{k=1}^n a_k\, |\delta_{x_k}|\Big|(x^*) = \Big|\sum_{k=1}^n \varepsilon_k a_k x^*(x_k)\Big| \leqslant \Big\|\sum_{k=1}^n \varepsilon_k a_k x_k\Big\| \sim \Big\|\sum_{k=1}^n a_k x_k\Big\|.$$

Taking sup over $x^* \in B_{E^*}$, we get $\Big\|\sum_{k=1}^n a_k\, |\delta_{x_k}|\Big\| \lesssim \Big\|\sum_{k=1}^n a_k x_k\Big\|$. □

Example 6.5.16 shows that the hypothesis of unconditionality in Proposition 6.6.4 cannot be removed. If "basis" is replaced by "basic sequence" in Proposition 6.6.3, the situation, again, is dramatically different:

Proposition 6.6.5. *Suppose Ω is a compact Hausdorff space and (x_k) is a sequence in $C(\Omega)$ equivalent to the ℓ_2 basis. Then for all $p \in [1,\infty]$ the sequence $\big(|\delta_{x_k}|\big)$ in $\mathrm{FBL}^{(p)}[C(\Omega)]$ is equivalent to the ℓ_2 basis.*

Proof. We assume that (x_k) is normalized and C-equivalent to the ℓ_2 basis.

Fix $a_1, \dots, a_n \in \mathbb{R}$. Using Proposition 6.6.4 and the fact that the $\mathrm{FBL}^{(\infty)}$-norm is the weakest of the $\mathrm{FBL}^{(p)}$-norms, we get

$$\Big\| \sum_k a_k |\delta_{x_k}| \Big\|_{\mathrm{FBL}^{(p)}[C(\Omega)]} \geq \Big\| \sum_k a_k |\delta_{x_k}| \Big\|_{\mathrm{FBL}^{(\infty)}[C(\Omega)]} \gtrsim \Big(\sum_k |a_k|^2 \Big)^{1/2}.$$

To establish the converse, it suffices to work with the FBL-norm as it is the strongest of the $\mathrm{FBL}^{(p)}$-norms. Note that

$$\Big| \sum_k a_k |\delta_{x_k}| \Big| \leq \sum_k |a_k| |\delta_{x_k}|,$$

hence it suffices to prove that there exists a universal constant K such that

$$\Big\| \sum_k a_k |\delta_{x_k}| \Big\|_{\mathrm{FBL}[C(\Omega)]} \leq K \Big(\sum_k a_k^2 \Big)^{1/2}$$

whenever $a_1, \dots, a_n \geq 0$.

By Proposition 6.6.1, $\big\| \sum_k a_k |\delta_{x_k}| \big\| = \pi_1(T)$, where $T : C(\Omega)^* \to \ell_1^n$ takes the measure μ to $(a_k \mu(x_k))_{k=1}^n$. Write $T = (jT_0)^*$, where

$$T_0 : \ell_\infty^n \to E = \mathrm{span}[x_1, \dots, x_n] : e_k \mapsto a_k x_k,$$

and j is the embedding of E into $C(\Omega)$. Then $\pi_1(T) \leq \|T_0\| \pi_1(j^*)$. As (x_k) is C-equivalent to the ℓ_2 basis, we obtain $\|T_0\| \leq C \big(\sum_k a_k^2 \big)^{1/2}$. Further, the domain of j^* is an AL-space, hence, by Grothendieck's Theorem, $\pi_1(j^*) \leq K_G d(E, \ell_2^n) \leq K_G C^2$ (here $d(\cdot,\cdot)$ stands for the Banach-Mazur distance). We conclude that $\pi_1(T) \leq K_G C^3 \big(\sum_k a_k^2 \big)^{1/2}$, which is what we need. $\square$

Corollary 6.6.6. *Suppose Ω is a compact Hausdorff space and (y_k) is a sequence in $C(\Omega)$, dominated by the ℓ_2 basis, and admitting a bounded sequence of biorthogonal functionals. Then for all $p \in [1, \infty)$ the sequence $\left(\left|\delta_{y_k}\right|\right)$ in $\mathrm{FBL}^{(p)}[C(\Omega)]$ is equivalent to the ℓ_2 basis.*

The above conditions are verified, for instance, if (y_k) is a semi-normalized basic sequence, dominated by the ℓ_2 basis.

Proof. By Proposition 6.5.12, $\left(\left|\delta_{y_k}\right|\right)$ dominates the ℓ_2 basis. To establish the converse, find a basic sequence (x_k) in $C(\Omega')$, equivalent to the ℓ_2 basis. It is well-known (see, e.g., [333, Section 4]) that $C(\Omega)^{**}$ is injective, hence there exists $T \in B(C(\Omega'), C(\Omega)^{**})$ so that $Tx_k = y_k$, for every k. This T extends to a lattice homomorphism $\overline{T} : \mathrm{FBL}^{(p)}[C(\Omega')] \to \mathrm{FBL}^{(p)}[C(\Omega)^{**}]$. By Proposition 6.6.5, ℓ_2 dominates $\left(\left|\delta_{y_k}\right|\right) \subseteq \mathrm{FBL}^{(p)}[C(\Omega)^{**}]$. To complete the proof, invoke Proposition 6.4.10. $\square$

Remark 6.6.7. In Proposition 6.6.5, one can replace $C(\Omega)$ with an arbitrary $\mathcal{L}_{\infty,\lambda}$ space (the equivalence constant will then depend not only on C but also on λ). Likewise, Corollary 6.6.6 works for $\mathcal{L}_{\infty,\lambda}$ spaces as well (combine [228, Theorem 4.1] with [333, Theorem 4.2]).

One can also describe sequences (x_k) which are equivalent to $\left(\left|\delta_{x_k}\right|\right)$ via regular operators.

Proposition 6.6.8. *Let (x_k) be a 1-unconditional basic sequence in a Banach space E, and view $[x_k]$ as a Banach lattice with the coordinate order induced by the basis. Let $j : [x_k] \to \mathrm{FBL}^{(p)}[E]$ be the canonical inclusion $x_k \mapsto \delta_{x_k}$. Then the following are equivalent:*

(i) $(|\delta_{x_k}|) \sim (x_k)$;

(ii) j *is regular;*

(iii) j *is pre-regular.*

Recall that an operator T between Banach lattices is called *pre-regular* if T^* is regular; we set $\|T\|_{pre-reg} = \|T^*\|_r$ (the regular norm). For more information on this class of operators, the reader is referred to [89, Section 4]. This class also coincides with that of $(1, 1)$-regular operators considered in [295].

Proof. (i)⇒(ii): Consider the linear extension of the map $T_+(x_k) = (\delta_{x_k})_+$ and $T_-(x_k) = (\delta_{x_k})_-$. These maps have well-defined extensions to $[x_k]$ because

$$\begin{aligned}\left\| T_+\left(\sum_{k=1}^n a_k x_k\right)\right\|_{\mathrm{FBL}^{(p)}[E]} &= \left\|\sum_{k=1}^n a_k(\delta_{x_k})_+\right\|_{\mathrm{FBL}^{(p)}[E]} \le \left\|\sum_{k=1}^n |a_k\delta_{x_k}|\right\| \\ &\sim \left\|\sum_{k=1}^n |a_k| x_k\right\| \sim \left\|\sum_{k=1}^n a_k x_k\right\|,\end{aligned}$$

and a similar estimate holds for T_-. Clearly these extensions are positive and $j = T_+ - T_-$.

(ii)⇒(iii) is trivial.

(iii)⇒(i): By Proposition 6.5.13, $(|\delta_{x_k}|)$ dominates (x_k). On the other hand,

$$\left|\sum_{k=1}^n a_k|\delta_{x_k}|\right| \le \sum_{k=1}^n |a_k\delta_{x_k}| = \sum_{k=1}^n |j(a_k x_k)|.$$

By [89, Theorem 4.40] we conclude that

$$\left\|\sum_{k=1}^n a_k|\delta_{x_k}|\right\| \le \|j\|_{pre-reg}\left\|\sum_{k=1}^n |a_k x_k|_{[x_k]}\right\| = \|j\|_{pre-reg}\left\|\sum_{k=1}^n a_k x_k\right\|.$$

Here, $|\cdot|_{[x_k]}$ denotes the modulus in $\overline{\mathrm{span}}[x_k : k \in \mathbb{N}]$, arising from the order determined by the unconditional basis (x_k). □

We now answer the question of when $\overline{\mathrm{span}}[|\delta_{x_k}| : k \in \mathbb{N}]$ is complemented in $\mathrm{FBL}[E]$:

Corollary 6.6.9. *Let E be a Banach space with a normalized unconditional basis (x_k). The following statements are equivalent:*

(i) $\overline{\mathrm{span}}[|\delta_{x_k}| : k \in \mathbb{N}]$ *is complemented in* $\mathrm{FBL}[E]$*;*

(ii) There is an unconditional sequence of biorthogonal functionals (u_k^*) to $(|\delta_{x_k}|) \subseteq \mathrm{FBL}[E]$ *such that* $\overline{\mathrm{span}}[|\delta_{x_k}| : k \in \mathbb{N}]$ *is normed by* $\overline{\mathrm{span}}[u_k^* : k \in \mathbb{N}]$*;*

(iii) (x_k) is equivalent to the ℓ_1 basis.

Here, for a Banach space E, and subspaces $F \subseteq E$, $G \subseteq E^*$, we say that F *is normed by* G if there is $K > 0$ such that for every $x \in F$, there is $x^* \in G$ with $\|x^*\| = 1$ and $|x^*(x)| \ge \|x\|/K$.

Proof. By Proposition 6.5.3, $\big(|\delta_{x_k}|\big)$ is an unconditional basic sequence in $\mathrm{FBL}[E]$. The implication (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3): If (u_k^*) is an unconditional sequence of biorthogonal functionals to $(|\delta_{x_k}|) \subseteq \mathrm{FBL}[E]$ such that $\overline{\mathrm{span}}[|\delta_{x_k}| : k \in \mathbb{N}]$ is normed by $\overline{\mathrm{span}}[u_k^* : k \in \mathbb{N}]$, then the argument in the proof of [231, Theorem 1.d.6(ii)] would go through and we would have that for any scalars $(a_k)_{k=1}^n$,

$$\Big\|\sum_{k=1}^n a_k|\delta_{x_k}|\Big\| \sim \Big\|\Big(\sum_{k=1}^n |a_k\delta_{x_k}|^2\Big)^{1/2}\Big\|.$$

On the other hand, applying [231, Theorem 1.d.6] (and its proof) to (δ_{x_k}) yields

$$\Big\|\Big(\sum_{k=1}^n |a_k\delta_{x_k}|^2\Big)^{1/2}\Big\| \lesssim \Big\|\sum_{k=1}^n a_k\delta_{x_k}\Big\|.$$

Thus, for any scalars $(a_k)_{k=1}^n$,

$$\begin{aligned}\Big\|\sum_{k=1}^n a_k\delta_{x_k}\Big\| \le \Big\|\sum_{k=1}^n |a_k||\delta_{x_k}|\Big\| &\sim \Big\|\sum_{k=1}^n a_k|\delta_{x_k}|\Big\| \\ &\sim \Big\|\Big(\sum_{k=1}^n |a_k\delta_{x_k}|^2\Big)^{1/2}\Big\| \lesssim \Big\|\sum_{k=1}^n a_k\delta_{x_k}\Big\|.\end{aligned}$$

Hence (x_k) and $(|\delta_{x_k}|)$ are equivalent. By Proposition 6.6.3 (x_k) is equivalent to the unit vector basis of ℓ_1.

(3) $\Rightarrow$ (1): Suppose (e_k) is the unit vector basis of ℓ_1; we will show that $(|\delta_{e_k}|)$ is complemented in $\mathrm{FBL}[\ell_1]$. Denote by (e_k^*) the canonical sequence in $\ell_\infty = \ell_1^*$ biorthogonal to (e_k). Define the map $T : \mathrm{FBL}[\ell_1] \to \ell_1$ by $f \mapsto (f(e_j^*))$. From the definition of the $\mathrm{FBL}[\ell_1]$ norm, T is contractive. Also, $T(|\delta_{e_k}|) = e_k$. Since $(|\delta_{e_k}|) \sim (e_k)$ we can define $S : \phi(\ell_1) \to \overline{\mathrm{span}}[|\delta_{e_k}| : k \in \mathbb{N}] : \delta_{e_k} \mapsto |\delta_{e_k}|$. Then $S \circ \phi \circ T : \mathrm{FBL}[\ell_1] \to \overline{\mathrm{span}}[|\delta_{e_k}| : k \in \mathbb{N}]$ is our desired projection. $\square$

For $p \in (1,\infty)$, the situation is considerably different.

Proposition 6.6.10. *Let E be a Banach space with an unconditional basis (x_k). Then $\overline{\mathrm{span}}[|\delta_{x_k}| : k \in \mathbb{N}]$ is not complemented in $\mathrm{FBL}^{(p)}[E]$ for any $p \in (1,\infty)$.*

Proof. Without loss of generality, (x_k) is normalized. Suppose, for the sake of contradiction, that $\overline{\mathrm{span}}[|\delta_{x_k}| : k \in \mathbb{N}]$ is complemented in $\mathrm{FBL}^{(p)}[E]$. As in the proof of Corollary 6.6.9, we conclude that (x_k) and $(|\delta_{x_k}|)$ should be equivalent to the ℓ_1 basis. By [231, Theorem 1.d.7 and the remark after], this is a contradiction with the p-convexity of $\mathrm{FBL}^{(p)}[E]$. $\square$

Note that Proposition 6.6.10 fails for $p = \infty$, as the moduli of the c_0 basis will be complemented in $\mathrm{FBL}^{(\infty)}[c_0]$. Moreover, the converse holds as well:

Proposition 6.6.11. *Let (x_k) be a semi-normalized unconditional basis of a Banach space E. Then $(|\delta_{x_k}|)$ is complemented in* $\mathrm{FBL}^{(\infty)}[E]$ *if and only if $(x_k) \sim c_0$.*

Proof. Since $\mathrm{FBL}^{(\infty)}[E]$ is generated by E as a lattice, it is separable. By Proposition 6.6.4, $(|\delta_{x_k}|) \sim (x_k)$. Hence, if $(x_k) \sim c_0$, then $(|\delta_{x_k}|) \sim c_0$, and Sobczyk's theorem applies.

For the converse, we note that by [97, p. 74] the only complemented semi-normalized unconditional basic sequences in $\mathcal{L}_\infty$-spaces are those equivalent to the unit vector basis of c_0. Since AM-spaces are $\mathcal{L}_\infty$, it follows that if $(|\delta_{x_k}|)$ is complemented in $\mathrm{FBL}^{(\infty)}[E]$ then $(|\delta_{x_k}|) \sim c_0$. Hence, if (x_k) is not c_0, then by Proposition 6.6.4, $(|\delta_{x_k}|)$ is not c_0, so $(|\delta_{x_k}|)$ is not complemented in $\mathrm{FBL}^{(\infty)}[E]$. $\square$

In the above three results (Corollary 6.6.9-Proposition 6.6.11) we assumed that (x_k) was an unconditional basis; it is unclear how to characterize complementation of $(|\delta_{x_k}|)$ when the unconditionality assumption on (x_k) is dropped.

Remark 6.6.12. Throughout this and the previous section we have dealt with the sequence of moduli of a basis. As noted in [30] one could consider other lattice expressions; for example, $((\delta_{x_k})_+)$ or $((\delta_{x_k})_-)$. Moreover, in [30] it is shown that $((\delta_{x_k})_+)$ and $((\delta_{x_k})_-)$ are 1-equivalent to each other. For their relation to $(|\delta_{x_k}|)$, note that

$$\Big\|\sum_{k=1}^n a_k|\delta_{x_k}|\Big\| = \Big\|\sum_{k=1}^n a_k(\delta_{x_k})_+ + \sum_{k=1}^n a_k(\delta_{x_k})_-\Big\| \leq 2\Big\|\sum_{k=1}^n a_k(\delta_{x_k})_+\Big\|.$$

This shows that $(|\delta_{x_k}|) \lesssim ((\delta_{x_k})_+)$. The converse domination is easily seen to be true for unconditional bases.

Basic sequences in $\mathrm{FBL}[L_1]$

Let us say that a sequence (x_k) in a Banach space E is *C-minimal* if it admits biorthogonal functionals of norm not exceeding C; (x_k) is said to be *uniformly minimal* if it is C-minimal for some C.

Proposition 6.6.13. *Suppose (x_k) is a normalized C-minimal sequence in L_1. Then for all $n \in \mathbb{N}$ and $a_1, \ldots, a_n \geq 0$, we have*

$$\sum_{k=1}^n a_k \geq \Big\| \sum_{k=1}^n a_k \big|\delta_{x_k}\big| \Big\|_{\mathrm{FBL}[L_1]} \geq \frac{1}{C} \sum_{k=1}^n a_k .$$

Proof. The left hand side follows from the triangle inequality. Proposition 6.6.1 states that

$$\Big\| \sum_{k=1}^n a_k \big|\delta_{x_k}\big| \Big\| = \pi_1(T), \text{ with } T : L_\infty \to \ell_1^n : f \mapsto \sum_k a_k f(x_k) e_k$$

(here $f(x_k)$ represents the dual action of L_∞ on L_1, and $e_1, \ldots, e_n$ form the canonical basis of ℓ_1^n). The domain of T is L_∞, hence $\pi_1(T) = \iota_1(T)$ (the integral norm, cf. [97, Corollary 5.8]). By the trace duality [97, Theorem 6.16],

$$\iota_1(T) = \sup \big\{ \mathrm{tr}(TS) : S \in B(\ell_1^n, L_\infty), \|S\| \leq 1 \big\}.$$

By C-minimality, there exist $f_1, \ldots, f_n \in L_\infty$ so that $f_j(x_k) = \delta_{jk}$ (the Kronecker's delta), and $\max_j \|f_j\| \leq C$. Then $Tf_j = a_j e_j \in \ell_1^n$. The operator $S : \ell_1^n \to L_\infty : e_j \mapsto C^{-1} f_j$ is contractive, and $TSe_j = C^{-1} a_j e_j$. Thus, $\pi_1(T) \geq \mathrm{tr}(TS) = C^{-1} \sum_j a_j$, as desired. □

We can now easily deduce the following:

Corollary 6.6.14. *Suppose (x_k) is a normalized uniformly minimal sequence (in particular, a normalized basic sequence) in L_1. Then the sequence $\big(\big|\delta_{x_k}\big|\big)$ is unconditional in* FBL$[L_1]$ *if and only if it is equivalent to the ℓ_1 basis.*

Corollary 6.6.14 may be useful for answering Question 6.5.5 in the special case that $E = L_1$ and $p = 1$. Here is another particular case when the above corollary is applicable:

Corollary 6.6.15. *If (x_k) is a sequence of normalized independent random variables in L_1, then $\big(\big|\delta_{x_k}\big|\big)$ is equivalent to the ℓ_1 basis in* FBL$[L_1]$.

Sketch of a proof. For $A \subseteq \mathbb{N}$, denote by Σ_A the σ-algebra generated by the random variables $\{x_k\}_{k \in A}$. Let Q_A be the conditional expectation from L_1 onto $L_1(\Sigma_A)$. Clearly Q_A is contractive, and $Q_A x_k = x_k$ if $k \in A$, $Q_A x_k = 0$ otherwise. Thus, Q_A generates a contractive lattice homomorphism (denoted by $\overline{Q_A}$) on FBL$[L_1]$, with $\overline{Q_A} \big|\delta_{x_k}\big| = \big|\delta_{x_k}\big|$ if $k \in A$, and 0 otherwise – that is, the sequence $\big(\big|\delta_{x_k}\big|\big)$ is 1-suppression unconditional. □

Remark 6.6.16. In [30] it was shown that the sequence $(|\delta_{e_k}|)$ in $\mathrm{FBL}[\ell_2]$ is equivalent to the standard ℓ_1 basis. In contrast to Example 6.3.13, Corollary 6.6.15 shows that the sequence $\big(|\delta_{r_k}|\big)$ (r_k are independent Rademachers) in $\mathrm{FBL}[L_1[0,1]]$ is equivalent to the standard ℓ_1 basis. Proposition 6.5.13 implies that the sequence $(|\delta_{r_k}|)$ in $\mathrm{FBL}^{(p)}[L_\infty[0,1]]$ is equivalent to the standard ℓ_1 basis for all $p \in [1,\infty]$, but we don't know how $(|\delta_{r_k}|)$ behaves in $\mathrm{FBL}^{(p)}[L_1[0,1]]$ for $p \in (1,\infty)$. For $p = \infty$ we obtain a copy of ℓ_2, see Proposition 6.6.4.

Normalized Haar functions form another notable sequence in L_1. These were investigated in Section 6.5, above.

Creating copies of ℓ_2 in free Banach lattices

In earlier sections we proved several results in which we assumed knowledge about a basis (x_k) of E and analysed the behaviour of the basic sequence $(|\delta_{x_k}|)$ in $\mathrm{FBL}^{(p)}[E]$. In particular, Corollary 6.5.11 shows that, if $(x_k) \subseteq E$ is equivalent to the c_0 basis, then $(|\delta_{x_k}|) \subseteq \mathrm{FBL}^{(p)}[E]$ is equivalent to the ℓ_2 basis. It is also of interest to study the opposite question. Specifically, suppose we have an (unconditional) basis (x_k) of E and we know the behaviour of $(|\delta_{x_k}|)$ in $\mathrm{FBL}^{(p)}[E]$. Can we deduce from this the behaviour of (x_k)? When $(|\delta_{x_k}|)$ is equivalent to the ℓ_2 basis, the answer is yes:

Theorem 6.6.17. *Suppose E is a Banach space with an unconditional basis (x_k), such that the sequence $\big(|\delta_{x_k}|\big)$ in $\mathrm{FBL}^{(p)}[E]$ (for some $1 \le p < \infty$) is equivalent to the ℓ_2 basis. Then (x_k) is equivalent to the c_0 basis.*

The rest of the subsection is devoted to proving this result. First we fix some notation and conventions.

Since $(|\delta_{x_k}|)$ is equivalent to the ℓ_2 basis, it is semi-normalized. Renorming if necessary, we can and do assume that the basis (x_k) is normalized and 1-unconditional in E, so that E is a Banach lattice when equipped with the order induced by the basis. In view of Proposition 6.5.12 and [30, Theorem 5], what we are really assuming is existence of a $C > 0$ so that, for any $a_k \in \mathbb{R}$,

$$\Big\| \sum_k a_k |\delta_{x_k}| \Big\| \le C \big(\sum_k a_k^2 \big)^{1/2}. \tag{6.6.3}$$

We shall denote by (x_k^*) the corresponding biorthogonal functionals in E^*, and let E' be the subspace of E^* spanned by them.

Denote by (e_k) the canonical basis on a space ℓ_p. If Y and Z are spaces with semi-normalized unconditional bases (y_k) and (z_k) respectively, and $\alpha = (a_1, a_2, \ldots) \in \ell_p$, we denote by $\Delta_\alpha : Y \to Z$ the diagonal operator which takes y_k to $a_k z_k$. Further, $\|\alpha\|_p$ refers to the ℓ_p norm of α like this.

Abusing the notation slightly, we often identify finitely supported elements of these spaces with their sequence representation. For instance, we identify $f = \sum_k b_k x_k^* \in E'$ with $\sum_k b_k e_k \in \ell_p$, and use $\|f\|_{E'}$ and $\|f\|_p$ as a shorthand for $\|\sum_k b_k x_k^*\|_{E'}$ and $\|\sum_k b_k e_k\|_p$, respectively.

For the rest of this section, we use the notation introduced above $(E, (x_k), C, \ldots)$; the proof of Theorem 6.6.17 begins with a lemma:

Lemma 6.6.18. *For any $\alpha = (a_k) \in c_{00}$, we have*

$$\Big(\sum_k a_k^2\Big)^{1/2} \sim \big\|\Delta_\alpha : E \to \ell_2\big\|.$$

Proof. By unconditionality, we can assume that $a_k \geq 0$ for any k. We clearly have

$$\Big(\sum_k a_k^2\Big)^{1/2} \geq \big\|\Delta_\alpha : E \to \ell_2\big\|$$

(compare with $\Delta_\alpha : c_0 \to \ell_2$), hence we only need to show that

$$\Big(\sum_k a_k^2\Big)^{1/2} \lesssim \big\|\Delta_\alpha : E \to \ell_2\big\|.$$

To establish this, consider the isomorphic embedding $J : \ell_2 \to L_p$ induced by the Rademacher functions (r_k), i.e., $Je_k = r_k$. Then $J\Delta_\alpha$ has a lattice homomorphic extension $T = \widehat{J\Delta_\alpha}$: $\mathrm{FBL}^{(p)}[E] \to L_p$, with $\|T\| \leq \|J\Delta_\alpha\| \leq \|J\|\|\Delta_\alpha\|$. Clearly $T|\delta_{x_k}| = a_k|r_k| = a_k 1$. Let now $u = \sum_k a_k |\delta_{x_k}|$, then $\|u\| \sim \big(\sum_k a_k^2\big)^{1/2}$, while $\|Tu\| = \sum_k a_k^2$, which implies $\big(\sum_k a_k^2\big)^{1/2} \lesssim \|T\|$. Then

$$\big\|\Delta_\alpha : E \to \ell_2\big\| \geq \frac{\|T\|}{\|J\|} \gtrsim \Big(\sum_k a_k^2\Big)^{1/2}.$$

□

Proof of Theorem 6.6.17. By Lemma 6.6.18, there exists a constant C so that the inequality $\|\Delta_\alpha : \ell_2 \to E'\| \geq C^{-1}\big(\sum_k |a_k|^2\big)^{1/2}$ holds for any $\alpha = (a_k) \in c_{00}$. Suppose, for the sake of contradiction, that the formal identity $\ell_1 \to E' : e_k \mapsto x_k^*$ is not bounded below. Then there exists a finitely supported $f = (f_k) \in E'_+$ so that $\|f\|_1 = \sum_k f_k = 1$, yet $\|f\|_{E'} < C^{-2}$. By Lemma 6.6.18, $\Delta_{\sqrt{f}}$ (where $\sqrt{f} = (\sqrt{f_k})$) defines an operator from ℓ_2 to E', with norm at least C^{-1}. Consequently, we can find a finitely supported norm one $g = (g_k) \in \big(\ell_2\big)_+$, so that

$$\big\|\Delta_{\sqrt{f}} g\big\|_{E'} = \big\|\big(\sqrt{f_k} g_k\big)\big\|_{E'} \geq C^{-1}.$$

However, using [231, Proposition 1.d.2] we conclude that

$$\big\|\big(\sqrt{f_k} g_k\big)\big\|_{E'} \leq \big\|\big(f_k\big)\big\|_{E'}^{1/2} \big\|\big(g_k^2\big)\big\|_{E'}^{1/2} < C^{-1} \|g_k^2\|_1^{1/2} = C^{-1},$$

which yields the desired contradiction. Thus, (x_k^*) is equivalent to the ℓ_1 basis, which implies that (x_k) is equivalent to the c_0 basis. $\square$

Remark 6.6.19. Proposition 6.6.5 shows that Theorem 6.6.17 fails if (x_k) is assumed to be not a basis, but merely an unconditional basic sequence, in E. We do not know whether the unconditionality assumption in Theorem 6.6.17 can be dropped.

A general question in this direction is:

Question 6.6.20. Given a basis (x_k), does there exists a basis (y_k) (possibly with some nice additional properties) such that $(x_k) \sim (|\delta_{y_k}|) \subseteq \mathrm{FBL}^{(p)}[\overline{\mathrm{span}}[y_k]]$? If so, classify/analyse such (y_k).

6.7 Bibasic and absolute sequences in free Banach lattices

Recall that a sequence of non-zero vectors (x_k) in a Banach lattice is *bibasic* if there exists a constant $M \geq 1$ such that for every $m \in \mathbb{N}$ and scalars $a_1, \dots, a_m$, the following *bibasis inequality* is satisfied:

$$\Big\| \bigvee_{n=1}^{m} \Big| \sum_{k=1}^{n} a_k x_k \Big| \Big\| \leq M \Big\| \sum_{k=1}^{m} a_k x_k \Big\|. \tag{6.7.1}$$

The least value of the constant M is called the *bibasis constant* of (x_k). Clearly, every bibasic sequence is basic. Indeed, to arrive at the bibasis inequality (6.7.1), one begins with the usual basis inequality

$$\bigvee_{n=1}^{m} \Big\| \sum_{k=1}^{n} a_k x_k \Big\| \leq K \Big\| \sum_{k=1}^{m} a_k x_k \Big\|,$$

and brings the supremum inside the norm. In general, a basic sequence need not be bibasic; however, this implication does hold in AM-spaces. For further details on bibasic sequences and their equivalent characterizations we refer the reader to [144, 314]. The importance of bibasic sequences stems from two places. The first is that there are several natural examples, including martingale difference sequences in $L_p(P)$ (P a probability measure and $p > 1$), the Walsh basis, unconditional blocks of the Haar in $L_1[0,1]$, and the trigonometric basis. The second important fact is [314, Theorem 2.1], which shows that several forms of convergence are equivalent for bibasic sequences. To set notation, for a basic sequence (x_k), we let $P_n : [x_k] \to [x_k]$ be the n-th canonical basis projection. Here, $[x_k]$ denotes the closed span of (x_k), and for $x = \sum_{k=1}^{\infty} a_k x_k$ we have $P_n x := \sum_{k=1}^{n} a_k x_k$. By definition, $P_n x \xrightarrow{\|\cdot\|} x$.

Theorem 6.7.1 (Bibasis Theorem)**.** *Let X be a Banach lattice and (x_k) a basic sequence in X. The following statements are equivalent:*

(i) (x_k) is bibasic;

(ii) For all $x \in [x_k]$, $P_n x \xrightarrow{u} x$;

(iii) For all $x \in [x_k]$, $P_n x \xrightarrow{o} x$;

(iv) For all $x \in [x_k]$, $(P_n x)$ is order bounded in X;

(v) For all $x \in [x_k]$, $(\bigvee_{n=1}^{m} |P_n x|)_{m=1}^{\infty}$ is norm bounded.

Above, we use several modes of convergence. The norm convergence is denoted by $\xrightarrow{\|\cdot\|}$, while $\xrightarrow{u}$ and $\xrightarrow{o}$ stand for the uniform and order convergence respectively. Specifically, $z_k \xrightarrow{u} 0$ if there exists $e \geq 0$ with the property that for every $\varepsilon > 0$ there exists N such that $|z_k| \leq \varepsilon e$ for any $k \geq N$. The condition $z_k \xrightarrow{o} 0$ is significantly weaker: There exists a net (y_α), decreasing to 0, with the property that for every α there exists N such that $|z_k| \leq y_\alpha$ for any $k \geq N$. The reader is referred to e.g. [314] for more details.

A subspace of a Banach lattice without bibasic sequences

In this subsection we show that the unit vector basis of c_0 is not bibasic in $\mathrm{FBL}^{(p)}[c_0]$ for finite p, and we use this to answer a question from [314], by exhibiting a subspace of a Banach lattice without a bibasic sequence. Let us begin with the following observation:

Lemma 6.7.2. *Let (x_k) be a basis of a Banach space E. If (δ_{x_k}) is bibasic in $\mathrm{FBL}^{(p)}[E]$, then (x_k) is also bibasic in any p-convex Banach lattice where E linearly isomorphically embeds.*

Proof. Let X be a p-convex Banach lattice and (y_k) a basic sequence in X equivalent to (x_k). Then there is a linear isomorphic embedding $T: E \to X$ with $Tx_k = y_k$. Extend this map to a lattice homomorphism $\widehat{T}: \mathrm{FBL}^{(p)}[E] \to X$. Then

$$\Big\| \bigvee_{n=1}^{m} \Big| \sum_{k=1}^{n} a_k y_k \Big| \Big\| = \Big\| \widehat{T} \bigvee_{n=1}^{m} \Big| \sum_{k=1}^{n} a_k \delta_{x_k} \Big| \Big\| \le \|\widehat{T}\| \Big\| \bigvee_{n=1}^{m} \Big| \sum_{k=1}^{n} a_k \delta_{x_k} \Big| \Big\| \le$$

$$\|\widehat{T}\| M \Big\| \sum_{k=1}^{m} a_k x_k \Big\| \le \|T\| \|T^{-1}\| M^{(p)}(X) M \Big\| \sum_{k=1}^{m} a_k y_k \Big\|,$$

where M is the bibasis constant of (δ_{x_k}) in $\mathrm{FBL}^{(p)}[E]$. $\square$

Remark 6.7.3. For general p, it is therefore of interest to know which normalized bases (x_k) of E are such that (δ_{x_k}) is bibasic in $\mathrm{FBL}^{(p)}[E]$ - when $p = \infty$ this is true for every basis (x_k).

Theorem 6.7.4. *The canonical copy of the c_0 basis (δ_{e_k}) in $\mathrm{FBL}^{(p)}[c_0]$ is not bibasic as long as $1 \le p < \infty$.*

Proof. Suppose, for the sake of contradiction, that (δ_{e_k}) is bibasic in $\mathrm{FBL}^{(p)}[c_0]$ with bibasis constant M. Fix m. Let H_m be the $m \times m$ Hilbert matrix defined by

$$H_m = \begin{bmatrix} \frac{1}{m-1} & \frac{1}{m-2} & \dots & 1 & 0 \\ \frac{1}{m-2} & \frac{1}{m-3} & \dots & 0 & -1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & -\frac{1}{m-3} & -\frac{1}{m-2} \\ 0 & -1 & \dots & -\frac{1}{m-2} & -\frac{1}{m-1} \end{bmatrix}$$

Here the (i,j)-th entry is $\frac{1}{m+1-i-j}$ when $i + j \neq m+1$ and zero otherwise. We view H_m as an operator from ℓ_∞^m to ℓ_p^m. Clearly, H_m^+ has the same upper-left quadrant as H_m but zeros in the lower-right quadrant. In the proof of Proposition 1.2 in [216], it is shown that $\|H_m^+\| \geqslant C \ln m \, \|H_m\|$, where C is an absolute constant.

Identifying ℓ_∞^m with $\mathrm{span}[e_1, \dots, e_m]$ in c_0, we extend H_m to an operator from c_0 to ℓ_p^m by setting $H_m e_k = 0$ whenever $k > m$. Let $\widehat{H_m}: \mathrm{FBL}^{(p)}[c_0] \to \ell_p^m$ be the extension of H_m to a lattice homomorphism with $\|\widehat{H_m}\| = \|H_m\|$. Applying the bibasis inequality to δ_{e_k}'s with

$a_1 = \cdots = a_m = 1$ and using the fact that $H_m e_k = \widehat{H_m}\delta_{e_k}$ for all k, we get

$$\Big\| \bigvee_{n=1}^m \Big|\sum_{k=1}^n H_m e_k\Big|\Big\| = \Big\| \bigvee_{n=1}^m \Big|\sum_{k=1}^n \widehat{H_m}\delta_{e_k}\Big|\Big\| = \Big\|\widehat{H_m}\Big(\bigvee_{n=1}^m \Big|\sum_{k=1}^n \delta_{e_k}\Big|\Big)\Big\| \leqslant \big\|\widehat{H_m}\big\| \Big\|\bigvee_{n=1}^m \Big|\sum_{k=1}^n \delta_{e_k}\Big|\Big\|$$
$$\leqslant M\,\|H_m\| \Big\|\sum_{k=1}^m \delta_{e_k}\Big\| = M\,\|H_m\| \Big\|\sum_{k=1}^m e_k\Big\| = M\,\|H_m\|\,.$$

Fix $j \leq m$. Then, clearly,

$$\bigvee_{n=1}^m \Big|\sum_{k=1}^n H_m e_k\Big| \geq \Big|\sum_{k=1}^{m-j} H_m e_k\Big|.$$

The j-th entry of the vector on the right hand side is $1+\frac{1}{2}+\cdots+\frac{1}{m-j}$. This number is also the j-th entry of $H_m^+\mathbb{1}$. It follows that $\bigvee_{n=1}^m |\sum_{k=1}^n H_m e_k| \geq H_m^+\mathbb{1}$, so that

$$\Big\|\bigvee_{n=1}^m \Big|\sum_{k=1}^n H_m e_k\Big|\Big\| \geq \big\|H_m^+\mathbb{1}\big\| = \big\|H_m^+\big\| \geq C\ln m\,\|H_m\|\,,$$

which is a contradiction because m is arbitrary. $\square$

It was asked in [314, Remark 4.4] whether every subspace of a Banach lattice contains a bibasic sequence. We next provide a negative answer to this question:

Theorem 6.7.5. *The subspace $\phi(c_0)$ in $\mathrm{FBL}^{(p)}[c_0]$ does not contain any bibasic sequence as long as $1 \leq p < \infty$.*

In particular, for every finite p there exists a p-convex Banach lattice containing a subspace without any bibasic sequence. This result is sharp, since, as noted above, any basic sequence in an AM-space is bibasic.

Proof. Suppose the contrary, that there is a sequence (y_k) in c_0 such that (δ_{y_k}) is bibasic in $\mathrm{FBL}^{(p)}[c_0]$. Let (x_k) be a block sequence of (y_k) which is equivalent to the c_0 basis, and complemented in c_0 (cf. [230, Propositions 1.a.12 and 2.a.2]). As a blocking of (δ_{y_k}), (δ_{x_k}) is bibasic in $\mathrm{FBL}^{(p)}[c_0]$ by [314, Corollary 4.1].

Let P be a projection from c_0 onto $E = \overline{\mathrm{span}}[x_k : k \in \mathbb{N}]$. By the results of Section 6.3, for every finite sequence (a_k), and every $m \in \mathbb{N}$, we have

$$\Big\|\bigvee_{n=1}^m \Big|\sum_{k=1}^n a_k\delta_{x_k}\Big|\Big\|_{\mathrm{FBL}^{(p)}[c_0]} \leq \Big\|\bigvee_{n=1}^m \Big|\sum_{k=1}^n a_k\delta_{x_k}\Big|\Big\|_{\mathrm{FBL}^{(p)}[E]}$$
$$\leq \|P\|\Big\|\bigvee_{n=1}^m \Big|\sum_{k=1}^n a_k\delta_{x_k}\Big|\Big\|_{\mathrm{FBL}^{(p)}[c_0]}.$$

Define now an isomorphism $T : E \to c_0 : x_k \mapsto e_k$. Then $\overline{T}$ is a lattice isomorphism, and, for (a_k) and m as above,

$$\begin{aligned}\Big\| \bigvee_{n=1}^{m} \Big| \sum_{k=1}^{n} a_k \delta_{e_k} \Big| \Big\|_{\mathrm{FBL}^{(p)}[c_0]} &= \Big\| \overline{T}\Big(\bigvee_{n=1}^{m} \Big| \sum_{k=1}^{n} a_k \delta_{x_k} \Big| \Big) \Big\|_{\mathrm{FBL}^{(p)}[c_0]} \\ &\le \|T\| \Big\| \bigvee_{n=1}^{m} \Big| \sum_{k=1}^{n} a_k \delta_{x_k} \Big| \Big\|_{\mathrm{FBL}^{(p)}[E]} \\ &\le \|P\| \|T\| \Big\| \bigvee_{n=1}^{m} \Big| \sum_{k=1}^{n} a_k \delta_{x_k} \Big| \Big\|_{\mathrm{FBL}^{(p)}[c_0]}.\end{aligned}$$

Now, using that (δ_{x_k}) is bibasic in $\mathrm{FBL}^{(p)}[c_0]$ and equivalent to the c_0 basis, it follows easily from the above that (δ_{e_k}) is as well. This contradicts Theorem 6.7.4. □

Remark 6.7.6. The above argument shows that the subspace $\phi(\ell_q)$ of $\mathrm{FBL}^{(p)}[\ell_q]$ does not contain a bibasic sequence, as long as the canonical copy of the ℓ_q basis is not bibasic in $\mathrm{FBL}^{(p)}[\ell_q]$. For what values of p and q is the latter condition satisfied? By Theorem 6.7.4, we know the answer for c_0. Furthermore, the canonical copy of the ℓ_2 basis (δ_{e_k}) is not bibasic in $\mathrm{FBL}^{(p)}[\ell_2]$, for $1 \le p \le 2$: use [314, Example 6.2] to get an orthonormal basis of $L_2[0,1]$ which is not a bibasis, then apply Lemma 6.7.2. On the other hand, it was shown in [314] that every copy of the ℓ_1 basis is bibasic in every Banach lattice; in particular, the canonical copy of the ℓ_1 basis is bibasic in $\mathrm{FBL}^{(p)}[\ell_1]$. This leads to the following conjecture: Let (x_k) be a normalized (unconditional) basis of a Banach space E, and fix $1 \le p < \infty$; if (δ_{x_k}) is bibasic in $\mathrm{FBL}^{(p)}[E]$ then (x_k) is equivalent to the unit vector basis of ℓ_1.

Being a basis is critical for Lemma 6.7.2 - Proposition 6.6.5 shows that, for any basic sequence (x_k) in $C(\Omega)$, equivalent to the ℓ_2 basis, the sequence (δ_{x_k}) is absolute in $\mathrm{FBL}^{(p)}[C(\Omega)]$ (see Section 6.7 below for a discussion on absoluteness), hence in particular is bibasic.

Note that, as in Proposition 6.5.12, c_0 gives a lower bound on the growth of the left hand side of the bibasis inequality. In other words, the inequality in the statement of Proposition 6.5.12 admits a natural "bibasis" analogue, where one places sups inside the norms.

A connection with majorizing maps

Let E be a normed space and X an Archimedean vector lattice. A linear map $T : E \to X$ is called *majorizing* (see [298, Chapter IV]) if for every norm null sequence (x_k) in E, the sequence (Tx_k) is order bounded. There are several equivalent characterizations of majorizing operators, which we record in the following proposition. The equivalence (i)$\Leftrightarrow$(v) is well known, see, for example, [298, Proposition IV.3.4], but we include the simple proof for the sake of analogy with Theorem 6.7.1. Our proof also does not require linearity, only positive homogeneity (and, in particular, it works for sublinear mappings, which seem to be underrepresented in the vector lattice literature, but are important in applications):

Proposition 6.7.7. *Let $T : E \to X$ be a positively homogeneous map, where E is a normed space and X is an Archimedean vector lattice. The following are equivalent:*

(i) T is majorizing;

(ii) $x_k \xrightarrow{\|\cdot\|} 0$ implies $Tx_k \xrightarrow{u} 0$ for all sequences (x_k) in E;

(iii) $x_k \xrightarrow{\|\cdot\|} 0$ implies $Tx_k \xrightarrow{o} 0$ for all sequences (x_k) in E.

Moreover, if X is a Banach lattice then these statements are further equivalent to:

(iv) $x_k \xrightarrow{\|\cdot\|} 0$ implies $(\bigvee_{k=1}^n |Tx_k|)$ is norm bounded for all sequences (x_k) in E;

(v) T is ∞-convex in the sense of [231, Definition 1.d.3], i.e., there exists $M \geq 1$ such that for each $x_1, \ldots, x_n$ in E,

$$\Big\| \bigvee_{k=1}^n |Tx_k| \Big\| \leq M \bigvee_{k=1}^n \|x_k\|.$$

Proof. Clearly, (ii)$\Rightarrow$(iii)$\Rightarrow$(i); we show (i)$\Rightarrow$(ii). Suppose (x_k) is a sequence in E and $x_k \xrightarrow{\|\cdot\|} 0$. Then there exists a sequence $0 \leq \lambda_k \uparrow \infty$ in $\mathbb{R}$ such that $\lambda_k x_k \xrightarrow{\|\cdot\|} 0$. Hence, $T(\lambda_k x_k)$ is order bounded in X, so there exists $0 < x \in X$ with $|T(\lambda_k x_k)| \leq x$ for all k. It follows that $|Tx_k| \leq \frac{1}{\lambda_k} x$, so that $Tx_k \xrightarrow{u} 0$ in X.

(i)$\Rightarrow$(iv) is obvious.

(iv)$\Rightarrow$(ii) Assume $x_k \xrightarrow{\|\cdot\|} 0$ implies $(\bigvee_{k=1}^n |Tx_k|)$ is norm bounded for all sequences (x_k) in E. Then, since X^{**} is monotonically complete, (Tx_k) is order bounded in X^{**}. Now, viewing T as a map from E to X^{**}, T satisfies (i), and hence (ii). Hence, $Tx_k \xrightarrow{u} 0$ in X^{**} and hence in X since X is a closed sublattice of X^{**} (use [314, Proposition 2.12]).

(v)$\Rightarrow$(iv): Suppose $x_k \xrightarrow{\|\cdot\|} 0$. Then (x_k) is norm bounded so there exists K such that $\bigvee_{k=1}^n \|x_k\| < K$ for each n. Then $\| \bigvee_{k=1}^n |Tx_k| \| \leq KM$, and we get (iv).

(iv)⇒(v): Suppose (v) fails. Then by positive homogeneity of the inequality, for every m there exists $x_1^m, \dots, x_{n_m}^m$ with $\bigvee_{k=1}^{n_m} \|x_k^m\| < \frac{1}{2^m}$ and $\| \bigvee_{k=1}^{n_m} |Tx_k^m| \| > m$. Then the sequence $x_1^1, \dots, x_{n_1}^1, x_1^2, \dots$ is $\|\cdot\|$-null but after applying T, (iv) fails. □

The infimum over all such numbers M as above is called the majorizing norm of T; it is denoted by $\|T\|_m$.

Proposition 6.7.8. *If, in the notation of Proposition 6.7.7, E is finite dimensional, and T is linear, then $\vee_{\|z\|\le 1}|Tz|$ exists, and $\|T\|_m = \| \vee_{\|z\|\le 1} |Tz|\|$.*

Proof. First we show that $T(B_E)$ is order bounded. To this end, let $(e_i)_{i=1}^n$ be an Auerbach basis of E. Let $x = \sum_{i=1}^n |Te_i|$. Any $z \in B_E$ admits a decomposition $z = \sum_i a_i e_i$, with $\vee_i |a_i| \le 1$. Then

$$|Tz| = \Big| \sum_i a_i Te_i \Big| \le \sum_i |a_i||Te_i| \le x.$$

Find a nested sequence of finite subsets $S_1 \subseteq S_2 \subseteq \dots \subseteq B_E$, so that, for every m, S_m is a 2^{-m}-net in B_E. Let $x_m = \vee_{z\in S_m}|Tz|$. Then clearly $x_1 \le x_2 \le \dots$. On the other hand, for any m, any $z \in B_E$ can be written as $z = u + v$, with $u \in S_m$ and $\|v\| \le 2^{-m}$. Then $|Tz| \le |Tu| + |Tv| \le x_m + 2^{-m}x$. In particular, $x_{m+1} \le x_m + 2^{-m}x$, and therefore, $\|x_{m+1} - x_m\| \le 2^{-m}\|x\|$. Thus, the sequence (x_m) is increasing to its limit (in the norm topology), call it x_∞. As we have seen, the inequality $|Tz| \le x_m + 2^{-m}x \le x_\infty + 2^{-m}x$ holds for any m, hence $x_\infty = \vee_{\|z\|\le 1}|Tz|$. Clearly, $\|T\|_m \le \|x_\infty\|$. On the other hand, $\|T\|_m \ge \sup_m \|x_m\| = \|x_\infty\|$. □

Remark 6.7.9. In general, Proposition 6.7.8 fails for non-linear maps, as the following map $T : \ell_2^2 \to \ell_2$ shows. For $n \in \mathbb{N}$ let $T\big(\cos\frac{\pi}{2n}, \sin\frac{\pi}{2n}\big) = e_n/\sqrt{n}$, where (e_i) is the canonical basis for ℓ_2; let $T(1,0) = 0$. Extend T to be continuous and homogeneous on ℓ_2^2. Then $\big\{|Tz| : z \in B_{\ell_2^2}\big\} \supseteq \{n^{-1/2}e_n : n \in \mathbb{N}\}$, and the latter set is not order bounded.

On the other hand, one can show that the formula $\|T\|_m = \|\vee_{\|z\|\le 1}|Tz|\|$ holds for various non-linear maps T, if $T : E \to X$ takes a finite dimensional E into an AM-space X. For this, it suffices to show that, if $S \subseteq X$ is relatively compact, then $x_\infty = \vee_{x\in S}|x|$ exists in X, and moreover, for any $\varepsilon > 0$ there exists a finite set $F \subseteq S$ so that $\|\vee_{x\in F}|x| - x_\infty\| < \varepsilon$. Imitating the proof of Proposition 6.7.8, find a nested sequence of finite subsets $S_1 \subseteq S_2 \subseteq \dots \subseteq S$, so that, for every m, S_m is a 2^{-m}-net in S. Let $x_m = \vee_{x\in S_m}|x|$. Then clearly $x_1 \le x_2 \le \dots$. Consider a lattice isometric embedding $J : X \to C(K)$, for some Hausdorff compact K. For each m, $Jx_{m+1} \le Jx_m + 2^{-m}1_K$, hence the sequence (Jx_m) converges in norm. As J is isometric, the sequence (x_m) converges to some $x_\infty \in X$, hence no upper bound on S can be strictly less than x_∞. On the other hand, for any $x \in S$ and $m \in \mathbb{N}$, we have

$$|Jx| \leq Jx_m + 2^{-m}1_K \leq Jx_\infty + 2^{-m}1_K,$$

and so $|x| \leq x_\infty$, due to m being arbitrary.

We now specialize to operators of the form $S = S_{\overline{x}} : \ell_q^n \to X : e_k \mapsto x_k$, where X is a Banach lattice, and $(e_k)_{k=1}^n$ is the canonical basis of ℓ_q^n. Then

$$\|S_{\overline{x}}\|_m = \Big\| \bigvee_{\sum_k |b_k|^q \leq 1} \sum_k b_k x_k \Big\| = \Big\| \Big(\sum_k |x_k|^p \Big)^{1/p} \Big\|. \tag{6.7.2}$$

In particular, given an operator $S = S_{\overline{x}}\colon \ell_q^n \to E : e_k \mapsto x_k$, where E is a Banach space, we can compose it with ϕ_E to get an operator of the above form. Applying (6.7.2) to $\phi_E S_{\overline{x}}$, we get

$$\Big\| \Big(\sum_k |\delta_{x_k}|^p \Big)^{1/p} \Big\| = \|\phi_E S_{\overline{x}}\|_m .$$

This gives a connection with (6.6.1).

Proposition 6.7.7 presents several equivalent characterizations of operators that map norm convergent sequences to uniformly convergent sequences. In [314, Propositions 5.3 and 5.4] the authors study operators that map uniformly convergent sequences to uniformly convergent sequences. We now present an analogue of the equivalence of statements (2) and (5) in Proposition 6.7.7 for such operators:

Proposition 6.7.10. *Let $T : E \subseteq X \to Y$ be a positively homogeneous map where X and Y are Banach lattices and E is a subspace of X. The following are equivalent:*

(i) T is sequentially uniformly continuous; i.e., $x_k \xrightarrow{u} 0$ implies $Tx_k \xrightarrow{u} 0$ for all sequences (x_k) in E;

(ii) T is (∞,∞)-regular in the sense of [295]; i.e., there exists M such that for any n and any $x_1, \dots, x_n$ in E,

$$\Big\| \bigvee_{k=1}^n |Tx_k| \Big\| \leq M \Big\| \bigvee_{k=1}^n |x_k| \Big\|.$$

Proof. (2)⇒(1): Suppose $x_k \xrightarrow{u} 0$. Then (x_k) is order bounded so there exists K such that $\|\bigvee_{k=1}^n |x_k|\| < K$ for each n. By (2), $\|\bigvee_{k=1}^n |Tx_k|\| \le KM$, and we can apply [314, Proposition 5.4]. Formally, [314, Proposition 5.4] is stated for linear maps, but the proof works for positively homogeneous maps.

(1)⇒(2): Suppose (2) fails. Then by positive homogeneity of the inequality, for every m there exists $x_1^m, \dots, x_{n_m}^m$ with $\|\bigvee_{k=1}^{n_m} |x_k^m|\| < \frac{1}{2^m}$ and $\|\bigvee_{k=1}^{n_m} |Tx_k^m|\| > m$. Then the sequence $x_1^1, \dots, x_{n_1}^1, x_1^2, \dots$ is u-null but after applying T [314, Proposition 5.4] fails. □

Remark 6.7.11. Sequentially uniformly continuous operators also appear in the theory of multinormed spaces, and it is known that a bounded operator between Banach lattices is ∞-multi-bounded if and only if it is 1-multi-bounded if and only if it is pre-regular; see Sections 4.2 and 4.5 in [89]. If X and Y are Banach lattices and E a *subspace* of X then one can also show that a linear map $T : E \subseteq X \to Y$ is sequentially uniformly continuous if and only if it is order bounded (in the sense of [314]) when viewed as a map $T : E \subseteq X \to Y^{**}$. Here, an operator $T : A \subseteq B \to C$ defined on a subspace A of a vector lattice B and taking values in a vector lattice C is *order bounded* if for any $b \in B_+$, the image of $[-b, b] \cap A$ under T is order bounded in C.

Indeed, if $T : E \subseteq X \to Y^{**}$ is order bounded, then it is sequentially uniformly continuous as a map into Y^{**} (see [315, Proposition 24.1]), and hence into Y since uniform convergence of sequences passes between closed sublattices ([314, Proposition 2.12]). Conversely, let $B \subseteq E$ be order bounded. Then there exists $x \in X$ with $|b| \le x$ for each $b \in B$. Now note that for $b_1, \dots, b_n$ in B we have $\||b_1| \vee \cdots \vee |b_n|\| \le \|x\|$ so that by Proposition 6.7.10

$$\left\| \bigvee_{k=1}^{n} |Tb_k| \right\| \le M\|x\|.$$

Now let $\mathcal{F} = \mathcal{P}_f(B)$ be the family of finite subsets of B, directed by inclusion. For $F \in \mathcal{F}$ set $y_F = \bigvee\{|Ty| : y \in F\}$. Then (y_F) is an increasing and norm bounded net in Y, hence has supremum in Y^{**}. Hence, $T(B)$ is order bounded in Y^{**} and we are done. Note the only property of Y^{**} we need is monotonically bounded, so we can replace Y^{**} by any monotonically bounded Banach lattice containing Y as a closed sublattice.

As a corollary, since order bounded embeddings map absolute sequences to absolute sequences (this follows directly from [314, Proposition 7.5]; see Section 6.7 below for more information on absolute sequences), so do sequentially uniformly continuous embeddings.

Given the fact that from every u-null net one can extract a u-null sequence, one may wonder if sequences can be replaced with nets in Proposition 6.7.7. This is *not* true. Indeed, in [137] the authors study strongly majorizing and Carleman operators. A linear map $T : E \to F$ from a normed space E to an Archimedean vector lattice F is *strongly majorizing* if T maps the unit ball of E into an order interval in F - one can easily show that these are exactly the operators which satisfy conditions (2) and (3) in Proposition 6.7.7 when sequences are replaced by nets. Note, for instance, that the identity map on c_0 is majorizing, but not strongly so. The reason that there are sequentially uniformly continuous but not uniformly continuous operators (even though uniform convergence is a sequential convergence) stems from the fact that uniform convergence is not topological.

For *Carleman operators* (operators mapping the unit ball into an order interval in the universal completion of the range), another nice characterization is available: T is Carleman if and only if T maps norm null nets to *uo*-null nets. We refer the reader to [122, 198, 317] for information on *uo*-convergence and its applications. In summary, many of the operators defined via "boundedness" in the literature are in fact merely continuous operators, if one finds the right notions of convergence. Moreover, many of the fundamental results hold if the operator is merely defined on a subspace of the lattice.

Absolute bases

Recall that the bibasis inequality (6.7.1) arises by commuting the supremum with the norm in the usual basis inequality. If one instead begins with the inequality

$$\bigvee_{\varepsilon_k=\pm1}\Big\|\sum_{k=1}^m \varepsilon_k a_k x_k\Big\| \leq M\Big\|\sum_{k=1}^m a_k x_k\Big\|,$$

characterizing unconditional sequences, brings the sup inside the norm, and notes that $\bigvee_{\varepsilon_k=\pm1}|\sum_{k=1}^m \varepsilon_k a_k x_k| = \sum_{k=1}^m |a_k x_k|$, one arrives at the notion of an absolute sequence from [314]. More formally, we say that a basic sequence (x_k) in a Banach lattice X is *absolute* if there exists a constant $A \geq 1$ such that

$$\Big\|\sum_{k=1}^m |a_k x_k|\Big\| \leq A\Big\|\sum_{k=1}^m a_k x_k\Big\|$$

for all $m \in \mathbb{N}$ and scalars $a_1, \ldots, a_m$. [314, Theorem 7.2] shows that (x_k) is absolute if and only if the convergence of $\sum_{k=1}^{\infty} a_k x_k$ is equivalent to the convergence of $\sum_{k=1}^{\infty} |a_k x_k|$. Note that $\left\| \sum_k a_k x_k \right\| \leq \left\| \sum_k |a_k x_k| \right\|$. Consequently, for any absolute basic sequence we have $\left\| \sum_k a_k x_k \right\| \sim \left\| \sum_k |a_k x_k| \right\|$. As is easy to see, absolute sequences must be both unconditional and bibasic.

By [314, Proposition 7.8], any sequence (x_k) in a Banach lattice, equivalent to the ℓ_1 basis (on the Banach space level), is absolute. On the other hand, from the above discussion it is clear that in AM-spaces a sequence is absolute if and only if it is unconditional. In this short subsection, we examine conditions on the sequence $(x_k) \subseteq E$ so that its canonical image (δ_{x_k}) is absolute in $\mathrm{FBL}^{(p)}[E]$.

Proposition 6.7.12. *If (x_k) is a normalized basis of E, and (δ_{x_k}) is an absolute basic sequence in $\mathrm{FBL}^{(p)}[E]$ for $p < \infty$, then (x_k) is equivalent to the ℓ_1 basis.*

Proof. As noted above, the sequence (δ_{x_k}) (equivalently, (x_k)) is unconditional. Proposition 6.5.13 shows that, for any finite sequence of scalars (a_k),

$$\Big\| \sum_k a_k x_k \Big\| = \Big\| \sum_k a_k \delta_{x_k} \Big\| \lesssim \Big\| \sum_k a_k |\delta_{x_k}| \Big\| \leq \Big\| \sum_k |a_k| |\delta_{x_k}| \Big\| \sim \Big\| \sum_k a_k x_k \Big\|.$$

Now apply Proposition 6.6.3. $\square$

Remark 6.7.13. Above we showed that, for a semi-normalized basis (x_k) of E, the following two statements are equivalent: (i) (x_k) is equivalent to the ℓ_1 basis; (ii) for any sequence of scalars (a_k), $\sum_k a_k x_k$ converges if and only if $\sum_k |a_k J x_k|$ converges for any embedding $J : E \to Z$, where Z is a Banach lattice. This provides a converse to [314, Proposition 7.8], which states that any sequence in a Banach lattice which is equivalent to the ℓ_1 basis is absolute - we now know that ℓ_1 is the only normalized basis with this property. One can also notice the similarity with the well-known fact that, for a normalized basic sequence (x_k), the norm convergence of $\sum_k a_k x_k$ is equivalent to the convergence of $\sum_k \|a_k x_k\|$ if and only if (x_k) is equivalent to the ℓ_1 basis.

For positive sequences, being absolute is the same as being unconditional. Hence, for every unconditional basis (x_k) of E, $(|\delta_{x_k}|)$ is absolute in $\mathrm{FBL}^{(p)}[E]$. However, Proposition 6.5.18 shows that $(|\delta_{x_k}|)$ need not be absolute if (x_k) is a conditional basis. Is it at least true that $(|\delta_{x_k}|)$ is bibasic? If $p = \infty$ then this is clear, as $\mathrm{FBL}^{(\infty)}[E]$ is an AM-space.

However, when $p \in [1, \infty)$ the situation is not as transparent. For example, we do not know whether $(|\delta_{s_k}|)$ is bibasic in $\mathrm{FBL}^{(p)}[c_0]$ (here (s_k) is the summing basis of c_0, and $p \in [1, \infty)$). Example 6.5.16 shows that the dual to the summing basis is not bibasic in $\mathrm{FBL}^{(p)}[\ell_1]$ for any finite p, though its modulus is absolute.

Remark 6.7.14. The proof of Lemma 6.7.2 can be easily adapted to show that if (x_k) is a normalized basis of E such that $(\delta_{x_k}) \subseteq \mathrm{FBL}^{(p)}[E]$ is absolute, then every isomorphic embedding T from E to any p-convex Banach lattice X maps (x_k) to an absolute sequence. In fact, if $(\delta_{x_k}) \subseteq \mathrm{FBL}^{(p)}[E]$ is absolute and $p < \infty$, then (x_k) is equivalent to the unit vector basis of ℓ_1 by Proposition 6.7.12, hence so is (Tx_k), which implies that (Tx_k) is absolute. On the other hand, for $p = \infty$, we note that every unconditional basic sequence in an AM-space is automatically absolute.

6.8 Sublattices of free Banach lattices

In this section we investigate the sublattice structure of $\mathrm{FBL}^{(p)}[E]$. Let us begin with some necessary conditions for a Banach lattice to be a sublattice of $\mathrm{FBL}^{(p)}[E]$ (compare with [31], where conditions under which E embeds into $\mathrm{FBL}[E]$ as a lattice-complemented sublattice are explored). First, recall that a Banach lattice X satisfies the *σ-bounded chain condition* (*σ-bcc*) if there is a countable decomposition $X_+ \setminus \{0\} = \bigcup_{n \geq 2} \mathcal{F}_n$ such that for every n, every subset $\mathcal{G} \subseteq \mathcal{F}_n$ of size n contains a pair of non-disjoint elements. This is stronger than the countable chain condition, meaning that every uncountable family in X_+ contains a pair of non-disjoint elements (see [25]).

Proposition 6.8.1. *If $1 \leq p \leq \infty$, and F is a closed sublattice of $\mathrm{FBL}^{(p)}[E]$ for some Banach space E, then:*

(i) F is p-convex;

(ii) F satisfies the σ-bcc;

(iii) The real-valued lattice homomorphisms separate the points of F.

Proof. (1) is clear. (2) follows from the fact that $C_{ph}(B_{E^*})$, the lattice of positively homogeneous weak*-continuous functions on B_{E^*}, satisfies the σ-bcc [25, Theorem 1.3], and this is transferred to (not necessarily closed) sublattices. By construction, recall that $\mathrm{FBL}^{(p)}[E]$ can be seen as a (in general, non-closed) sublattice of $C_{ph}(B_{E^*})$.

(3) is analogous to [26, Corollary 2.7]: For every $x^* \in E^*$, the evaluation functional $\widehat{x^*} : \mathrm{FBL}^{(p)}[E] \to \mathbb{R}$ given by $\widehat{x^*}(f) = f(x^*)$ is a lattice homomorphism and clearly $f = g$ in $\mathrm{FBL}^{(p)}[E]$ if and only if $\widehat{x^*}(f) = \widehat{x^*}(g)$ for every $x^* \in E^*$. It follows that for every sublattice F of $\mathrm{FBL}^{(p)}[E]$, the real-valued lattice homomorphisms (obtained by restricting $\widehat{x^*}$ to F) separate the points of F. □

Remark 6.8.2. As a direct consequence of Proposition 6.8.1 we see that $L_p(\mu)$ can possibly embed as a sublattice of $\mathrm{FBL}^{(p)}[E]$ only when μ is purely atomic (with countably many atoms).

We will actually see next that ℓ_q always embeds as a sublattice of $\mathrm{FBL}^{(p)}[\ell_q]$ if $q \geq p$. Specifically, we show:

Theorem 6.8.3. *Suppose E is a Banach lattice, p-convex with constant 1, with the order induced by a 1-unconditional basis. Then $\mathrm{FBL}^{(p)}[E]$ contains an isometric copy of E as a sublattice. Moreover, there exists a contractive lattice homomorphic projection onto this sublattice.*

This provides an alternative approach to [31, Theorem 4.1], valid for arbitrary $p \in [1, \infty]$.

Throughout, we work with a fixed E from Theorem 6.8.3. For this proof, we change our notational conventions slightly and denote the normalized 1-unconditional basis for E by (e_i). The corresponding (normalized) biorthogonal functionals shall be denoted by $(e_i^*)_{i\in\mathbb{N}}$. For $N \in \mathbb{N}$ with $N \leq \dim E$, denote by P_N the canonical (contractive) projection from E onto $\mathrm{span}[e_i : 1 \leq i \leq N]$. Then P_N^* projects E^* onto $\mathrm{span}[e_i^* : 1 \leq i \leq N]$, with $P_N^* e_i^* = e_i^*$ if $i \leq N$, $P_N^* e_i^* = 0$ if $i > N$. We thus observe that $\mathrm{span}[e_i : 1 \leq i \leq N]$ and $\mathrm{span}[e_i^* : 1 \leq i \leq N]$ are in duality with each other.

We need to establish a technical lemma:

Lemma 6.8.4. *Denote by $\mathcal{P}$ the set of all finite sequences (β_i) so that $\sum_i |\beta_i\gamma_i|^p \leq 1$ whenever $\|\sum_i \gamma_i e_i\| \leq 1$. Then for every choice of scalars (α_i) we have*

$$\sup\Big\{\Big(\sum_i |\alpha_i\beta_i|^p\Big)^{1/p} : (\beta_i) \in \mathcal{P}\Big\} = \Big\|\sum_i \alpha_i e_i\Big\|.$$

Proof. Denote by $E_{(p)}$ the p-concavification of E, as described in e.g. [231, Section 1.d]. If (t_i) is a finite sequence, then we can view $\sum_i t_i e_i$ as an element of $E_{(p)}$, with the norm

$$\Big\|\sum_i t_i e_i\Big\|_{E_{(p)}} = \Big\|\sum_i t_i^{1/p} e_i\Big\|_E^p, \text{ where } t^{1/p} = \operatorname{sign} t \cdot |t|^{1/p}.$$

Such finitely supported sequences are dense in $E_{(p)}$.

Due to the unconditionality of the basis (e_i), we can assume $\alpha_i \geq 0$, and there exists $N \in \mathbb{N}$ so that $\alpha_i = 0$ whenever $i > N$. Projecting onto $\mathrm{span}[e_i : 1 \leq i \leq N]$ and $\mathrm{span}[e_i^* : 1 \leq i \leq N]$, we can assume that $\beta_i = 0 = \gamma_i$ for $i > N$. By unconditionality, we also assume $\beta_i, \gamma_i \geq 0$. Now let $t_i = \alpha_i^p$, $s_i = \beta_i^p$, $z_i = \gamma_i^p$. Then $(\beta_i) \in \mathcal{P}$ if and only if $\sum_i s_i z_i \leq 1$ whenever

$$\big\| \sum_i z_i^{1/p} e_i \big\|_E^p = \big\| \sum_i z_i e_i \big\|_{E_{(p)}} \leq 1.$$

By duality, $\big\| \sum_i s_i e_i^* \big\|_{(E_{(p)})^*} \leq 1$ if and only if $(\beta_i) \in \mathcal{P}$, and therefore,

$$\begin{aligned} \sup\Big\{ \big(\sum_i |\alpha_i \beta_i|^p \big)^{1/p} : (\beta_i) \in \mathcal{P} \Big\} &= \sup\Big\{ \big(\sum_i t_i s_i \big)^{1/p} : \big\| \sum_i s_i e_i^* \big\|_{(E_{(p)})^*} \leq 1 \Big\} \\ &= \big\| \sum_i t_i e_i \big\|_{E_{(p)}}^{1/p} = \big\| \sum_i \alpha_i e_i \big\|_E, \end{aligned}$$

which is what we want. $\square$

Proof of Theorem 6.8.3. We present a proof in the case of infinite dimensional E. Only minor adjustments are needed to handle the finite dimensional setting.

We find a sequence of disjoint functions $f_i \in \mathrm{FBL}^{(p)}[E]_+$, so that, for any finite sequence of positive numbers $(\alpha_i)_i$,

$$\big\| \sum_i \alpha_i f_i \big\|_{\mathrm{FBL}^{(p)}[E]} = \big\| \sum_i \alpha_i e_i \big\|_E. \tag{6.8.1}$$

Once this is established, we conclude that E is lattice isometric to $F = \mathrm{span}[f_i : i \in \mathbb{N}]$.

For $k \in \mathbb{N}$, define $f_k \in H[E]_+$:

$$f_k = \Big(|\delta_{e_k}| - 2^{2k} \big(\sum_{i<k} |\delta_{e_i}| + \sum_{i>k} 2^{-i} |\delta_{e_i}| \big) \Big)_+.$$

As $\sum_i 2^{-i} |\delta_{e_i}|$ converges in norm, f_k actually belongs to $\mathrm{FBL}^{(p)}[E]$; in fact,

$$f_k = \lim_N \Big(|\delta_{e_k}| - 2^{2k} \big(\sum_{i<k} |\delta_{e_i}| + \sum_{i=k+1}^{N} 2^{-i} |\delta_{e_i}| \big) \Big)_+.$$

Moreover, the functions f_k are disjoint. Indeed, suppose $i < j$, and $e^* \in E^*$ is such that both $f_i(e^*), f_j(e^*) > 0$. We will derive a contradiction. To this end, observe that

$$f_i(e^*) \le \big|e^*(e_i)\big| - 2^{2i-j}\big|e^*(e_j)\big| \text{ and } f_j(e^*) \le \big|e^*(e_j)\big| - 2^{2j}\big|e^*(e_i)\big|.$$

Since both $f_i(e^*)$ and $f_j(e^*)$ are strictly positive, we have

$$\big|e^*(e_i)\big| > 2^{2i-j}\big|e^*(e_j)\big| > 2^{2i+j}\big|e^*(e_i)\big|,$$

which is impossible.

To establish (6.8.1), let $f = \sum_{k=1}^N \alpha_k f_k$ (recall that $\alpha_k \ge 0$). By scaling, assume $\big\| \sum_k \alpha_k e_k \big\|_E = 1$. Recall that $\|f\|_{\mathrm{FBL}^{(p)}[E]}$ is the supremum of $\big(\sum_i |f(x_i^*)|^p\big)^{1/p}$, given $\sup_{x \in B_E} \sum_i |x_i^*(x)|^p \le 1$ (with finite sums).

We first obtain a lower estimate on $\|f\|$. By Lemma 6.8.4, we can find $(\beta_i)_{i=1}^N \subseteq [0,\infty)$ so that $\big(\sum_i (\alpha_i\beta_i)^p\big)^{1/p} = \| \sum_i \alpha_i e_i \| = 1$, and $\sum_i |\beta_i \gamma_i|^p \le 1$ whenever $\| \sum_i \gamma_i e_i \| \le 1$. Let $x_i^* = \beta_i e_i^*$ $(1 \le i \le N)$. Then clearly $\sup_{x \in B_E} \sum_i |x_i^*(x)|^p \le 1$, and

$$\|f\| \ge \Big(\sum_{i=1}^n |f(x_i^*)|^p \Big)^{1/p} = \Big(\sum_i (\alpha_i \beta_i)^p \Big)^{1/p} = 1.$$

Next we need to establish an upper bound for $\|f\|_{\mathrm{FBL}^{(p)}[E]}$. Specifically, we suppose $\sup_{x \in B_E} \sum_i |x_i^*(x)|^p \le 1$, and show that $\big(\sum_{i=1}^n |f(x_i^*)|^p\big)^{1/p} \le 1$. As $f(x^*) = f(-x^*)$ for every $x^* \in E^*$, we can and do assume that $f(x_j^*) \ge 0$ for any j.

Define an auxiliary function $g : E^* \to [0,\infty)$ (not necessarily continuous) in the following manner. For $x^* \in E^*$ let $\mathcal{I}(x^*) = \{k : f_k(x^*) \ne 0\}$. As the functions f_k are disjointly supported, $\mathcal{I}(x^*)$ is either empty or a singleton. If $\mathcal{I}(x^*) = \emptyset$, let $g(x^*) = 0$. If $\mathcal{I}(x^*) = \{i\}$, let $g(x^*) = \alpha_i \big|x^*(e_i)\big|$.

Now define the disjoint sets $S_i = \{j : f_i(x_j^*) \ne 0\}$ (that is, $j \in S_i$ if and only if $\mathcal{I}(x_j^*) = \{i\}$). For $j \in S_i$, we have $|f(x_j^*)| \le \alpha_i \big|x_j^*(e_i)\big| = g(x_j^*)$, hence it suffices to show that

$$\sum_j \big|g(x_j^*)\big|^p \le 1.$$

Find (positive) scalars (t_j) so that $\sum_j |t_j|^q = 1$ and $\left(\sum_{j=1}^n |g(x_j^*)|^p\right)^{1/p} = \sum_j t_j g(x_j^*)$. For $j \in S_i$, let $\varepsilon_j = |x_j^*(e_i)|/x_j^*(e_i)$ if $x_j^*(e_i) \neq 0$ and $\varepsilon_j = 0$ otherwise, and set

$$y_i^* = \kappa_i^{-1} \sum_{j \in S_i} t_j \varepsilon_j x_j^*, \text{ where } \kappa_i = \big(\sum_{j \in S_i} |t_j|^q\big)^{1/q}.$$

Note first that, for $x \in B_E$, $\sum_i |y_i^*(x)|^p \leq 1$. Indeed, find scalars (s_i) so that $\sum_i |s_i|^q = 1$, and $\left(\sum_i |y_i^*(x)|^p\right)^{1/p} = \sum_i s_i y_i^*(x)$. Write $u_j = s_i/\kappa_i$ if $j \in S_i$. Then

$$\sum_i s_i y_i^*(x) = \sum_j u_j t_j \varepsilon_j x_j^*(x) \leq \big(\sum_j |u_j|^q |t_j|^q\big)^{1/q} \big(\sum_j |x_j^*(x)|^p\big)^{1/p}.$$

We have

$$\sum_j |u_j|^q |t_j|^q = \sum_i |s_i|^q \kappa_i^{-q} \sum_{j \in S_i} |t_j|^q = \sum_i |s_i|^q = 1,$$

so $\sum_i s_i y_i^*(x) \leq 1$.

Next we show that $\sum_i \left|g(y_i^*)\right|^p \geq \sum_j \left|g(x_j^*)\right|^p$. To this end, note first that, by the definition of f_i, $\mathcal{I}(x^*) = \{i\}$ if and only if

$$\left|x^*(e_i)\right| > 2^{2i}\big(\sum_{k<i} \left|x^*(e_k)\right| + \sum_{k>i} 2^{-k}\left|x^*(e_k)\right|\big).$$

For any $j \in S_i$, we have

$$|x_j^*(e_i)| > 2^{2i}\big(\sum_{k<i} \left|x_j^*(e_k)\right| + \sum_{k>i} 2^{-k}\left|x_j^*(e_k)\right|\big).$$

By the convexity of the absolute value, it follows that

$$|y_i^*(e_i)| = \frac{1}{\kappa_i} \sum_{j \in S_i} t_j |x_j^*(e_i)| > 2^{2i}\big(\sum_{k<i} \left|y_i^*(e_k)\right| + \sum_{k>i} 2^{-k}\left|y_i^*(e_k)\right|\big)$$

as well. Therefore, $g(y_i^*) = \alpha_i |y_i^*(e_i)| = \kappa_i^{-1} \sum_{j \in S_i} t_j g(x_j^*)$. However, $\sum_i \kappa_i^q = 1$, hence

$$\left(\sum_i \left|g(y_i^*)\right|^p\right)^{\frac{1}{p}} \geq \sum_i \kappa_i g(y_i^*) = \left(\sum_j \left|g(x_j^*)\right|^p\right)^{\frac{1}{p}}.$$

The reasoning above implies that it suffices to show that

$$\sum_i \alpha_i^p \left|y_i^*(e_i)\right|^p \leq 1 \text{ whenever } \sup_{x \in B_E} \sum_i \left|y_i^*(x)\right|^p = 1. \tag{6.8.2}$$

By projecting, we can assume that $y_1^*, \dots, y_N^*$ "live" in $E_N^* = \mathrm{span}[e_i^* : 1 \le i \le N]$; this space can be interpreted as the dual of $E_N = \mathrm{span}[e_i : 1 \le i \le N]$. Consider the operator

$$T : E_N \to \ell_p^N : x \mapsto \sum_{i=1}^{N} y_i^*(x)\sigma_i,$$

where (σ_i) is the canonical basis for ℓ_p^N. The condition $\sup_{x \in B_E} \sum_i \left|y_i^*(x)\right|^p = 1$ is equivalent to T being contractive. Tong's argument [230, Proposition 1.c.8] shows that the diagonalization of T – that is, the operator

$$T' : E_N \to \ell_p^N : x \mapsto \sum_{i=1}^{N} y_i^*(e_i)e_i^*(x)\sigma_i,$$

is contractive as well. Taking $x = \sum_{i=1}^N \alpha_i e_i$, we conclude that

$$\|T'x\| = \Big(\sum_i \alpha_i^p \big|y_i^*(e_i)\big|^p\Big)^{1/p} \le \|x\| = 1,$$

which implies (6.8.2).

It remains to show that there exists a contractive lattice homomorphic projection from $\mathrm{FBL}^{(p)}[E]$ onto $\mathrm{span}[f_i : i \in \mathbb{N}]$. To this end, recall that the identity map $I : E \to E$ has a unique lattice homomorphic contractive extension $\widehat{I} : \mathrm{FBL}^{(p)}[E] \to E$. In particular, we have

$$\begin{aligned} &\widehat{I}\Big(\big|\delta_{e_k}\big| - 2^{2k}\big(\sum_{i<k}\big|\delta_{e_i}\big| + \sum_{i=k+1}^{p} 2^{-i}\big|\delta_{e_i}\big|\big)\Big)_+ \\ &= \Big(\big|Ie_k\big| - 2^{2k}\big(\sum_{i<k}\big|Ie_i\big| + \sum_{i=k+1}^{p} 2^{-i}\big|Ie_i\big|\big)\Big)_+ = e_k, \end{aligned}$$

hence by continuity, $\widehat{I}f_k = e_k$ for any k.

Note that the map $U : E \to \mathrm{FBL}^{(p)}[E] : e_k \mapsto f_k$ is a lattice isometry, and $\widehat{I}U$ is the identity on E. Therefore, $U\widehat{I}$ is a contractive projection onto $\mathrm{span}[f_i : i \in \mathbb{N}]$, and a lattice homomorphism. □

Remark 6.8.5. Note that if E is a p-convex Banach lattice, then the identity on E extends to a lattice homomorphism $\beta_E : \mathrm{FBL}^{(p)}[E] \to E$, that is $\beta_E \phi_E = id_E$. This allows us to see

E as a complemented subspace of $\mathrm{FBL}^{(p)}[E]$. There is a partial converse to this: Suppose E is a Banach lattice which is isomorphic to a complemented subspace of $\mathrm{FBL}^{(p)}[E]$ for some $1 \le p \le 2$, then E must be itself p-convex (see [231, Theorem 1.d.7] and the remark after it).

For the next proposition, we need some notation. Let us denote by $\mathrm{FBL}^{(p)n}[E]$ the n-fold iterate of $\mathrm{FBL}^{(p)}$'s, i.e., $\mathrm{FBL}^{(p)1}[E] = \mathrm{FBL}^{(p)}[E]$, $\mathrm{FBL}^{(p)2}[E] = \mathrm{FBL}^{(p)}[\mathrm{FBL}^{(p)}[E]]$, etc. Following the same ideas in [31] we have:

Proposition 6.8.6. *For every Banach space E and $n \ge 1$, there is a lattice isometric embedding $S : \mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)n}[E]$ and a contractive lattice projection onto $S(\mathrm{FBL}^{(p)}[E])$.*

Proof. By convention, let us set $\mathrm{FBL}^{(p)0}[E] = E$ and for $k \in \mathbb{N}$ let $\phi_k : \mathrm{FBL}^{(p)k-1}[E] \to \mathrm{FBL}^{(p)k}[E]$ be the canonical embedding. Let

$$T = \phi_n \circ \cdots \circ \phi_1 : E \to \mathrm{FBL}^{(p)n}[E].$$

Since $\mathrm{FBL}^{(p)n}[E]$ is p-convex, there is a lattice homomorphism $\widehat{T} : \mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)n}[E]$ extending T, that is, $\widehat{T} \circ \phi_1 = T$, with $\|\widehat{T}\| = \|T\| = 1$.

Now for $k \in \mathbb{N}$ let $\beta_k := \widehat{I_k} : \mathrm{FBL}^{(p)k+1}[E] \to \mathrm{FBL}^{(p)k}[E]$ be the extension of the identity map $I_k : \mathrm{FBL}^{(p)k}[E] \to \mathrm{FBL}^{(p)k}[E]$, i.e., $\beta_k \circ \phi_{k+1} = I_k$ and $\|\beta_k\| = 1$. Finally, define

$$\beta := \beta_1 \circ \cdots \circ \beta_{n-1} : \mathrm{FBL}^{(p)n}[E] \to \mathrm{FBL}^{(p)}[E].$$

We claim that $\beta\widehat{T} = I_{\mathrm{FBL}^{(p)}[E]}$. Indeed, given $x \in E$ we have

$$\beta\widehat{T}\phi_1(x) = \beta_1 \circ \cdots \circ \beta_{n-1} \circ \phi_n \circ \cdots \circ \phi_1(x) = \phi_1(x).$$

Since $\beta\widehat{T}$ is a lattice homomorphism and $\mathrm{FBL}^{(p)}[E]$ is lattice-generated by the elements of the form $\phi_1(x)$ with $x \in E$, it follows that $\beta\widehat{T} = I_{\mathrm{FBL}^{(p)}[E]}$, as claimed. $\square$

Remark 6.8.7. Proposition 6.8.6 is related to the fact that the pair $(E, \mathrm{FBL}^{(p)}[E])$ has the 1-POE-p. Actually, $(E, \mathrm{FBL}^{(p)}[E])$ has the 1-POE-q for every $q \ge p$.

Finally, we remark that some other results from [31] are valid in the p-convex category. For example, if a p-convex Banach lattice P is projective for p-convex lattices (in the sense of [185]) then it embeds as a lattice-complemented sublattice of $\mathrm{FBL}^{(p)}[P]$.

6.9 Encoding properties of E as properties of the free space

In this section we begin to build a dictionary between Banach space properties of E and Banach lattice properties of $\mathrm{FBL}^{(p)}[E]$. There are several results already known in this direction:

(i) E is isomorphic to a complemented subspace of a p-convex Banach lattice if and only if $\phi(E)$ is complemented in $\mathrm{FBL}^{(p)}[E]$;

(ii) E is C-linearly projective for p-convex lattices if and only if $\mathrm{FBL}^{(p)}[E]$ is C-projective for p-convex lattices [220];

(iii) More generally, we have characterized some relations between an operator $T : F \to E$ and the induced operator $\overline{T} : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$. See Proposition 6.3.2.

In this section we significantly expand this list.

Finite dimensionality corresponds to strong units and separability to quasi-interior points

Recall that when E is finite dimensional, $\mathrm{FBL}^{(p)}[E]$ is lattice isomorphic to the space of continuous functions $C(S_{E^*})$, where S_{E^*} is the unit sphere of E^*. In particular, it is ∞-convex. The situation is completely different when E is infinite dimensional. Indeed, we now show that when $\dim E = \infty$, $\mathrm{FBL}^{(p)}[E]$ never has a strong unit. Moreover, in Section 6.9 we show that $\mathrm{FBL}[E]$ can never be more than 2-convex.

Proposition 6.9.1. *For any $1 \leq p \leq \infty$, $\mathrm{FBL}^{(p)}[E]$ has a strong unit if and only if E is finite dimensional.*

Proof. When E is finite dimensional, all the $\mathrm{FBL}^{(p)}[E]$ are lattice isomorphic to each other, and to a $C(K)$-space, so in particular they have a strong unit. Conversely, suppose there exists $e \in \mathrm{FBL}^{(p)}[E]$ such that $|\delta_x| \leq e$ for all $x \in B_E$. Now, for each $x^* \in S_{E^*}$ and $\varepsilon > 0$, we can find $x \in B_E$ such that $|x^*(x)| \geq 1 - \varepsilon$. Hence, from $|\delta_x| \leq e$ and the pointwise ordering we get $1 - \varepsilon \leq e(x^*)$. Hence, e takes at least the value one on S_{E^*}. However, $\mathrm{FBL}^{(p)}[E]$ is the closure of the sublattice generated by $\{\delta_x : x \in E\}$. By [12, p. 204, Exercise 8(b)], a typical element of the (non-closed) sublattice generated by $\{\delta_x : x \in E\}$ is of the form $f = \bigvee_{k=1}^n \delta_{x_k} - \bigvee_{k=1}^n \delta_{y_k}$ with $n \in \mathbb{N}$, $x_1, \dots, x_n, y_1 \dots, y_n \in E$. Since E is infinite dimensional, we can find $x^* \in S_{E^*}$ such that $x^*(x_k) = x^*(y_k) = 0$ for $k = 1, \dots, n$. Then

$$\|e - f\| \geq |e(x^*) - f(x^*)| \geq 1.$$

Hence, e is not in the closure of this sublattice, so is not in $\mathrm{FBL}^{(p)}[E]$. □

Remark 6.9.2. By contrast, $\mathrm{FBL}^{(\infty)}[E]$ may be linearly isomorphic to a Banach lattice with strong unit: Proposition 6.10.32 shows that, if E separable, then $\mathrm{FBL}^{(\infty)}[E]$ is isomorphic to $C[0, 1]$. Separability is essential here, per Remark 6.10.33.

Remark 6.9.3. The proof of Proposition 6.9.1 can be adapted to show that for infinite dimensional E, $\mathrm{FBL}^{(\infty)}[E]$ cannot be monotonically bounded (increasing norm bounded nets are order bounded). Indeed, if $\mathrm{FBL}^{(\infty)}[E]$ were monotonically bounded, we could order the finite subsets of B_E by inclusion, and consider the net $\{x_1, \dots, x_n\} \mapsto |\delta_{x_1}| \vee \cdots \vee |\delta_{x_n}|$. By definition of the $\mathrm{FBL}^{(\infty)}$-norm, this net is norm bounded, hence order bounded. Hence, there exists $e \in \mathrm{FBL}^{(\infty)}[E]$ such that $|\delta_x| \leq e$ for all $x \in B_E$ and we proceed as in the proof of Proposition 6.9.1 to reach a contradiction. Note, however, that it *is* possible for $\mathrm{FBL}[E]$ to be monotonically bounded - even strong Nakano (cf. [26, Theorem 4.11]). On the other hand, there are Banach spaces E for which $\mathrm{FBL}[E]$ does not even have the Fatou property ([26, Theorem 4.13]), which is a non-trivial weakening of the strong Nakano property. A characterization of when $\mathrm{FBL}[E]$ has these, or related, properties in terms of properties of E is not currently known.

We now characterize when $\mathrm{FBL}^{(p)}[E]$ has a quasi-interior point. Recall that an element e of a Banach lattice X is a *quasi-interior point* if the closed ideal generated by e is the whole of X. The *center* of X, denoted $Z(X)$, is the space of all linear operators T on X for which there is a real number $\lambda > 0$ satisfying $|Tx| \leq \lambda |x|$ for all $x \in X$. The center is *trivial* if the only elements of $Z(E)$ are the scalar multiples of the identity operator; the center is called *topologically full* if for each $x, y \in X$ with $0 \leq x \leq y$ there is a sequence (T_n) in $Z(X)$ with $T_n y \to x$ in norm.

Proposition 6.9.4. *Let E be a non-zero Banach space and $p \in [1, \infty]$. The following are equivalent:*

(i) E is separable;

(ii) $\mathrm{FBL}^{(p)}[E]$ has a quasi-interior point;

(iii) $Z(\mathrm{FBL}^{(p)}[E])$ *is topologically full;*

(iv) $Z(\mathrm{FBL}^{(p)}[E])$ *is non-trivial.*

Proof. If E is separable then $\mathrm{FBL}^{(p)}[E]$ is separable and, therefore, has a quasi-interior point. Conversely, suppose $\mathrm{FBL}^{(p)}[E]$ has a quasi-interior point, say e. Without loss of generality, $e \geqslant 0$. If $x^* \in B_{E^*}$ satisfies $e(x^*) = 0$ then $f(x^*) = 0$ for every $f \in I_e$ and, therefore, for all $f \in \mathrm{FBL}^{(p)}[E]$. Thus, e only vanishes at 0. For every n, let $U_n = \{x^* \in B_{E^*} : e(x^*) < \frac{1}{n}\}$. Then U_n is a weak*-open subset of B_{E^*} and $\bigcap_{n=1}^{\infty} U_n = \{0\}$. Now the relevant direction of the proof of [87, Theorem 5.1, p. 134] shows that E is separable. This shows (1) $\Leftrightarrow$ (2).

The rest of the proof is inspired by [269, Theorem 8.4]. From the proof of [269, Theorem 8.4], every Banach lattice with a quasi-interior point has a topologically full center. Since E is non-zero, the center being topologically full implies it is non-trivial. For the implication (4)$\Rightarrow$(1), suppose that the center is non-trivial. By [327, Theorem 3.1], $\{0\}$ is a G_δ set which, as before, implies that E is separable. $\square$

Remark 6.9.5. Unlike with strong units and quasi-interior points, $\mathrm{FBL}^{(p)}[E]$ always has a weak unit, as was noted in Proposition 6.2.12.

Number of generators

For a Banach lattice X, we denote by $\mathbf{n}(X)$ the smallest cardinality of a set S which generates X as a Banach lattice. For general information on this parameter, see [298, Section V.2].

Proposition 6.9.6. *Suppose E is a Banach space.*

(i) If $\dim E = n \in \mathbb{N}$, *then* $\mathbf{n}(\mathrm{FBL}^{(p)}[E]) = n$.

(ii) If $\dim E = \infty$, *then* $\mathbf{n}(\mathrm{FBL}^{(p)}[E]) = \infty$.

Proof. If $\dim E = n$, and $e_1, \ldots, e_n \in E$ form a basis of E, then $\delta_{e_1}, \ldots, \delta_{e_n}$ also generate $\mathrm{FBL}^{(p)}[E]$ as a Banach lattice.

Now consider $f_1, \ldots, f_m \in \mathrm{FBL}^{(p)}[E]$, with $m < \dim E$. By Borsuk-Ulam Theorem, these functions cannot separate points of the sphere of E^*: There exist distinct $e_1^*, e_2^* \in S_{E^*}$ such that $f_i(e_1^*) = f_i(e_2^*)$ for $1 \leq i \leq m$. As point evaluations are continuous on $\mathrm{FBL}^{(p)}[E]$, $f(e_1^*) = f(e_2^*)$ for any f in the Banach lattice L generated by $f_1, \ldots, f_m$ inside of $\mathrm{FBL}^{(p)}[E]$.

There exists $e \in E$ so that $e_1^*(e) \neq e_2^*(e)$, or equivalently, $\delta_e(e_1^*) \neq \delta_e(e_2^*)$. Consequently, L is a proper sublattice of $\mathrm{FBL}^{(p)}[E]$. $\square$

Weakly compactly generated spaces

Free Banach lattices played a fundamental role in solving a problem regarding weak compact generation, raised by J. Diestel in a conference in La Manga (Spain) in 2011 (see [26]). Recall that a Banach space E is weakly compactly generated (WCG) provided there is a weakly compact set $K \subseteq E$ whose linear span is dense. The Diestel question was first analyzed in [36], where the following terminology was introduced: A Banach lattice X is weakly compactly generated as a lattice (LWCG, for short) if there is a weakly compact set $K \subseteq X$ so that the sublattice generated by K is dense in X. Diestel asked whether, for Banach lattices, the notions of LWCG and WCG are equivalent. A few years later, this was answered in the negative: [26] shows that $\mathrm{FBL}[\ell_p(\Gamma)]$ (for $1 < p \leq 2$) is LWCG but not WCG as long as the index set Γ is uncountable.

The following observation was made in [26] for $p = 1$: If E is a p-convex Banach lattice, and $\mathrm{FBL}^{(p)}[E]$ is LWCG, then so is E. This is because the identity $I : E \to E$ extends to a surjective lattice homomorphism $\widehat{I} : \mathrm{FBL}^{(p)}[E] \to E$, and a lattice homomorphic image of an LWCG lattice is again LWCG.

Clearly, if E is WCG, then $\mathrm{FBL}^{(p)}[E]$ is LWCG. We can also establish a partial converse:

Proposition 6.9.7. *Suppose $1 \leq p \leq \infty$, and E is either a p-convex order continuous Banach lattice, or an AM-space. Then $\mathrm{FBL}^{(p)}[E]$ is LWCG if and only if E is WCG.*

Proof. As noted in the above paragraph, we only need to show that, if E is a p-convex order continuous Banach lattice or an AM-space, and $\mathrm{FBL}^{(p)}[E]$ is LWCG, then E is WCG. The reasoning above shows that, if $\mathrm{FBL}^{(p)}[E]$ is LWCG, then so is E. Consequently, E is WCG (apply [36, Theorem 2.2] for AM-spaces, and [36, Theorem 3.1] in the order continuous case). $\square$

Remark 6.9.8. Let $\mathrm{FBL}^{(p)}[E]$ be LWCG and let $K \subseteq \mathrm{FBL}^{(p)}[E]$ be a weakly compact set generating $\mathrm{FBL}^{(p)}[E]$ as a lattice. Note that if $K \subseteq \phi(E)$, then E is WCG. Indeed, let F denote the closed linear span of K, and suppose $F \subsetneq \phi(E)$. Let $x \in E$ be such that $\delta_x \notin F$. By Hahn-Banach we can take $x^* \in B_{E^*}$ such that $y(x^*) = x^*(\phi^{-1}(y)) = 0$ for $y \in K$

and $x^*(x) > 0$. As $y(x^*) = 0$ for every $y \in K$, and K generates $\mathrm{FBL}^{(p)}[E]$ it follows that $f(x^*) = 0$ for every $f \in \mathrm{FBL}^{(p)}[E]$. This is a contradiction with $\delta_x(x^*) > 0$. Therefore, $F = \phi(E)$ and E is WCG. However, there seems to be a priori no reason to guarantee that when $\mathrm{FBL}^{(p)}[E]$ is LWCG, there is a generating weakly compact set lying in $\phi(E)$.

We can characterize when $\mathrm{FBL}^{(p)}[E]$ is LWCG as follows:

Proposition 6.9.9. *Given a Banach space E, the following are equivalent:*

(i) $\mathrm{FBL}^{(p)}[E]$ *is LWCG.*

(ii) There exist a WCG Banach space F and a lattice homomorphism $T : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ with $\|T\| = 1$ and dense range.

Proof. Suppose first that $\mathrm{FBL}^{(p)}[E]$ is LWCG. Let $K \subseteq \mathrm{FBL}^{(p)}[E]$ be a weakly compact set whose lattice span is dense. Let $F \subseteq \mathrm{FBL}^{(p)}[E]$ be the closed linear span of K. Clearly F is WCG, and the formal inclusion $\iota : F \to \mathrm{FBL}^{(p)}[E]$ extends to a lattice homomorphism $T = \widehat{\iota} : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ with $\|T\| = 1$. Since the lattice span of F is dense in $\mathrm{FBL}^{(p)}[E]$ it follows that T has dense image.

For the converse implication, note that if F is WCG, then $\mathrm{FBL}^{(p)}[F]$ is LWCG. The result follows directly from [36, Proposition 2.1]. $\square$

Remark 6.9.10. Note that for $p \leq q \leq \infty$, the formal inclusion $\mathrm{FBL}^{(p)}[E] \hookrightarrow \mathrm{FBL}^{(q)}[E]$ has dense image. Hence, if $\mathrm{FBL}^{(p)}[E]$ is LWCG, then so is $\mathrm{FBL}^{(q)}[E]$ for every $p \leq q \leq \infty$.

Remark 6.9.11. Recall that a Banach space E is a subspace of a WCG space if and only if its dual unit ball B_{E^*} in the weak* topology is an Eberlein compact. Moreover, $C(K)$ is WCG if and only if K is Eberlein compact. See, for example, [108] and [110, Theorem 14.9].

Although at this point we do not know whether E must be WCG whenever $\mathrm{FBL}^{(p)}[E]$ is LWCG, we can at least show that if $\mathrm{FBL}^{(p)}[E]$ is LWCG, then E must be a subspace of a WCG space. We will use the following result, which actually proves quite a bit more than what is needed to deduce that E must be a subspace of a WCG space:

Proposition 6.9.12. *If* $\mathrm{FBL}^{(p)}[E]$ *is LWCG then there is a positively homogeneous homeomorphism between B_{E^*} with its weak* topology and a weakly compact set in a Banach space mapping weakly p-summable sequences to weakly p-summable sequences.*

Proof. Suppose $\mathrm{FBL}^{(p)}[E]$ is LWCG. By Proposition 6.9.9, there exist a WCG Banach space F and a lattice homomorphism $T : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ with dense image. By Proposition 6.10.2 below, it follows that the induced map $\Phi_T : B_{E^*} \to B_{F^*}$ is in particular injective, and since it is weak* to weak* continuous, we deduce that B_{E^*} is homeomorphic to $\Phi_T(B_{E^*})$ both with the weak* topology.

As F is WCG, [110, Theorem 13.20] guarantees the existence of an injective weak*-weak continuous bounded linear operator $S : F^* \to c_0(\Gamma)$ for some Γ. The composition $S\Phi_T$ is positively homogeneous, injective, and weak*-weak continuous, so it defines a homeomorphism between B_{E^*} and its image (a weakly compact set in $c_0(\Gamma)$). Further, by Proposition 6.10.1, Φ_T sends weakly p-summable sequences to weakly p-summable sequences, hence so does $S\Phi_T$; this completes the proof. □

Corollary 6.9.13. *If* $\mathrm{FBL}^{(p)}[E]$ *is LWCG then* E *is a subspace of a WCG space.*

Proof. By Proposition 6.9.12, B_{E^*} in its weak* topology is homeomorphic to a weakly compact set in a Banach space. Thus, B_{E^*} is Eberlein compact. Now use Remark 6.9.11. □

Remark 6.9.14. We outline an alternative approach to Corollary 6.9.13, which is simpler but less informative than going through Proposition 6.9.12. Suppose $\mathrm{FBL}^{(p)}[E]$ is LWCG. As $\mathrm{FBL}^{(p)}[E]$ is dense in $\mathrm{FBL}^{(\infty)}[E]$, the latter lattice is also LWCG, hence WCG by [36, Theorem 2.2].

Above, we noted that, if E is WCG, then $\mathrm{FBL}^{(p)}[E]$ is LWCG. The WCG assumption on E cannot be weakened to E being LWCG (provided it is a Banach lattice). Indeed, take $E = \mathrm{FBL}[\ell_2(\Gamma)]$ for any uncountable Γ. Clearly, E is LWCG; however, it contains a subspace isomorphic to $\ell_1(\Gamma)$ by [26, Theorem 5.4]. Therefore, if $\mathrm{FBL}^{(p)}[E]$ were LWCG, Corollary 6.9.13 would imply that $\ell_1(\Gamma)$ embeds into a WCG space. This is impossible (see e.g. [26, proof of Corollary 5.5]).

In a similar fashion, one can show that, for $p \in [1, \infty)$, it may happen that E is WCG, while $\mathrm{FBL}^{(p)}[E]$ does not embed into a WCG space. Indeed, take $E = \ell_2(\Gamma)$. Modifying the proof of [26, Theorem 5.4] with the help of Example 6.5.15, we conclude that $\mathrm{FBL}^{(p)}[E]$ contains a copy of $\ell_1(\Gamma)$, hence it cannot embed into a WCG space when Γ is uncountable.

Observe that the converse of Corollary 6.9.13 does not hold in general. Indeed, it is well-known that $L_1(\mu)$ is WCG, for any σ-finite measure μ (indeed, it suffices to consider the case of μ being a probability measure; then the unit ball of $L_2(\mu)$ is relatively weakly compact, and generates $L_1(\mu)$). [290] gives an example of a non-WCG subspace $X_{\mathcal{R}}$ of $L_1(\mu)$, which has a long unconditional basis. Thus, $X_{\mathcal{R}}$ is an order continuous Banach lattice, so $\mathrm{FBL}[X_{\mathcal{R}}]$ is not LWCG, by Proposition 6.9.7.

As noted above, for $p \in [1, \infty)$, it is in general false that if E is a subspace of a WCG space, then so is $\mathrm{FBL}^{(p)}[E]$. However, this is true for $p = \infty$:

Proposition 6.9.15. *Let E be a Banach space. The following are equivalent:*

(i) (B_{E^*}, w^*) *is Eberlein compact.*

(ii) $(B_{\mathrm{FBL}^{(\infty)}[E]^*}, w^*)$ *is Eberlein compact.*

(iii) $\mathrm{FBL}^{(\infty)}[E]$ *is a sublattice of a WCG Banach lattice.*

Furthermore, if E and F are Banach spaces such that (B_{E^*}, w^*) is Eberlein compact and $\mathrm{FBL}^{(\infty)}[F]$ *is a subspace of* $\mathrm{FBL}^{(\infty)}[E]$*, then (B_{F^*}, w^*) is Eberlein compact.*

Proof. We use Remark 6.9.11. The implication (3)$\Rightarrow$(2) follows immediately from it. If (2) holds, then $\mathrm{FBL}^{(\infty)}[E]$ is a subspace of a WCG space, and since E is a subspace of $\mathrm{FBL}^{(\infty)}[E]$, (1) follows.

(1)$\Rightarrow$(3): If (B_{E^*}, w^*) is Eberlein compact, then $C(B_{E^*})$ is WCG. Since $\mathrm{FBL}^{(\infty)}[E]$ is a sublattice of $C(B_{E^*})$, it is a sublattice of a WCG Banach lattice.

To address the "furthermore" statement, suppose (B_{E^*}, w^*) is Eberlein compact, and $\mathrm{FBL}^{(\infty)}[F]$ embeds as a subspace into $\mathrm{FBL}^{(\infty)}[E]$. By (1)$\Rightarrow$(3), $\mathrm{FBL}^{(\infty)}[E]$ embeds into a WCG space, hence the same is true for $\mathrm{FBL}^{(\infty)}[F]$. By (2)$\Rightarrow$(1), (B_{F^*}, w^*) is Eberlein compact. $\square$

Remark 6.9.16. Proposition 6.9.15 implies that for any uncountable set Γ, $\mathrm{FBL}^{(\infty)}[\ell_1(\Gamma)]$ does not embed (isomorphically) into $\mathrm{FBL}^{(\infty)}[\ell_r(\Gamma')]$ for any $r \in (1, \infty)$ and any set Γ'. However – as a special case of the results in the next section – we will see that $\mathrm{FBL}^{(\infty)}[\ell_1]$ and $\mathrm{FBL}^{(\infty)}[\ell_r]$ are lattice isometric.

Complemented copies of ℓ_1

In this subsection, we show that E contains a complemented subspace isomorphic to ℓ_1 if and only if $\mathrm{FBL}[E]$ contains ℓ_1 in various ways. In preparation, we need a lemma relating T and $\widehat{T}$, which complements the various relations between T and $\overline{T}$ discussed in Section 6.3.

Lemma 6.9.17. *Let X be a Banach lattice not containing c_0. Given a Banach space E, an operator $T : E \to X$ is weakly compact if and only if $\widehat{T} : \mathrm{FBL}[E] \to X$ is weakly compact.*

Remark 6.9.18. Lemma 6.9.17 fails if no restrictions on X are assumed. Indeed, suppose E is 2-dimensional, $X = \mathrm{FBL}[E]$ (which is lattice isomorphic to $C(S^1)$, where S^1 is the unit circle), and $T = \phi_E$. Then $\widehat{T}$ is the identity on $\mathrm{FBL}[E]$, which is not weakly compact, since the latter lattice is not reflexive.

Proof. We will make use of the Davis-Figiel-Johnson-Pełzcyński factorization method, and in particular its version for Banach lattices explained in [12, Theorems 5.37 & 5.41].

Suppose $T : E \to X$ is weakly compact. Let W denote the convex solid hull of $T(B_E)$, which by [12, Theorems 4.39 & 4.60] is a relatively weakly compact set. Let Ψ be the reflexive Banach lattice induced by W as in [12, Theorem 5.37 & 5.41]. This means that we have a commutative diagram

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "S"'] & & X \\ & \Psi \arrow[ur, "J"'] & \end{tikzcd}$$

where J is a lattice homomorphism. Let $\widehat{S} : \mathrm{FBL}[E] \to \Psi$ be the lattice homomorphism such that $\widehat{S}\phi_E = S$. Note that $J\widehat{S} : \mathrm{FBL}[E] \to X$ is a lattice homomorphism with the property that $J\widehat{S}\phi_E = T$. Hence, we must have $\widehat{T} = J\widehat{S}$, which implies that $\widehat{T}$ factors through the reflexive Banach lattice Ψ, so $\widehat{T}$ is weakly compact as claimed. The converse is clear. $\square$

Remark 6.9.19. The method of proof of Lemma 6.9.17 is quite general. For example, with Remark 6.2.5 in mind, a similar argument to Lemma 6.9.17 shows that an operator $T : E \to X$ from a Banach space E to a Banach lattice X is p-convex if and only if $\widehat{T} : \mathrm{FBL}[E] \to X$ is p-convex.

Theorem 6.9.20. *For a Banach space E, the following are equivalent:*

(i) E contains a complemented subspace isomorphic to ℓ_1.

(ii) FBL$[E]$ *contains a lattice complemented sublattice isomorphic to* FBL$[\ell_1]$.

(iii) FBL$[E]$ *contains a lattice complemented sublattice isomorphic to* ℓ_1.

(iv) ℓ_1 *is a lattice quotient of* FBL$[E]$.

(v) ℓ_1 *is a sublattice of* FBL$[E]$.

(vi) ℓ_1 *is a complemented subspace of* FBL$[E]$.

[12, Theorem 4.69] provides more equivalent characterizations of Banach lattices containing a lattice copy of ℓ_1.

Proof. (1) $\Rightarrow$ (2) follows from [26, Corollary 2.8] together with the observation that if $P : E \to E$ is a projection onto a subspace isomorphic to ℓ_1, then $\overline{P} : \mathrm{FBL}[E] \to \mathrm{FBL}[E]$ is a lattice projection onto the corresponding sublattice isomorphic to FBL$[\ell_1]$. (2) $\Rightarrow$ (3) follows from Theorem 6.8.3. (3) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5) are straightforward. (5) $\Leftrightarrow$ (6) comes from [12, Theorem 4.69].

(4) $\Rightarrow$ (5): Let P be a lattice quotient from FBL$[E]$ onto ℓ_1. By [269, Theorem 11.11], there exists a lattice isomorphic embedding $T : \ell_1 \to \mathrm{FBL}[E]$ such that $PT = id_{\ell_1}$.

(5) $\Rightarrow$ (1): By [244, Theorem 2.4.14], (5) holds if and only if FBL$[E]^*$ is not order continuous, if and only if there is $\varphi_0 \in \mathrm{FBL}[E]^*_+$ and $\varepsilon_0 > 0$ such that for any $f \in \mathrm{FBL}[E]_+$,

$$B_{\mathrm{FBL}[E]} \not\subseteq [-f, f] + \{g \in \mathrm{FBL}[E] : \varphi_0(|g|) \leq \varepsilon_0\}. \tag{6.9.1}$$

Let $N_0 = \{f \in \mathrm{FBL}[E] : \varphi_0(|f|) = 0\}$, which is an ideal in FBL$[E]$ so that $\varphi_0(|\cdot|)$ defines an AL-norm on the quotient FBL$[E]/N_0$. By Kakutani representation theorem (cf. [231, Theorem 1.b.2]), its completion is lattice isometric to $L_1(\mu)$ for some (not necessarily σ-finite) measure space. Let $Q : \mathrm{FBL}[E] \to L_1(\mu)$ be the dense range lattice homomorphism induced by the corresponding quotient map.

We claim that $Q\phi_E : E \to L_1(\mu)$ is not a weakly compact operator. Indeed, if it were, by Lemma 6.9.17, Q would also be weakly compact. Hence, by [9, Theorem 5.2.9] for every $\varepsilon > 0$, there is $h \in L_1(\mu)$ such that

$$Q(B_{\mathrm{FBL}[E]}) \subseteq [-h, h] + \varepsilon B_{L_1(\mu)}.$$

Since Q has dense range, we can find $f' \in \mathrm{FBL}[E]_+$ such that $\|Qf' - h\|_1 < \varepsilon$, which implies that

$$Q(B_{\mathrm{FBL}[E]}) \subseteq [-Qf', Qf'] + 2\varepsilon B_{L_1(\mu)}.$$

Also, it follows from [298, Proposition II.2.5] and the construction of Q that $Q[-f', f']$ must be dense in $[-Qf', Qf']$. Thus, for every $f \in B_{\mathrm{FBL}[E]}$ there exists $|f''| \leq f'$ and $h \in L_1(\mu)$ such that $\|h\|_1 \leq 3\varepsilon$ and

$$Qf = Qf'' + h.$$

Or equivalently, $\varphi_0(|f - f''|) \leq 3\varepsilon$. This means that

$$B_{\mathrm{FBL}[E]} \subseteq [-f', f'] + \{g \in \mathrm{FBL}[E] : \varphi_0(|g|) \leq 3\varepsilon\},$$

so taking $\varepsilon < \varepsilon_0/3$ we reach a contradiction with (6.9.1).

Therefore, $Q\phi_E$ is not weakly compact as claimed. It follows from [9, Theorem 5.2.9] that $Q\phi_E(B_E)$ contains a complemented basic sequence (h_n) equivalent to the canonical basis of ℓ_1. As a consequence, E must contain a complemented subspace isomorphic to ℓ_1. Indeed, let $(x_n) \subseteq B_E$ be such that $Q\phi_E(x_n) = h_n$ and let $P : L_1(\mu) \to L_1(\mu)$ denote a projection onto the span of (h_n); it is straightforward to check that (x_n) must be equivalent to the canonical basis of ℓ_1, so the linear map $U : [h_n] \to E$ given by $U(h_n) = x_n$ is bounded, and $UPQ\phi_E$ defines a projection of E onto the span of (x_n). □

Upper p-estimates and the local theory of free Banach lattices

In this section we characterize when $\mathrm{FBL}[E]$ satisfies an upper p-estimate. We then use this to study the structure of finite dimensional subspaces and sublattices of free Banach lattices, as well as to find upper p-estimate variants of classical theorems on p-convexity.

Recall that a Banach lattice X satisfies an *upper p-estimate with constant C* (resp. *lower p-estimate with constant C*) if, for any $x_1, \dots, x_n \in X$, we have

$$\Big\| \bigvee_{k=1}^{n} |x_k| \Big\| \leq C \Big(\sum_{k=1}^{n} |x_k|^p \Big)^{1/p}$$

(resp. $\big\| \sum_{k=1}^{n} |x_k| \big\| \geq C^{-1} \big(\sum_{k=1}^{n} |x_k|^p \big)^{1/p}$). By [231, Proposition 1.f.6], it suffices to verify these inequalities when $x_1, \dots, x_n$ are disjoint. Further, [231, Proposition 1.f.5] shows that X has an upper (resp. lower) p-estimate if and only if X^* has a lower (resp. upper) q-estimate, with the same constant. Here, $\frac{1}{p}+\frac{1}{q}=1$.

Upper and lower estimates are deeply connected to the convexity and concavity of a Banach lattice, as well as to its type and cotype. In particular, p-convexity clearly implies upper p-estimates; conversely, upper p-estimates imply r-convexity for $r<p$ [231, Theorem 1.f.7]. More information about this can be found in [231, Section 1.f].

Recall that an operator $T : F \to E$ is (q,p)*-summing* (cf. [97, Chapter 10]) if there is a $K>0$ such that for every choice of $(x_k)_{k=1}^n \subseteq F$ we have

$$\Big(\sum_{k=1}^n \|Tx_k\|^q\Big)^{1/q} \le K \sup_{x^*\in B_{F^*}} \Big(\sum_{k=1}^n |x^*(x_k)|^p\Big)^{1/p}.$$

We use $\pi_{q,p}(T)$ to denote the least possible constant K in this inequality.

Theorem 6.9.21. *Let E be a Banach space and $1\le p,q\le\infty$ with $\frac{1}{p}+\frac{1}{q}=1$. The following are equivalent:*

(i) id_{E^*} *is* $(q,1)$*-summing.*

(ii) $\mathrm{FBL}[E]$ *satisfies an upper p-estimate.*

In this case, the upper p-estimate constant of $\mathrm{FBL}[E]$ *and* $\pi_{q,1}(id_{E^*})$ *coincide.*

Proof. (1) $\Rightarrow$ (2): We shall show that if $f_1,\dots,f_n\in \mathrm{FBL}[E]_+$ are disjoint then $\|\sum_{j=1}^n f_j\| \le \pi_{q,1}(id_{E^*})\big(\sum_{j=1}^n \|f_j\|^p\big)^{1/p}$.

We view elements of $\mathrm{FBL}[E]$ as positively homogeneous functions on the unit ball of E^*. Let $g=\sum_{j=1}^n f_j$. Then $\|g\|$ is the supremum of $\sum_k |g(x_k^*)|$, where the finite sequence $(x_k^*)\subseteq E^*$ is such that $\|\sum_k \pm x_k^*\|\le 1$ for any choice of $\pm$. Fix (x_k^*) as above. For any j let $S_j=\{k : f_j(x_k^*)\neq 0\}$. These sets are disjoint (due to the disjointness of f_j's themselves), and

$$\sum_k |g(x_k^*)| \le \sum_k\sum_j |f_j(x_k^*)| = \sum_j\sum_{k\in S_j} |f_j(x_k^*)|.$$

For each j let $\alpha_j=\max_\pm \|\sum_{k\in S_j} \pm x_k^*\|$, and $y_j^* = \mathrm{argmax}_\pm \|\sum_{k\in S_j}\pm x_k^*\|$. Then

$$\sum_{k\in S_j} |f_j(x_k^*)| \le \|f_j\|\alpha_j = \|f_j\|\|y_j^*\|.$$

Note that

$$\max_{\pm} \| \sum_j \pm y_j^* \| \leq \max_{\pm} \| \sum_k \pm x_k^* \| \leq 1.$$

Let $\kappa = \pi_{q,1}(id_{E^*})$, then

$$1 \geq \max_{\pm} \| \sum_j \pm y_j^* \| \geq \kappa^{-1} \big(\sum_j \|y_j^*\|^q \big)^{1/q} = \kappa^{-1} \big(\sum_j \alpha_j^q \big)^{1/q}.$$

Therefore,

$$\begin{aligned}\sum_k |g(x_k^*)| &\leq \sum_j \sum_{k \in S_j} |f_j(x_k^*)| \leq \sum_j \|f_j\| \alpha_j \leq \big(\sum_j \|f_j\|^p \big)^{1/p} \big(\sum_j \alpha_j^q \big)^{1/q} \\ &\leq \kappa \big(\sum_j \|f_j\|^p \big)^{1/p}.\end{aligned}$$

We obtain the desired estimate on $\|g\|$ by taking the supremum over all suitable sequences (x_k^*).

(2) $\Rightarrow$ (1): Suppose FBL$[E]$ has an upper p-estimate with constant C (clearly $C \geq 1$). We fix $x_1^*, \ldots, x_n^* \in E^*$, and aim to show that

$$\Big(\sum_{i=1}^n \|x_i^*\|^q \Big)^{\frac{1}{q}} \leq C \sup_{x \in B_E} \sum_{i=1}^n \big| x_i^*(x) \big|. \tag{6.9.2}$$

If $\dim E = 1$, then it is easy to see that $\pi_{q,1}(id_{\mathbb{R}}) = 1$, hence (6.9.2) is satisfied (in fact, [269, Theorem 8.1] shows that FBL$[\mathbb{R}] = \ell_\infty^2$, hence its upper p-estimate constant equals 1 for all p).

If $\dim E > 1$, then, by a small perturbation argument, we may assume that no two of the vectors $x_1^*, \ldots, x_n^*$ are proportional. Since $\widehat{x_1^*}, \ldots, \widehat{x_n^*}$ are lattice homomorphisms in FBL$[E]^*$, they are atoms by [12, p. 111 Exercise 5] and, therefore, disjoint. Fix $\varepsilon > 0$. For each i, pick $f_i \in$ FBL$[E]_+$ with $\|f_i\| \leqslant 1$ and $\widehat{x_i^*}(f_i) > (1-\varepsilon) \left\| \widehat{x_i^*} \right\|$ or, equivalently, $f_i(x_i^*) > (1-\varepsilon) \|x_i^*\|$. Applying Proposition 1.4.13 in [244] to the normalized functionals, we may assume that the f_i's are disjoint. We have

$$(1-\varepsilon) \Big(\sum_{i=1}^n \|x_i^*\|^q \Big)^{\frac{1}{q}} \leqslant \Big(\sum_{i=1}^n f_i(x_i^*)^q \Big)^{\frac{1}{q}} = \sum_{i=1}^n \lambda_i f_i(x_i^*)$$

for some $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_+$ with $\sum_{i=1}^n \lambda_i^p = 1$. Put $f = \sum_{i=1}^n \lambda_i f_i$. Since FBL$[E]$ has the upper p-estimate, we get $\|f\| \leqslant C \Big(\sum_{i=1}^n \|\lambda_i f_i\|^p \Big)^{\frac{1}{p}} \leqslant C$. Using the definition of the FBL norm, we get

$$\sum_{i=1}^{n} \lambda_i f_i(x_i^*) = \sum_{i=1}^{n} f(x_i^*) \leqslant \|f\| \sup_{x\in B_E} \sum_{i=1}^{n} \bigl|x_i^*(x)\bigr| \leqslant C \sup_{x\in B_E} \sum_{i=1}^{n} \bigl|x_i^*(x)\bigr|.$$

Since ε is arbitrary, we conclude that (6.9.2) holds, thus completing the proof. □

Remark 6.9.22. If E is infinite dimensional, then, by Dvoretzky Theorem, id_{E^*} cannot be $(q,1)$-summing for $q < 2$. Therefore, $\mathrm{FBL}[E]$ can only have an upper p-estimate for $p \leq 2$. In the next section, we shall see that this estimate is sharp, and $\mathrm{FBL}[E]$ can even be 2-convex (see, for instance, Corollary 6.9.47). For more general information about possible q-convexity of $\mathrm{FBL}^{(p)}[E]$, see Proposition 6.9.30. On the other hand, recall that if E is finite dimensional, then $\mathrm{FBL}^{(p)}[E]$ is lattice isomorphic to $C(S_{E^*})$, hence it satisfies an upper r-estimate for every $r \in [1,\infty]$.

Remark 6.9.23. Although Theorem 6.9.21 is stated for $\mathrm{FBL}[E]$, in Theorem 6.9.40 we will prove an extrapolation result which allows us to characterize when $\mathrm{FBL}^{(p)}[E]$ has non-trivial convexity.

Corollary 6.9.24. *Suppose F is a subspace of a Banach space E, so that (F,E) has the POE-1. Fix $q \in [1,\infty]$. If id_{E^*} is $(q,1)$-summing, then so is id_{F^*}.*

Note that, if $\dim F < \infty$, then id_{F^*} is $(q,1)$-summing for any q. If $\dim F = \infty$, then, by Remark 6.9.22, we must have $q \in [2,\infty]$.

Proof. By Theorem 6.9.21, $\mathrm{FBL}[E]$ has an upper p-estimate, with $1/p + 1/q = 1$. Denote the canonical embedding $F \to E$ by ι. By the POE-1, $\bar{\iota} : \mathrm{FBL}[F] \to \mathrm{FBL}[E]$ is a lattice isomorphic embedding, hence $\mathrm{FBL}[F]$ has an upper p-estimate as well. Apply Theorem 6.9.21 again to reach the desired conclusion about id_{F^*}. □

We next present the following "local" version of Theorem 6.9.20. We use the shorthand "E has trivial cotype" to mean that no non-trivial cotype is present.

Corollary 6.9.25. *For an infinite dimensional Banach space E, the following statements are equivalent.*

(i) E contains uniformly complemented subspaces isomorphic to ℓ_1^n.

(ii) $\mathrm{FBL}[E]$ contains uniformly lattice-complemented sublattices isomorphic to ℓ_1^n.

(iii) $\mathrm{FBL}[E]$ contains sublattices ℓ_1^n uniformly.

(iv) E^* *has trivial cotype.*

(v) $\mathrm{FBL}[E]$ *fails to be* p*-convex for any* $p > 1$*.*

(vi) $\mathrm{FBL}[E]$ *contains uniformly complemented subspaces isomorphic to* ℓ_1^n*.*

(vii) $\mathrm{FBL}[E]^*$ *has trivial cotype.*

Recall that a Banach lattice X is said to contain sublattices ℓ_1^n uniformly if there exist lattice isomorphisms $u_n : \ell_1^n \to X_n \subseteq X$ so that $\sup_n \|u_n\|\|u_n^{-1}\| < \infty$. By Krivine's theorem (see [299]), the uniform lattice copies of ℓ_1^n in $\mathrm{FBL}[E]$ can be taken to be $(1+\varepsilon)$-uniform whenever they exist. In this case one can select u_n's in such a way that $\lim_n \|u_n\|\|u_n^{-1}\| = 1$.

The following lemma is known, but we include it for the sake of completeness.

Lemma 6.9.26. *For a Banach space* E*, the following statements are equivalent:*

(i) E *contains uniformly complemented copies of* ℓ_1^n*.*

(ii) E^* *contains copies of* ℓ_∞^n *uniformly.*

(iii) E^* *has trivial cotype.*

Proof. (2)⇔(3) is given by [97, Theorem 14.1], and duality gives us (1)⇒(2). To establish (2)⇒(1), suppose E^* contains copies of ℓ_∞^n uniformly. By [265, Theorem 2.5], we can assume that the said copies of ℓ_∞^n are complemented via weak* continuous projections, with uniformly bounded norms. Passing to the predual, we conclude that E satisfies (1). □

Proof of Corollary 6.9.25. (1)⇒(2) is similar to Theorem 6.9.20, where we make use of Theorem 6.8.3 to see that $\mathrm{FBL}[\ell_1^n]$ contains ℓ_1^n as a nicely complemented sublattice. The implications (2)⇒(3)⇒(5) and (2)⇒(6) are trivial. (5)⇒(3) is a consequence of the Banach lattice version of Krivine's Theorem [299]. Lemma 6.9.26 contains (1)⇔(4).

(5)⇒(4): If E^* has non-trivial cotype, then by [97, Theorem 14.1] id_{E^*} is $(q,1)$-summing for some q. Then, by Theorem 6.9.21, $\mathrm{FBL}[E]$ has a non-trivial upper estimate, which implies non-trivial convexity [231, Section 1.f].

(6)⇔(7) follows from Lemma 6.9.26.

(7)⇒(5): If (7) holds, then by [231, Section 1.f], $\mathrm{FBL}[E]^*$ cannot be q-concave for any finite q. By duality, $\mathrm{FBL}[E]$ cannot be p-convex for any $p > 1$. □

The next remark puts the above results in a broader context:

Remark 6.9.27. Let X be a Banach lattice. By [97, Chapter 16] (see also [231, Section 1.f] and [307, 308, 309]; the relevant results are neatly summarized in [54]), we have the following general implications and no others:

(i) For $2 < q < \infty$, q-concavity $\Rightarrow$ cotype q $\Leftrightarrow$ id_X is $(q,1)$-summing $\Leftrightarrow$ X has a lower q-estimate;

(ii) For $q = 2$, 2-concavity $\Leftrightarrow$ cotype 2 $\Rightarrow$ id_X is $(2,1)$-summing $\Rightarrow$ X has a lower 2-estimate.

By duality [231, Proposition 1.f.5], if E is a Banach lattice and $1 < p < 2$ we conclude that id_{E^*} is $(q,1)$-summing $(1/p + 1/q = 1)$ if and only if E^* has a lower q-estimate if and only if E has an upper p-estimate. Combining these observations with Theorem 6.9.21 we see that:

Corollary 6.9.28. *Suppose E is a Banach lattice and $1 < p < 2$. The following are equivalent:*

(i) E satisfies an upper p-estimate;

(ii) $\mathrm{FBL}[E]$ *satisfies an upper p-estimate.*

Proposition 6.9.30 shows that the above equivalence fails for $p > 2$.

Corollary 6.9.24 immediately implies an upper p-estimate version of [231, Theorem 1.d.7]:

Corollary 6.9.29. *Suppose $p \in (1,2)$, E and F are Banach lattices, and $\iota : F \to E$ is a linear isomorphic embedding, so that $\iota(F)$ is complemented in E or, more generally, that $(\iota(F), E)$ has POE-1. Then, if E has an upper p-estimate, then the same is true for F.*

The existence of a complemented copy of L_2 inside of L_p shows that Corollary 6.9.29 fails for $2 < p < \infty$. For $p = 2$, the proof only shows that, if id_{E^*} is $(2,1)$-summing, then F has an upper 2-estimate; we do not know if the assumption on E can be relaxed to it merely having an upper 2-estimate. In connection to this, we should also mention a "dual" analogue of Corollary 6.9.29, discussed on [231, p. 98-99]. Namely, suppose a Banach lattice F embeds isomorphically into a Banach lattice E with a lower p-estimate. If $p \in (2,\infty)$, then F has a lower p-estimate as well; this is no longer true for $p=2$.

As mentioned previously, if E is finite dimensional then $\mathrm{FBL}[E]$ is lattice isomorphic to a $C(K)$-space, so is in particular ∞-convex. The situation is different in the infinite dimensional setting.

Proposition 6.9.30. *Suppose E is an infinite dimensional Banach space. If $\mathrm{FBL}^{(p)}[E]$ is q-convex, then $q \le \max\{2,p\}$.*

Proof. Fix $n \in \mathbb{N}$. Use Dvoretzky Theorem to find norm 2 vectors $x_1^*, \dots, x_n^* \in E^*$, so that the inequality

$$\Big(\sum_j |a_j|^2\Big)^{1/2} \le \Big\| \sum_j a_j x_j^* \Big\| \le 3\Big(\sum_j |a_j|^2\Big)^{1/2}$$

holds for any scalars $a_1, \dots, a_n$. Use Local Reflexivity to find $x_1, \dots, x_n \in E$, of norm not exceeding 1, and biorthogonal to the x_j^*'s. We will establish that

$$\|f\|_{\mathrm{FBL}^{(p)}[E]} \gtrsim n^{1/r}, \text{ where } r = \max\{2,p\}, \text{ and } f = \Big(\sum_j |\delta_{x_j}|^q\Big)^{1/q}.$$

We shall achieve this by testing f against $x_1^*, \dots, x_n^*$. Let $F = \mathrm{span}[x_1^*, \dots, x_n^*]$. By applying Local Reflexivity, and then passing from E^{**} to $E^{**}/F^\perp \sim F^*$, we obtain

$$\sup_{x \in B_E} \Big(\sum_{j=1}^n |x_j^*(x)|^p\Big)^{1/p} = \sup_{x \in B_{F^*}} \Big(\sum_{j=1}^n |x_j^*(x)|^p\Big)^{1/p} \le 3n^\gamma,$$

with

$$\gamma = \begin{cases} \frac{1}{p} - \frac{1}{2} & 1 \le p \le 2, \\ 0 & p \ge 2. \end{cases}$$

Note that, for any $x^* \in E^*$, $f(x^*) = \big(\sum_j |x^*(x_j)|^q\big)^{1/q}$, and therefore, $\big(\sum_j |f(x_j^*)|^p\big)^{1/p} = n^{1/p}$. By (6.1.1),

$$\|f\|_{\mathrm{FBL}^{(p)}[E]} \gtrsim \frac{n^{1/p}}{n^\gamma} = n^{1/r}$$

(with r as above). On the other hand, if $\mathrm{FBL}^{(p)}[E]$ is q-convex, then

$$\|f\|_{\mathrm{FBL}^{(p)}[E]} \lesssim \Big(\sum_j \|\delta_{x_j}\|^q_{\mathrm{FBL}^{(p)}[E]}\Big)^{1/q} \sim n^{1/q},$$

giving the desired estimate for q. $\square$

We finish this section with some applications to the local theory.

Recall that for a Banach lattice E, the *upper index of E* is

$$S(E) = \sup\{p \geq 1 : E \text{ satisfies an upper } p\text{-estimate}\}.$$

By [231, Section 1.f], "upper p-estimate" can be replaced by "p-convex" in the definition of $S(E)$. $S(E)$ is very important in the local theory of Banach lattices. Indeed, a theorem of Krivine [299] states that an infinite dimensional Banach lattice E contains, for all integers n and all $\varepsilon > 0$, a $(1+\varepsilon)$-lattice copy of ℓ_p^n when $p = S(E)$.

If E is a finite dimensional Banach space, then $\mathrm{FBL}^{(p)}[E]$ is lattice isomorphic to an AM-space, hence $S(\mathrm{FBL}^{(p)}[E]) = \infty$. For infinite dimensional E, Proposition 6.9.30 shows that $S(\mathrm{FBL}^{(p)}[E]) = p$ for $2 \leq p \leq \infty$, while $p \leq S(\mathrm{FBL}^{(p)}[E]) \leq 2$ for $1 \leq p \leq 2$. In particular, we conclude that for infinite dimensional Banach lattices, the indices are related as follows:

Corollary 6.9.31. *Suppose E is an infinite dimensional Banach lattice. Then*

$$S(E) \wedge 2 = S(\mathrm{FBL}[E]).$$

Remark 6.9.32. On the other end of the spectrum, note that $\mathrm{FBL}[E]$ always contains a lattice copy of c_0 as long as $\dim E \geq 2$, so in particular contains uniform lattice copies of ℓ_∞^n. Further, $\mathrm{FBL}[\ell_1^2] \simeq C(S^1)$ contains isomorphic copies of every separable Banach space, hence so does $\mathrm{FBL}[E]$ for every E with $\dim E \geq 2$. One should note, however, that $\mathrm{FBL}[E]$ being universal is restricted to separable spaces; for other density characters it is an interesting problem to classify the subspaces of $\mathrm{FBL}[E]$ up to isomorphism. For example, $\mathrm{FBL}[E]$ has the same density character as E ([26, Section 3]), but, as was shown in [30], when $1 \leq p \leq 2$ and Γ is uncountable, $\mathrm{FBL}[\ell_p(\Gamma)]$ does not embed into a weakly compactly generated Banach space, and in particular does not embed into $\mathrm{FBL}[\ell_q(\Gamma)]$, $2 < q < \infty$, which is WCG. These simple facts will play a role in the next section when we compare $\mathrm{FBL}^{(p)}[E]$ and $\mathrm{FBL}^{(q)}[F]$; in particular, when E and F are separable, we will aim to distinguish these spaces by showing that one does not linearly embed onto a complemented subspace of the other.

Remark 6.9.33. We also note that the disjoint sequence structure of $\mathrm{FBL}[E]$ can be very complicated. Indeed, when E is the complementably universal space for unconditional bases (see [230, Theorem 2.d.10] for the construction), then, by Theorem 6.8.3, $\mathrm{FBL}[E]$ contains lattice copies of every separable order continuous atomic lattice (i.e., every Banach lattice with lattice structure induced by an unconditional basis).

Automatic convexity, factorization theory and isomorphisms between $\mathrm{FBL}^{(p)}[E]$ and $\mathrm{FBL}^{(q)}[F]$

In this section, we characterize when $\mathrm{FBL}^{(p)}[E]$ is q-convex via strong factorizations (representing an operator as a composition of two or more, one of which is a lattice homomorphism), and then use $\mathrm{FBL}^{(p)}[E]$ as a tool to study the classical factorization theory. We also give various situations where we can prove that $\mathrm{FBL}^{(p)}[E]$ and $\mathrm{FBL}^{(q)}[F]$ are lattice isomorphic, and other situations where we can prove that one of these spaces does not even linearly embed as a complemented subspace of the other.

We begin with some preparation:

Proposition 6.9.34. *Suppose E is a Banach space, Z is a p-convex Banach lattice with constant 1, and $\iota : E \to Z$ is an isometric embedding with the following properties:*

(i) Z is generated by $\iota(E)$ as a Banach lattice.

(ii) There exists a constant C so that for every contraction $T : E \to L_p(\mu)$ there is a lattice homomorphism $T' : Z \to L_p(\mu)$ with $T'\iota = T$ and $\|T'\| \leq C$.

Then Z is C-lattice isomorphic to $\mathrm{FBL}^{(p)}[E]$. More precisely, the canonical extension $\widehat{\iota} : \mathrm{FBL}^{(p)}[E] \to Z$ is invertible and $\|\widehat{\iota}^{-1}\| \leq C$.

Proof. From the definition of $\mathrm{FBL}^{(p)}[E]$, there exists a unique lattice homomorphism

$$\widehat{\iota}\colon \mathrm{FBL}^{(p)}[E] \to Z$$

such that $\widehat{\iota}\phi_E = \iota$ and $\|\widehat{\iota}\| = 1$. Observe that $\widehat{\iota}$ has dense range. Indeed, fix $z \in Z$ and $\varepsilon > 0$. By assumption, there exists u in the sublattice generated by $\iota(E)$ such that $\|z - u\| < \varepsilon$. We can write u as a lattice-linear expression $u = F(\iota x_1, \dots, \iota x_n)$ for some $x_1, \dots, x_n \in E$. Then $u = \widehat{\iota}F(\delta_{x_1}, \dots, \delta_{x_n}) \in \operatorname{Range}\widehat{\iota}$.

Let $f \in \mathrm{FBL}^{(p)}[E]$ with $\|f\| > 1$. By the definition of the $\mathrm{FBL}^{(p)}$-norm, there exists $n \in \mathbb{N}$ and a contractive operator $T\colon E \to \ell_p^n$ such that $\left\|\widehat{T}f\right\| > 1$, where $\widehat{T}$ is the unique lattice homomorphism $\widehat{T}\colon \mathrm{FBL}^{(p)}[E] \to \ell_p^n$ such that $\widehat{T}\phi_E = T$. By assumption, there exists a lattice homomorphism $T'\colon Z \to \ell_p^n$ such that $T'\iota = T$ and $\|T'\| \leqslant C$. We have $T'\widehat{\iota}\phi_E(x) = T'\iota x = Tx = \widehat{T}\phi_E(x)$ for every $x \in E$. It follows that $T'\widehat{\iota}$ agrees with $\widehat{T}$ on $\phi(E)$ and, therefore, $T'\widehat{\iota} = \widehat{T}$. We now have $1 < \left\|\widehat{T}f\right\| = \|T'\widehat{\iota}f\| \leqslant C\,\|\widehat{\iota}f\|$. It follows that $\widehat{\iota}$ is bounded below. In particular, it is invertible and $\|\widehat{\iota}^{-1}\| \leqslant C$. $\square$

A standard direct sum argument implies:

Corollary 6.9.35. *Suppose E is a Banach space, Z is a p-convex Banach lattice with constant 1, and $\iota : E \to Z$ is an isometric embedding with the following properties:*

(i) Z is generated by $\iota(E)$ as a Banach lattice.

(ii) Every contraction $T : E \to L_p(\mu)$ extends to a lattice homomorphism $T' : Z \to L_p(\mu)$.

Then Z is lattice isomorphic to $\mathrm{FBL}^{(p)}[E]$. More precisely, the canonical extension $\widehat{\iota}$: $\mathrm{FBL}^{(p)}[E] \to Z$ is a surjective isomorphism.

Proof. By Proposition 6.9.34, it is enough to show that there is a uniform constant C such that every contraction $T : E \to L_p(\mu)$ extends to a lattice homomorphism $T' : Z \to L_p(\mu)$ with $\|T'\| \leq C$. Suppose this is not the case, and let $T_n : E \to L_p(\mu_n)$ be such that $\|T_n\| = 1$, but any lattice homomorphism $S : Z \to L_p(\mu_n)$ extending T_n has $\|S\| \geq 2^{n/p}n$.

Consider $L_p(\nu)$ to be the infinite ℓ_p sum of the spaces $L_p(\mu_n)$ and let $T : E \to \ell_p(L_p(\mu_n)) = L_p(\nu)$ be given by $Tx = (\frac{T_n x}{2^{n/p}})_{n=1}^{\infty}$. Note that

$$\|Tx\| = \Big(\sum_{n=1}^{\infty} \frac{\|T_n x\|^p}{2^n} \Big)^{\frac{1}{p}} \leq \|x\|.$$

Let $T' : Z \to L_p(\nu)$ be a lattice homomorphism extending T. Note that if $\pi_n : \ell_p(L_p(\mu_n)) \to L_p(\mu_n)$ denotes the canonical band projection, we have that the operator $T'_n = 2^{n/p}\pi_n T'$: $Z \to L_p(\mu_n)$ is a lattice homomorphism extending T_n. Hence, $2^{n/p}n \leq \|2^{n/p}\pi_n T'\|$, which yields $n \leq \|T'\|$. As this holds for every $n \in \mathbb{N}$, we get a contradiction with the fact that T' is bounded. □

Proposition 6.9.34 has a natural analogue for free Banach lattices satisfying an upper p-estimate. We first recall some facts on weak L_p-spaces and (p,∞)-convex operators:

For $f \in L_0(\mu)$ and $0 < p < \infty$, let

$$\|f\|_{p,\infty} = \{\sup_{t>0} t^p \mu(\{|f| > t\})\}^{1/p}.$$

The space $L_{p,\infty}(\mu)$ is the set of all $f \in L_0(\mu)$ such that $\|f\|_{p,\infty} < \infty$. It is well-known that when μ is σ-finite and $0 < r < p$ the expression

$$|||f|||_{p,\infty,[r]} := \sup_{0<\mu(E)<\infty} \mu(E)^{-\frac{1}{r}+\frac{1}{p}} \left(\int_E |f|^r d\mu \right)^{\frac{1}{r}}$$

satisfies

$$\|f\|_{p,\infty} \leq |||f|||_{p,\infty,[r]} \leq \left(\frac{p}{p-r}\right)^{\frac{1}{r}} \|f\|_{p,\infty}$$

(see, for example, [134, Exercise 1.1.12]).

If (X, μ) is a measure space with μ finite, $0 < q < p$ and $f \in L_{p,\infty}(\mu)$ then

$$\int_X |f(x)|^q d\mu(x) \leq \frac{p}{p-q} \mu(X)^{1-\frac{q}{p}} \|f\|_{p,\infty}^q, \tag{6.9.3}$$

i.e., $L_{p,\infty}(\mu)$ continuously injects into $L_q(\mu)$ with control of the constants (see [134, Exercise 1.1.11]). This will be used in the proof of Proposition 6.9.36 below to justify a certain multiplication operator being bounded by universal constants.

Below, we concern ourselves with $p \in (1, \infty)$. Equip $L_{p,\infty}(\mu)$ with the equivalent norm $|||\cdot|||_{p,\infty,[1]}$, or, for short, $|||\cdot|||_{p,\infty}$. This turns $L_{p,\infty}$ into a Banach lattice. Moreover, the space $(L_{p,\infty}, |||\cdot|||_{p,\infty})$ has an upper p-estimate with constant 1. To establish the latter fact, we show that the inequality $|||\vee_{i=1}^n |f_i| |||_{p,\infty} \leq \big(\sum_{i=1}^n |||f_i|||_{p,\infty}^p\big)^{1/p}$ holds for any $f_1, \ldots, f_n \in L_{p,\infty}(\Omega, \mu)$. In other words, we show that, for any $E \subseteq \Omega$, we have

$$\sup_{E \subseteq \Omega} \mu(E)^{1/p-1} \int_E \vee_i |f_i| \leq \big(\sum_i |||f_i|||_{p,\infty}^p\big)^{1/p}.$$

Represent E as a union of disjoint sets E_j $(1 \leq j \leq n)$, so that $\vee_i |f_i| = |f_j|$ on E_j. For the sake of convenience write $p' = p/(p-1)$ (so $1/p + 1/p' = 1$), $a_i = \int_{E_i} |f_i|$, and $b_i = \mu(E_i)^{1/p'}$ (by getting rid of "redundant" f_i's, we can assume that $b_i > 0$ for any i). Then $|||f_i|||_{p,\infty} \geq b_i^{-1} a_i$; therefore, it suffices to show that

$$\Big(\sum_i (b_i^{-1} a_i)^p\Big)^{1/p} \geq \big(\sum_i b_i^{p'}\big)^{-1/p'} \sum_i a_i.$$

The last inequality is equivalent to

$$\sum_i a_i \leq \Big(\sum_i (b_i^{-1} a_i)^p\Big)^{1/p} \big(\sum_i b_i^{p'}\big)^{1/p'},$$

which is an easy consequence of Hölder's Inequality.

Let X be a Banach lattice and E a Banach space. Recall that an operator $T : X \to E$ is (q,p)-concave if there is a constant C such that, for any $x_1, \ldots, x_n \in X$ we have

$$\left(\sum_{k=1}^{n} \|Tx_k\|^q \right)^{1/q} \leq C \left\| \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} \right\|.$$

The least constant that works is denoted $K_{q,p}(T)$. It is easy to see that if $p > q$ then the only (q,p)-concave operator is the zero operator. Moreover, (p,p)-concave operators are exactly the p-concave operators, and for $1 \leq p < q < \infty$ an operator is (q,p)-concave if and only if it is $(q,1)$-concave (see [97, Corollary 16.6]). An operator $S : E \to X$ is (p,q)-convex if there is a constant C such that for each $x_1, \ldots, x_n$ in E we have

$$\left\| \left(\sum_{k=1}^{n} |Sx_k|^q \right)^{1/q} \right\| \leq C \left(\sum_{k=1}^{n} \|x_k\|^p \right)^{1/p}.$$

There is a natural duality between (p,q)-convexity and (p',q')-concavity $(1/p + 1/p' = 1 = 1/q + 1/q')$; see [97, Theorem 16.21].

Following [185], we denote by $\mathrm{FBL}_K^{\uparrow p}[E]$ the free Banach lattice satisfying an upper p-estimate with constant K over E. This is the (necessarily unique) Banach lattice Z so that (i) Z satisfies an upper p-estimate with constant K; (ii) there is an isometric embedding $\psi : E \to Z$, generating Z as a lattice; (iii) for any linear operator $T : E \to X$, where X is a Banach lattice satisfying an upper p-estimate with constant K, there exists a lattice homomorphism $\widehat{T} : Z \to X$, with $\widehat{T}\psi = T$, and $\|\widehat{T}\| = \|T\|$. We write $\mathrm{FBL}^{\uparrow p}[E]$ for $\mathrm{FBL}_1^{\uparrow p}[E]$.

The existence and uniqueness of $\mathrm{FBL}_K^{\uparrow p}[E]$ was established in [185]. Moreover, the lattices $\mathrm{FBL}_K^{\uparrow p}[E]$ for different values of K are canonically lattice isomorphic: by [280, Remark 1.5 and its proof], a Banach lattice satisfying an upper p-estimate with constant K can be K-renormed to satisfy an upper p-estimate with constant one.

Many aspects of $\mathrm{FBL}^{\uparrow p}[E]$ remain mysterious. For instance, no functional representation of this lattice, and no explicit norm arising from it, are known (compare and contrast with Section 6.2). However, we have the following result:

Proposition 6.9.36. *Suppose E is a Banach space, Z is a Banach lattice, $1 < p < \infty$, and $i : E \to Z$ is an isometric embedding with the following properties:*

(i) Z is generated by $i(E)$ as a Banach lattice.

(ii) There exists a constant C so that every operator $T : E \to L_{p,\infty}(\mu)$ extends to a lattice homomorphism $T' : Z \to L_{p,\infty}(\mu)$ with $\|T'\| \le C\|T\|$ (μ is a finite measure).

Then for any Banach lattice X and any (p,∞)-convex operator $S : E \to X$ there exists a (necessarily unique) lattice homomorphism $S' : Z \to X$ satisfying $S'i = S$. Moreover, $\|S'\| \le \gamma C K^{(p,\infty)}(S)$, where $K^{(p,\infty)}(S)$ is the (p,∞)-convexity constant of S, and the constant γ depends on p only.

Proof. For brevity, we write $K = K^{(p,\infty)}(S)$. By the universality of $\mathrm{FBL}[E]$, we have a lattice homomorphism $\widehat{i} : \mathrm{FBL}[E] \to Z$ extending $i : E \to Z$. As $i(E)$ generates Z, it follows that $\widehat{i}$ has dense range. Also, let $\widehat{S} : \mathrm{FBL}[E] \to X$ be the lattice homomorphism such that $\widehat{S}\phi = S$. Consider the following:

Claim: There is a constant $\gamma > 0$ (depending only on p) such that

$$\|\widehat{S}f\|_X \le \gamma CK\|\widehat{i}f\|_Z \quad \forall f \in \mathrm{FBL}[E]. \tag{6.9.4}$$

Proof of claim. Given $f \in \mathrm{FBL}[E]$, choose $x^* \in X_+^*$ with $\|x^*\| = 1$ and $x^*\big(\big|\widehat{S}f\big|\big) = \Big\|\widehat{S}f\Big\|_X$. Let N_{x^*} denote the null ideal generated by x^*, that is, $N_{x^*} = \big\{x \in X : x^*\big(|x|\big) = 0\big\}$, and let Y be the completion of the quotient lattice X/N_{x^*} with respect to the norm $\|x + N_{x^*}\| := x^*\big(|x|\big)$. Since this is an abstract L_1-norm, Y is lattice isometric to $L_1(\Omega, \Sigma, \mu)$ for some measure space (Ω, Σ, μ) (see, e.g., [231, Theorem 1.b.2]). The canonical quotient map of X onto X/N_{x^*} induces a lattice homomorphism $Q \colon X \to L_1(\Omega, \Sigma, \mu)$ with $\|Q\| = 1$. For our purposes, we may without loss of generality assume that (Ω, Σ, μ) is σ-finite, passing for instance to the band generated by $Q(\widehat{S}f)$.

Since Q is a lattice homomorphism and S is (p,∞)-convex with constant K, we have

$$\Big\|\bigvee_{k=1}^{n}\big|QS(x_k)\big|\Big\|_{L_1(\mu)} \le \Big\|\bigvee_{k=1}^{n}\big|S(x_k)\big|\Big\|_X \le K\Big(\sum_{k=1}^{n}\|x_k\|_E^p\Big)^{\frac{1}{p}}$$

for every finite sequence (x_k) in E. Hence, by [280, Theorem 1.2], there exists $h \in L_1(\mu)_+$ with $\int_\Omega h\,d\mu \le 1$, yielding a factorization

$$\begin{tikzcd} E \arrow[rr, "QS"] \arrow[dr, "T"'] & & L_1(\mu) \\ & L_{p,\infty}(h\,d\mu) \arrow[ur, "R"'] & \end{tikzcd},$$

with R being a lattice homomorphism implemented by multiplication by h. (6.9.3) gives

$$\begin{aligned}\|Rf\|_{L_1(\mu)} &= \|hf\|_{L_1(\mu)} = \|f\|_{L_1(hd\mu)}\\ &\leq \frac{p}{p-1}\left(\int_\Omega hd\mu\right)^{1-\frac{1}{p}} \cdot \|f\|_{L_{p,\infty}(hd\mu)} \leq \frac{p}{p-1}\|f\|_{L_{p,\infty}(hd\mu)},\end{aligned}$$

hence $\|R\| \leq \frac{p}{p-1}$.

Moreover, in the above factorization h can be chosen in such a way that $\|T\| \leq \gamma_0 K$, where γ_0 depends only on p. To see this, we follow the proof of [280, Theorem 1.2]. In [280, Theorem 1.1], let us take $r = 1$, and choose our subset of $L_1(\mu)$ to be $\{QSx : \|x\|_E \leq 1\}$. We claim that statement (iii) of this theorem holds with C being K. Indeed,

$$\left\| \bigvee_{k=1}^n |\alpha_k QSx_k| \right\|_{L_1(\mu)} \leq K\left(\sum_{k=1}^n \|\alpha_k x_k\|_E^p\right)^{\frac{1}{p}} \leq K\left(\sum_{k=1}^n |\alpha_k|^p\right)^{\frac{1}{p}}.$$

Thus, tracing through the proof of [280, Theorem 1.1], statement (ii) holds with $K'' = K\left(1-\frac{1}{p}\right)^{\frac{1}{p}-1}$. This tells us (with a bit of a clash of notation - what one should do is avoid the appeal to Theorem 1.2, only appeal to Theorem 1.1, and use Theorem 1.1 to prove 1.2 with control of the constants) that [280, Theorem 1.2(iii)] holds, which is just a restatement of [280, Theorem 1.2(iv)]. In other words, $\|T\| \leq \gamma_0 K$, where $\gamma_0 = \left(1-\frac{1}{p}\right)^{\frac{1}{p}-1}$.

By hypothesis, there is a lattice homomorphism $T' : Z \to L_{p,\infty}(h\,d\mu)$ with $T'i = T$ and $\|T'\| \leq C\|T\|$. Let us consider the composition $RT'\widehat{i} : \mathrm{FBL}[E] \to L_1(\mu)$. Note this is a lattice homomorphism which for $x \in E$ satisfies

$$RT'\widehat{i}\phi_E(x) = RT'i(x) = RT(x) = QS(x).$$

It follows from the universality of FBL[E] that $RT'\widehat{i} = Q\widehat{S}$. In particular,

$$\begin{aligned}\|\widehat{S}f\|_X &= \|Q\widehat{S}f\|_{L_1} = \|RT'\widehat{i}f\|_{L_1}\\ &\leq \frac{p}{p-1}C\|T\|\|\widehat{i}f\|_Z \leq \gamma CK\|\widehat{i}f\|_Z, \text{ where } \gamma = \frac{p}{p-1}\gamma_0,\end{aligned}$$

as we wanted to show. □

Having proven the claim, for $f \in \mathrm{FBL}[E]$, put $S'(\widehat{i}f) := \widehat{S}f$. By (6.9.4), S' is well-defined and bounded on Range $\widehat{i}$; it is easy to see that it is a lattice homomorphism. Since Range $\widehat{i}$ is dense, S' extends to a lattice homomorphism on Z. We clearly have $S'i = S$, and $\|S'\| \leq \gamma CK$. □

Generally speaking, p-convexity and p-concavity are much better understood than upper and lower p-estimates. However, using free Banach lattice technology we can find upper p-estimate versions of classical theorems on p-convexity. Indeed, in Corollary 6.9.29 we were able to extend [231, Theorem 1.d.7]; we now show that [287, Theorem 3] has a natural analogue for upper p-estimates:

Corollary 6.9.37. *Suppose $p \in (1, \infty)$, E is a Banach space, X is a Banach lattice and $T : E \to X$ is any operator. The following statements are equivalent:*

(i) T is (p, ∞)-convex;

(ii) There exists a Banach lattice Y with an upper p-estimate, and a factorization $T = S\phi$, where $\phi : E \to Y$ is bounded, and $S : Y \to X$ is a lattice homomorphism.

Moreover, in (2) we can take ϕ to be the isometric embedding of E into $\mathrm{FBL}^{\uparrow p}[E]$, *$S = \widehat{T}$, and $\|S\| \leq \kappa K^{(p,\infty)}(S)$, with κ depending only on p.*

Proof. For (2) $\Rightarrow$ (1), suppose T factors through Y as above. As function calculus intertwines with lattice homomorphisms, for any $x_1, \ldots, x_n \in E$ we have

$$\left\| \bigvee_{k=1}^{n} |Tx_k| \right\|_X \leq \|S\| \left\| \bigvee_{k=1}^{n} |\phi(x_k)| \right\|_Y$$
$$\leq M\|S\| \left(\sum_{k=1}^{n} \|\phi x_k\|_Y^p \right)^{1/p} \leq M\|S\|\|\phi\| \left(\sum_{k=1}^{n} \|x_k\|_E^p \right)^{1/p}$$

(M is the upper p-estimate constant of Y), showing that T is (p, ∞)-convex.

For (1) $\Rightarrow$ (2), by Proposition 6.9.36, it suffices to extend an operator $T : E \to (L_{p,\infty}(\mu), \|\cdot\|_{p,\infty})$ to a lattice homomorphism from $\mathrm{FBL}^{\uparrow p}[E]$ to $L_{p,\infty}(\mu)$, with norm of the extension controlled. Let $S = IT$, where I is the identity $(L_{p,\infty}(\mu), \|\cdot\|_{p,\infty}) \to (L_{p,\infty}(\mu), \|\!|\cdot|\!\|_{p,\infty})$. Then $\|S\| \leq C_p\|T\|$. Extend S to $\widehat{S} : \mathrm{FBL}^{\uparrow p}[E] \to (L_{p,\infty}(\mu), \|\!|\cdot|\!\|_{p,\infty})$ with $\|\widehat{S}\| = \|S\|$. Now the map $T' := I^{-1}\widehat{S} : Z \to (L_{p,\infty}(\mu), \|\cdot\|_{p,\infty})$ is a lattice homomorphism extending T and satisfying $\|T'\| \leq C_p\|T\|$. $\square$

To characterize the spaces E for which $\mathrm{FBL}[E]$ and $\mathrm{FBL}^{(p)}[E]$ are lattice isomorphic, we need two definitions. Suppose E is a Banach space, $C \geq 0$, and Z, X are Banach lattices. We say that $T : E \to Z$ *C-strongly factors through* X if we can write $T = US$, where $S : E \to X$ is a contraction, and $U : X \to Z$ is a lattice homomorphism, with $\|U\| \leq C$.

If $\mathcal{X}$ is a class of Banach lattices, we say that $T : E \to Z$ *C-strongly factors through* $\mathcal{X}$ if it C-strongly factors through some $X \in \mathcal{X}$. If, in the preceding setting, X and Z are both spaces of functions on the same space, we say that T *C-multiplicatively factors through* X if U as above is implemented by a multiplication operator. We say that T factors *strongly* (or *multiplicatively*) if such factorization exists for some C. Obviously, Corollary 6.9.37 can be restated in this language.

Proposition 6.9.38. *Let E be a Banach space, $p > q \geq 1$, and $C \geq 1$. The following are equivalent:*

(i) $\mathrm{FBL}^{(p)}[E]$ *is lattice C-isomorphic to* $\mathrm{FBL}^{(q)}[E]$*;*

(ii) $\mathrm{FBL}^{(p)}[E]$ *is canonically lattice C-isomorphic to* $\mathrm{FBL}^{(q)}[E]$*, that is, the map taking δ_x ($x \in E$) to itself generates a lattice C-isomorphism between* $\mathrm{FBL}^{(p)}[E]$ *and* $\mathrm{FBL}^{(q)}[E]$*;*

(iii) Every contraction $T : E \to L_q(\mu)$ C-strongly factors through a p-convex Banach lattice with p-convexity constant 1;

(iv) Every contraction $T : E \to L_q(\mu)$ C-multiplicatively factors through $L_p(\mu)$;

(v) Every contraction $T : E \to L_q(\mu)$ is p-convex with constant C, i.e., for all finite sequences (x_k) in E we have

$$\left\| \left(\sum_{k=1}^{n} |T(x_k)|^p \right)^{\frac{1}{p}} \right\|_{L_q(\mu)} \leq C \left(\sum_{k=1}^{n} \|x_k\|^p \right)^{\frac{1}{p}}.$$

Corollaries 6.9.45 and 6.9.47 below provide examples of Banach spaces E which possess the equivalent properties described here.

Proof. (2)⇒(1) is straightforward.

(1)⇒(3) Suppose that there is a lattice isomorphism $V\colon \mathrm{FBL}^{(q)}[E] \to \mathrm{FBL}^{(p)}[E]$ such that $\|V\| = 1$ and $\|V^{-1}\| \leq C$. Let $T\colon E \to L_q(\mu)$ be a contraction. Consider $\widehat{T}\colon \mathrm{FBL}^{(q)}[E] \to L_q(\mu)$. Then $T = (\widehat{T}V^{-1})(V\phi_E)$ is a required factorization.

(3)⇒(2) We will use Proposition 6.9.34 with p replaced with q, $Z = \mathrm{FBL}^{(p)}[E]$, and $\iota = \phi_E\colon E \to \mathrm{FBL}^{(p)}[E]$. Let $T\colon E \to L_q(\mu)$ be a contraction. By assumption, we can factor T through a p-convex Banach lattice X with constant 1, $T\colon E \xrightarrow{S} X \xrightarrow{U} L_q(\mu)$ such that

$\|S\| \leqslant 1$, $\|U\| \leqslant C$, and U is a lattice homomorphism. Then $T' := U\widehat{S}\colon \mathrm{FBL}^{(p)}[E] \to L_q(\mu)$ extends T, is a lattice homomorphism, and $\|T'\| \leqslant C$. By Proposition 6.9.34, ϕ_E extends to a lattice C-isomorphism from $\mathrm{FBL}^{(q)}[E]$ to $\mathrm{FBL}^{(p)}[E]$.

To prove (3)⇒(5), we use the strong factorization $T = US$, with $\|U\| \leq C$ and $\|S\| \leq 1$. Then

$$\Big\| \Big(\sum_{k=1}^n |T(x_k)|^p\Big)^{\frac{1}{p}} \Big\|_{L_q(\mu)} = \Big\| \Big(\sum_{k=1}^n |US(x_k)|^p\Big)^{\frac{1}{p}} \Big\|_{L_q(\mu)} \leq$$

$$C\Big\| \Big(\sum_{k=1}^n |S(x_k)|^p\Big)^{\frac{1}{p}} \Big\|_{X} \leq C \Big(\sum_{k=1}^n \|Sx_k\|^p\Big)^{\frac{1}{p}} \leq C \Big(\sum_{k=1}^n \|x_k\|^p\Big)^{\frac{1}{p}}.$$

The first (in)equality is the factorization, the second is since U is a lattice homomorphism of norm at most C, the third by p-convexity of X, and the last since S is a contraction.

Clearly (4)⇒(3). The equivalence between (4) and (5) is essentially [329, p. 264]. □

We can also state an upper p-estimate variant of Proposition 6.9.38.

Proposition 6.9.39. *Let E be a Banach space and $p > q \geq 1$. The following are equivalent:*

(i) $\mathrm{FBL}^{\uparrow p}[E]$ *is lattice isomorphic to* $\mathrm{FBL}^{(q)}[E]$*;*

(ii) $\mathrm{FBL}^{\uparrow p}[E]$ *is canonically lattice isomorphic to* $\mathrm{FBL}^{(q)}[E]$*;*

(iii) There exists $C \geq 1$ such that for every Banach lattice Y with $M^{(q)}(Y) = 1$, every contraction $T : E \to Y$ C-strongly factors through a Banach lattice X which has an upper p-estimate with constant 1;

(iv) There exists $C \geq 1$ such that every contraction $T : E \to L_q(\mu)$ C-strongly factors through a Banach lattice X which has an upper p-estimate with constant 1;

(v) There exists $C \geq 1$ such that every contraction $T : E \to L_q(\mu)$ C-multiplicatively factors through $L_{p,\infty}(\mu)$;

(vi) There exists $C \geq 1$ such that every contraction $T : E \to L_q(\mu)$ is (p,∞)-convex with constant C.

Proof. The implications (2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (2) are similar to Proposition 6.9.38. (4)$\Rightarrow$(6) follows from the same factorization argument used in the proofs of Corollary 6.9.37 and the implication (3)$\Rightarrow$(5) in Proposition 6.9.38. (5)$\Leftrightarrow$(6) is [280, Theorem 1.2], and (5)$\Rightarrow$(4) follows because $L_{p,\infty}(\mu)$ (with an appropriate norm) satisfies an upper p-estimate with constant 1. $\square$

In statement (4) of Proposition 6.9.38 we require that *every* contraction $T : E \to L_q(\mu)$ factor multiplicatively through $L_p(\mu)$; similarly, in statement (5) we require that *every* operator verify a certain inequality. This makes statements (4) and (5) properties of the Banach space E. However, as was evident from the proof, statements (4) and (5) hold on an operator-by-operator basis. More precisely, a contraction $T : E \to L_q(\mu)$ factors multiplicatively through $L_p(\mu)$ if and only if it verifies the inequality in statement (5) of Proposition 6.9.38. Analogous reasoning (using [280]) shows that similar results hold true when $L_p(\mu)$ is replaced by $L_{p,\infty}(\mu)$. As we will now see, the fact that we quantify over all operators gives some interesting relations between the roles of $L_p(\mu)$, $L_q(\mu)$ and $L_{p,\infty}(\mu)$ in the above statements. More precisely, we have the following extrapolation theorem:

Theorem 6.9.40. *Suppose* $\mathrm{FBL}^{(p)}[E]$ *is q-convex for some* $1 \leq p < q$. *Then* $\mathrm{FBL}^{(r)}[E]$ *is q-convex for all* $1 \leq r \leq \infty$.

Proof. By Proposition 6.9.38, there exists a constant C so that any contraction $u : E \to L_p(\mu)$ C-strongly factors through $L_q(\mu)$. As suggested on [242, p. 42], consider the dual pair (E^*, E), where E^* is equipped with its weak* topology $\sigma(E^*, E)$; this turns E^* into a locally convex Hausdorff space, or "elcs" (espace localement convexe séparé) in the French language of [242]. The dual space is then E, with its norm topology.

Applying [242, Théorème 23, (c) $\Rightarrow$ (a)] to this dual pair (or, alternatively, using Exercise 2 on p. 286 of [329], and its solution on p. 336), we conclude that the inequality $\pi_p(T) \leq C\pi_q(T)$ holds for every $T : E^* \to \ell_q$ (we also used the fact that the (p, weak) summing norms of n-tuples in E^* can be computed using either B_E or $B_{E^{**}}$, cf. (6.1.2)). By [97, 3.17 Extrapolation Theorem], for any Banach space F we have $\Pi_q(E^*, F) = \Pi_1(E^*, F)$. In particular, any q-nuclear operator from E^* into ℓ_q is 1-summing. By [242, Théorème 23, (b) $\Rightarrow$ (c)] (or invoking [329, p. 270]), any $u : E \to L_1(\mu)$ strongly factors through $L_q(\mu)$. By Proposition 6.9.38, we conclude that $\mathrm{FBL}[E]$ is lattice isomorphic to $\mathrm{FBL}^{(q)}[E]$. Thus, $\mathrm{FBL}[E]$ is q-convex, and therefore, for any $r \in [1, q]$, $\mathrm{FBL}^{(r)}[E]$ is lattice isomorphic to $\mathrm{FBL}^{(q)}[E]$, hence q-convex. For $r > q$, the q-convexity is automatic. $\square$

We now use the preceding result to establish a few facts regarding factorizable and p-summing operators.

Corollary 6.9.41. *Suppose* $1 \leq q$, $p > \max\{2, q\}$, *and* E *is an infinite dimensional Banach space. There exist* $T, S \in B(E, L_q(\mu))$ *so that* T *(respectively,* S*) does not strongly factor through* $L_p(\mu)$ *(respectively,* $L_{p,\infty}(\mu)$*).*

Proof. (i) If any operator in $B(E, L_q(\mu))$ strongly factors through $L_p(\mu)$, then, by Proposition 6.9.38, $\mathrm{FBL}^{(q)}[E]$ is p-convex. This, however, contradicts Proposition 6.9.30.

(ii) If any operator in $B(E, L_q(\mu))$ strongly factors through $L_{p,\infty}(\mu)$, then, by Proposition 6.9.39, $\mathrm{FBL}^{(q)}[E]$ has an upper p-estimate. Consequently, $\mathrm{FBL}^{(q)}[E]$ is s-convex for any $s < p$ [231, Section 1.f], which contradicts Proposition 6.9.30 (one can take $s \in (\max\{q, 2\}, p)$). $\square$

The following result indicates the limits of extrapolation of summing maps.

Corollary 6.9.42. *Suppose* $1 \leq r < p \leq \infty$, *and* $\Pi_p(E, \ell_p) = \Pi_r(E, \ell_p)$, *for some infinite dimensional* E. *Then* $p \leq 2$.

The restriction $p \leq 2$ is sharp. For instance, $\Pi_r(H, \ell_2) = \Pi_2(H, \ell_2)$, for any Hilbert space H and $r \in [1, \infty)$.

Proof. Suppose, for the sake of contradiction, that E is infinite dimensional, $p \in (2, \infty]$, and $\Pi_p(E, \ell_p) = \Pi_r(E, \ell_p)$ for some $r \in [1, p)$.

(i) $p = \infty$. If $\Pi_r(E, \ell_\infty) = B(E, \ell_\infty)$, then $\Pi_r(E, \ell_\infty(I)) = B(E, \ell_\infty(I))$ for any index I. Now find I so large that E embeds into $\ell_\infty(I)$. Then id_E is r-summing, which is impossible.

(ii) $p < \infty$. If $\Pi_p(E, \ell_p) = \Pi_r(E, \ell_p)$, then, by Extrapolation Theorem [97, p. 3.17], $\Pi_p(E, \ell_p) = \Pi_1(E, \ell_p)$. Imitating the reasoning from the proof of Theorem 6.9.40, we apply [242, Théorème 23] to the dual pair (E, E^*) (E is equipped with its norm topology). We then conclude that any operator from E^* to $L_1(\mu)$ strongly factors through $L_p(\mu)$, which implies $p \leqslant 2$ by Corollary 6.9.41. $\square$

Returning to free Banach lattices, we prove:

Corollary 6.9.43. *If* E *is a Banach space, and* $1 \leq r < p \leq \infty$, *then the following statements are equivalent:*

(i) $\mathrm{FBL}^{(r)}[E]$ *has an upper* p*-estimate.*

(ii) $\mathrm{FBL}[E]$ *has an upper* p*-estimate.*

(iii) id_{E^*} *is* $(q, 1)$*-summing, with* $1/p + 1/q = 1$*.*

Proof. (2) $\Leftrightarrow$ (3) has been established in Theorem 6.9.21. To handle (1) $\Leftrightarrow$ (2), pick $s \in (r, p)$. From [231, Section 1.f], we know that an upper p-estimate implies s-convexity. If one of the lattices involved – either $\mathrm{FBL}^{(r)}[E]$ or $\mathrm{FBL}[E]$ – is s-convex, then the two coincide, by Theorem 6.9.40. $\square$

Note that Theorem 6.9.21 identifies the upper p-estimate constant of $\mathrm{FBL}[E]$ as the $(q, 1)$-summing norm of id_{E^*}; we make no claim that the upper p-estimate constant of $\mathrm{FBL}^{(r)}[E]$ agrees with that of $\mathrm{FBL}[E]$.

Combining Corollary 6.9.43 with Proposition 6.9.38, we obtain:

Corollary 6.9.44. *Suppose* id_{E^*} *is* $(q, 1)$*-summing,* $1/p + 1/q = 1$*, and* $1 \le r < s < p$*. Then any operator from* E *to* $L_r(\mu)$ *multiplicatively factors through* $L_s(\mu)$*.*

We now examine conditions guaranteeing, or precluding, lattice isomorphism between $\mathrm{FBL}^{(p)}[E]$ and $\mathrm{FBL}^{(q)}[E]$.

Corollary 6.9.45. *Suppose a Banach space* E *has type* $s \in (1, 2)$*. Then, for* $1 \le p < q < s$*,* $\mathrm{FBL}^{(p)}[E]$ *and* $\mathrm{FBL}^{(q)}[E]$ *are canonically lattice isomorphic.*

Proof. By Proposition 6.9.38, we need to show that there exists a constant C so that any contraction $T : E \to L_p(\mu)$ has a lattice homomorphic extension $T' : \mathrm{FBL}^{(q)}[E] \to L_p(\mu)$, with $\|T'\| \le C\|T\|$. Emulating the proof of Corollary 6.9.35, we see that it actually suffices to establish the existence of *some* extension T'; the norm will be controlled automatically.

To obtain the desired extension, we use Maurey-Nikishin Extension Theorem [329, III.H.12]: T can be factored through $L_q(\mu)$ as $T = uS$, where $u : L_q(\mu) \to L_p(\mu)$ is a lattice homomorphism. Then S has a lattice homomorphic extension $\widehat{S} : \mathrm{FBL}^{(q)}[E] \to L_q(\mu)$. Then $T' = u\widehat{S}$ is the extension we want. $\square$

Remark 6.9.46. An alternative argument for Corollary 6.9.45 is to note that if E has type s then the dual has cotype s' (for $\frac{1}{s} + \frac{1}{s'} = 1$), hence id_{E^*} is $(s', 1)$-summing, which characterizes upper s-estimates of $\mathrm{FBL}[E]$.

Using [329, III.H.16] instead of [329, III.H.12], we obtain:

Corollary 6.9.47. *For $2 \leq r \leq \infty$ and $1 \leq p \leq 2$, $\mathrm{FBL}^{(p)}[L_r(\mu)]$ and $\mathrm{FBL}^{(2)}[L_r(\mu)]$ are canonically lattice isomorphic.*

Remark 6.9.48. Suppose $r \geq 2$ and $1 \leq p \leq 2$. Corollary 6.9.47 implies that the moduli of the ℓ_r basis in $\mathrm{FBL}^{(p)}[\ell_r]$ and in $\mathrm{FBL}[\ell_r]$ are equivalent; by [30], both are equivalent to the ℓ_s basis, with $1/s = 1/r + 1/2$. We do not know what the span of these moduli is for $r, p \in (2, \infty)$.

On the other hand, it follows immediately from Proposition 6.9.30 that:

Corollary 6.9.49. *Suppose E, F are infinite dimensional Banach spaces, $p \in [1, \infty]$, $q \in (2, \infty]$, and $p \neq q$. Then $\mathrm{FBL}^{(p)}[E]$ is not lattice isomorphic to $\mathrm{FBL}^{(q)}[F]$.*

Remark 6.9.50. By Corollary 6.9.25, if E^* has finite cotype, then $\mathrm{FBL}[E]$ is p-convex for some $p > 1$. Using the fact that the r-convexification of a s-convex space is sr-convex, one can easily show that for such E, $\mathrm{FBL}^{(p)}[E]$ is not lattice isomorphic to the p-convexification of $\mathrm{FBL}[E]$.

We finish this section with a simple observation precluding $\mathrm{FBL}^{(q)}[F]$ from being isomorphic (in the Banach space sense) to a complemented subspace of $\mathrm{FBL}^{(p)}[E]$. Indeed, by combining Corollary 6.9.28 with Corollary 6.9.29, we improve [30, Theorem 9]:

Corollary 6.9.51. *Let $1 \leq p < \min\{2, q\} \leq \infty$. Then $\mathrm{FBL}[\ell_p]$ is not linearly isomorphic to a complemented subspace of $\mathrm{FBL}[\ell_q]$.*

Remark 6.9.52. A related result follows from [231, Theorem 1.d.7 and the remark after]: if $p \in (1, 2]$ and $\mathrm{FBL}^{(q)}[F]$ is not p-convex, then it does not linearly embed complementably into $\mathrm{FBL}^{(p)}[E]$ for any E. Here, complementation is key as $\mathrm{FBL}[E]$ contains isomorphic copies of every separable Banach space as long as $\dim E \geq 2$; see Remark 6.9.32, which also discusses the non-separable setting.

Remark 6.9.53. In this section, we focused on *strong* factorizations via lattices which are p-convex, or have upper p-estimates. Related factorizations (which were not assumed to involve lattice homomorphisms) are considered in [68] (positive factorizations via lattices with upper or lower estimates) and [288] (factorizations using operators with given convexity and concavity).

6.10 Isomorphism of free Banach lattices

In this section we give a partial resolution to the question of whether $\mathrm{FBL}^{(p)}[E]$ and $\mathrm{FBL}^{(p)}[F]$ can be lattice isomorphic (some negative results can be extracted from the r-convexity and r-upper estimate criterion presented in the previous section).

Representation of lattice homomorphisms

In this subsection, we represent lattice homomorphisms on free lattices as composition operators, and gather some consequences of this representation. The following proposition is similar to some results of [220].

Proposition 6.10.1. *Given Banach spaces E, F, $p \in [1, \infty]$ and a lattice homomorphism $T : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$, there exists a mapping $\Phi_T : E^* \to F^*$ so that $Tf = f \circ \Phi_T$ for every $f \in \mathrm{FBL}^{(p)}[F]$. Moreover, Φ_T satisfies the following properties:*

(i) For any $x^* \in E^*$ and $y \in F$, $\Phi_T x^*(y) = (T\delta_y)(x^*)$,

(ii) Φ_T is positively homogeneous,

(iii) Φ_T is weak to weak* continuous on bounded sets,*

(iv) For $y^* \in E^*$, we have $\|\Phi_T y^*\| \leq \|T\|\|y^*\|$. If $p < \infty$, then for every $(y_k^*)_{k=1}^m \subseteq E^*$ we have
$$\sup_{x\in B_F} \Big(\sum_{k=1}^m |[\Phi_T y_k^*](x)|^p \Big)^{\frac{1}{p}} \leq \|T\| \sup_{y\in B_E} \Big(\sum_{k=1}^m |y_k^*(y)|^p \Big)^{\frac{1}{p}} .$$

Proof. First recall that the atoms of $\mathrm{FBL}^{(p)}[E]^*$ are precisely the linear functionals which act on $\mathrm{FBL}^{(p)}[E]$ as lattice homomorphisms [12, p. 111], and these correspond to point evaluations ([26, Corollary 2.7] establishes this for $p = 1$, but the proof for other values of p works in the same way). For $x^* \in E^*$, denote the corresponding evaluation functional on $H[E]$ (and therefore, on $\mathrm{FBL}^{(p)}[E]$) by $\widehat{x^*}$. One can check that $\|\widehat{x^*}\|_{\mathrm{FBL}^{(p)}[E]^*} = \|x^*\|_{E^*}$, for every p, and, as $H[E]$ consists of positively homogeneous functions, we have $\widehat{\alpha x^*} = \alpha \widehat{x^*}$ for $\alpha \geq 0$.

If $T : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ is a lattice homomorphism, then T^* is interval preserving, and, in particular, maps atoms to atoms. Using the description of atoms given in the previous paragraph, we conclude that T^* induces a positively homogeneous map $\Phi_T : E^* \to F^*$, via $\Phi_T x^* = T^* \widehat{x^*} \circ \phi_F$ (that is, $\widehat{\Phi_T x^*} = T^* \widehat{x^*}$).

By construction, for every $f \in \mathrm{FBL}^{(p)}[F]$ we have $Tf = f \circ \Phi_T$. Indeed, for $x^* \in E^*$ let $y^* = \Phi_T x^*$. Then

$$(f \circ \Phi_T)(x^*) = f(y^*) = \widehat{y^*}(f) = [T^* \widehat{x^*}](f) = Tf(x^*).$$

Plugging in $f = \delta_y$, we obtain (1). This, in turn, implies (2): for $\lambda \geq 0$, $x^* \in E^*$ and $y \in F$,

$$\Phi_T(\lambda x^*)(y) = (T\delta_y)(\lambda x^*) = \lambda (T\delta_y)(x^*) = \lambda \Phi_T x^*(y).$$

To establish (3), note that, if $y^*_\alpha \xrightarrow{w^*} y^*$ is a bounded net in E^*, then for every $x \in F$ we have

$$[\Phi_T y^*_\alpha](x) = [T\delta_x](y^*_\alpha) \longrightarrow [T\delta_x](y^*) = [\Phi_T y^*](x),$$

as $T\delta_x \in \mathrm{FBL}^{(p)}[E]$ is weak* continuous on bounded sets.

To handle (4), let $(y_k^*)_{k=1}^m \subseteq E^*$. We have

$$\begin{aligned}
\sup_{x \in B_F} \sum_{k=1}^m |\Phi_T y_k^*(x)|^p &= \sup_{x \in B_F} \sum_{k=1}^m |T\delta_x(y_k^*)|^p \\
&\leq \sup_{x \in B_F} \|T\delta_x\|^p_{\mathrm{FBL}^{(p)}[E]} \sup_{y \in B_E} \sum_{k=1}^m |y_k^*(y)|^p \\
&\leq \|T\|^p \sup_{y \in B_E} \sum_{k=1}^m |y_k^*(y)|^p.
\end{aligned}$$

□

In certain cases, more can be said about the map Φ_T. The proof of the following proposition is straightforward.

Proposition 6.10.2. *In the notation of Proposition 6.10.1, we have:*

(i) Suppose T is surjective, so that, by Open Mapping Theorem, there exists $c > 0$ so that for every $g \in \mathrm{FBL}^{(p)}[E]$ there exists $f \in \mathrm{FBL}^{(p)}[F]$ with $Tf = g$ and $\|f\| \leq c^{-1}\|g\|$. Then $c\|x^*\| \leq \|\Phi_T x^*\|$ for every $x^* \in E^*$.

(ii) If T has dense range, then Φ_T is injective.

(iii) If T is a lattice isomorphism, then Φ_T is bijective, and $\Phi_{T^{-1}} = \Phi_T^{-1}$.

(iv) If T is a lattice isometry, then $\|\Phi_T x^*\| = \|x^*\|$ for any $x^* \in E^*$.

Remark 6.10.3. Suppose $T : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ is a lattice isometry, and F^* has the *weak** (or *dual*) *Kadec-Klee Property*, investigated in [102] and [149]. That is, if (x_n^*) is a sequence in F^* weak*-converging to $x^* \in F^*$, and such that $\|x_n^*\| \to \|x^*\|$, then $\|x_n^* - x^*\| \to 0$. Then we can further deduce that Φ_T is norm to norm continuous.

Remark 6.10.4. Lemma 6.3.1 shows that for $T : F \to E$, the induced map $\overline{T} : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ satisfies $\Phi_{\overline{T}} = T^*$.

Proposition 6.2.2 immediately implies that the converse of Proposition 6.10.1 is valid for $p = \infty$.

Corollary 6.10.5. *Suppose E and F are Banach spaces, and $\Phi : E^* \to F^*$ is a positively homogeneous map, weak* to weak* continuous on bounded sets, so that $C := \sup_{y^* \in E^* \setminus \{0\}} \dfrac{\|\Phi y^*\|}{\|y^*\|} < \infty$. Then there exists a lattice homomorphism $T : \mathrm{FBL}^{(\infty)}[F] \to \mathrm{FBL}^{(\infty)}[E]$ so that $\|T\| = C$, and $\Phi = \Phi_T$.*

Remark 6.10.6. In contrast, the converse of Proposition 6.10.1 fails for $p = 1$. Below we present a map Φ, satisfying Proposition 6.10.1(2,3,4) for $p = 1$, but not implementing a lattice homomorphism of $\mathrm{FBL}[\ell_1]$ to itself. Specifically, define

$$\Phi\big((a_i)_{i=1}^\infty\big) = \Big(|a_1| \wedge \big(\vee_{i\geq 2} \frac{|a_i|}{i}\big), 0, 0, \dots\Big).$$

Clearly Φ is positively homogeneous and weak* continuous (relative to the canonical identification $\ell_\infty = \ell_1^*$) on bounded sets, so (2) and (3) of Proposition 6.10.1 hold. To establish (4), consider a finite collection $(x_k) \subseteq \ell_\infty$, with $\max_\pm \|\sum_k \pm x_k\| \leq 1$. Write $x_k = (a_{ki})_{i=1}^\infty$. Then $\vee_i \sum_k |a_{ki}| \leq 1$. Consequently,

$$\max_\pm \|\sum_k \pm \Phi x_k\| \leq \sum_k |a_{k1}| \leq 1.$$

Let $e = (1, 0, 0, \dots) \in \ell_1$. Then $f = |\delta_e| : (a_i) \mapsto |a_1|$ belongs to $\mathrm{FBL}[\ell_1]$. Now consider $g : \ell_\infty \to \mathbb{R} : x^* \mapsto f(\Phi x^*)$ – that is,

$$g\big((a_i)\big) = |a_1| \wedge \big(\vee_{i\geq 2} \frac{|a_i|}{i}\big).$$

By [26, Example 2.11], $g \notin \mathrm{FBL}[\ell_1]$. This shows that the composition operator defined by Φ does not map $\mathrm{FBL}[\ell_1]$ to itself, as claimed.

The following statement is reminiscent of the notion of "dependence on finitely many coordinates" in [90].

Lemma 6.10.7. *Suppose $1 \leq p \leq \infty$, and $T : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ is a lattice homomorphism. Then for any $y \in F$ and $\varepsilon > 0$ there exist $N = N[y] \in \mathbb{N}$, $(x_i[y])_{i=1}^{N[y]} \subseteq E$, and a $\boldsymbol{F}[y] : \mathbb{R}^N \to \mathbb{R}$, represented by finitely many linear and lattice operations, so that*

$$\big|[\Phi_T x^*](y) - \boldsymbol{F}[y]((x^*(x_i[y]))_{i=1}^{N[y]})\big| \leq \varepsilon \|x^*\| \text{ for any } x^* \in E^*.$$

Proof. The function $T\delta_y : E^* \to \mathbb{R} : x^* \mapsto [\Phi_T x^*](y)$ belongs to $\mathrm{FBL}^{(p)}[E]$, hence it is the limit (in the $\mathrm{FBL}^{(p)}[E]$ norm, and, consequently, in the sup norm on B_{E^*}) of elements of $\mathrm{FVL}[E]$. Now recall that elements of $\mathrm{FVL}[E]$ can be written as $f(\delta_{x_1}, \ldots, \delta_{x_N})$, where f is a composition of finitely many linear and lattice operations. □

For future use (addressing the same setting), we state the following:

Corollary 6.10.8. *Suppose $1 \leq p \leq \infty$, and $T : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ is a lattice homomorphism. Let G be a finite dimensional subspace of F, and $\varepsilon > 0$. Then there exist $N \in \mathbb{N}$, and $x_1, \ldots, x_N \in E$, so that if $x^* \in E^*$, $\|x^*\| \leq 1$, and $x^*(x_i) = 0$ for $1 \leq i \leq N$, then $|[\Phi_T x^*](y)| \leq \varepsilon \|y\|$ for any $y \in G$.*

Proof. By scaling, assume $\|T\| \leq 1$. Let $(y_j)_{j=1}^M$ be an $\varepsilon/2$-net in the unit ball of G. By Lemma 6.10.7, there exist $x_1, \ldots, x_N \in E$, so that if $x^* \in E^*$, $\|x^*\| \leq 1$, and $x^*(x_i) = 0$ for $1 \leq i \leq N$, then $|[\Phi_T x^*](y_j)| \leq \varepsilon/2$ for $1 \leq j \leq M$. For an arbitrary y in the unit ball of G, find j so that $\|y - y_j\| < \varepsilon/2$. Then

$$\begin{aligned}\big|[\Phi_T x^*](y)\big| &= \big|[T\delta_y](x^*)\big| \leq \big|[T\delta_{y_j}](x^*)\big| + \|y - y_j\| \\ &= \big|[\Phi_T x^*](y_j)\big| + \|y - y_j\| \leq \varepsilon.\end{aligned}$$

□

Proposition 6.10.1 also allows us to study lattice transitivity of $\mathrm{FBL}^{(p)}$ in the following sense. We say that a Banach lattice X is *lattice almost transitive* if, for any norm one $x, y \in X_+$, and $\varepsilon > 0$, there exists a surjective lattice isometry $T : X \to X$ so that $\|Tx - y\| < \varepsilon$ (note that T^{-1} is a lattice isometry as well). The spaces $L_p(0,1)$ $(1 \leq p < \infty)$ are known to be lattice almost transitive (see e.g. the proof of [113, Theorem 12.4.3], or [112, Proposition 3.5]). Another example is the "Gurarij AM-space", recently constructed in [112]. Despite the fact that $\mathrm{FBL}^{(p)}$ lattices possess a large number of lattice homomorphisms, we will now show that such lattices fail to be lattice almost transitive.

Proposition 6.10.9. *For any non-trivial Banach space E, and any $p \in [1, \infty]$, the space* $\mathrm{FBL}^{(p)}[E]$ *is not lattice almost transitive.*

Proof. Fix a norm one $e \in E$, and let $f = \big[\delta_e\big]_+$, $g = \big|\delta_e\big|$. Note that $\|f\|_\infty \le \|f\| \le \|e\|$, hence $\|f\| = 1$. Similarly, $\|g\| = 1$. We shall show that $\|Tf - g\| \ge 1/3$ whenever T is a surjective lattice isometry on $\mathrm{FBL}^{(p)}[E]$.

Suppose, for the sake of contradiction, that $\gamma := \|Tf - g\| < 1/3$. By the preceding discussion, T is implemented by a positively homogeneous map $\Phi = \Phi_T : B_{E^*} \to B_{E^*}$, weak* continuous on bounded sets, which preserves norms; Φ^{-1} has the same properties, since it implements T^{-1}. Then, for any $x^* \in B_{E^*}$, we have

$$\big||x^*(e)| - [\Phi x^*(e)]_+\big| \le \gamma. \tag{6.10.1}$$

Let now

$$\begin{aligned} &U_+ = \{x^* \in B_{E^*} : x^*(e) \ge 1/3\},\ U_- = \{x^* \in B_{E^*} : x^*(e) \le -1/3\}, \\ &U = U_+ \cup U_-,\ V = \{x^* \in B_{E^*} : x^*(e) \ge 2/3\}. \end{aligned}$$

If $\Phi x^* \in V$, then, by (6.10.1), $|x^*(e)| \ge 2/3 - \gamma > 1/3$, hence $x^* \in U$. In other words, $V \subseteq \Phi U = \Phi U_+ \cup \Phi U_-$.

The sets U_+ and U_- are closed (in the relative weak* topology of B_{E^*}), hence the same is true of their images. Since V is a convex set, in particular it is path connected, hence there exists $\eta \in \{-1, +1\}$ so that $\Phi U_\eta \cap V = \emptyset$. Now take a norm one x^* so that $x^*(e) = \eta$. Then $|x^*(e)| = 1$, while $\Phi x^*(e) < 2/3 < 1 - \gamma$, contradicting (6.10.1). $\square$

For $1 \le p < \infty$, $\mathrm{FBL}^{(p)}$ lattices are often distinct

In this subsection, we establish that, for $p < \infty$, in certain cases $\mathrm{FBL}^{(p)}[E]$ and $\mathrm{FBL}^{(p)}[F]$ cannot be lattice isomorphic. As a tool, we need the "weak p" norms (see e.g. [97]). Recall that, for $(z_i)_{i=1}^N \subseteq Z$,

$$\|(z_i)\|_{p,\mathrm{weak}} = \sup_{z^* \in B_{Z^*}} \Big(\sum_i |z^*(z_i)|^p\Big)^{1/p} = \sup\Big\{\|\sum_i \alpha_i z_i\| : \sum_i |\alpha_i|^q \le 1\Big\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. For $(z_i^*)_{i=1}^N \subseteq Z^*$, moreover, (6.1.2) yields:

$$\|(z_i^*)\|_{p,\mathrm{weak}} = \sup_{z^{**} \in B_{Z^{**}}} \Big(\sum_i |z^{**}(z_i^*)|^p\Big)^{1/p} = \sup_{z \in B_Z} \Big(\sum_i |z_i^*(z)|^p\Big)^{1/p}.$$

By duality, $\|(z_i^*)\|_{p,\text{weak}}$ coincides with the norm of the operator $\ell_q^N \to Z^* : e_i \mapsto z_i^*$, where (e_i) is the canonical basis of ℓ_q^N, and $\frac{1}{p} + \frac{1}{q} = 1$.

Suppose E and F are Banach spaces, and fix $C > 0$ and $p \in [1, \infty)$. Define a (C, p)-game between two players as follows:

At the start of the n-th round, we have finite dimensional subspaces $F_1, \ldots, F_{n-1} \subseteq F$, $E_1, \ldots, E_{n-1} \subseteq E$, and norm one $y_i^* \in F_i^\perp$, $x_i^* \in E_i^\perp$ for $1 \le i \le n-1$ (here, for a subspace $G \subseteq F$, we denote $G^\perp = \{x^* \in F^* : x^*(x) = 0, \forall x \in G\}$).

Round n, step 1: Player 1 selects a finite dimensional $E_n \subseteq E$, then Player 2 picks a finite dimensional $F_n \subseteq F$.

Round n, step 2: Player 1 chooses $y_n^* \in S_{F_n^\perp}$ (the unit sphere of $F_n^\perp$), then Player 2 picks $x_n^* \in S_{E_n^\perp}$.

Player 1 wins the (C, p)-game after N rounds if there exist $\alpha_1, \ldots, \alpha_N \ge 0$ so that $\|(\alpha_i x_i^*)_{i=1}^N\|_{p,\text{weak}} > C\|(\alpha_i y_i^*)_{i=1}^N\|_{p,\text{weak}}$ (we say that $(\alpha_i, x_i^*, y_i^*)_{i=1}^N$ witnesses the win of Player 1).

We shall say that E^* *p-dominates* F^* (relative to preduals E and F, which we will omit if the duality is canonical) if Player 1 has a winning strategy for the (C, p) game for any $C > 0$ (that is, Player 1 can win, no matter what Player 2 does). Note that we can always assume that $E_1 \subseteq E_2 \subseteq \ldots$, and $F_1 \subseteq F_2 \subseteq \ldots$.

We need a simple observation combining duality with small perturbations.

Lemma 6.10.10. *Suppose Z is a Banach space, and $\varepsilon > 0$.*

(i) Suppose G is a subspace of Z. Then for any $z^* \in Z^*$, $\operatorname{dist}(z^*, G^\perp) = \sup\{|z^*(z)| : z \in G, \|z\| \le 1\}$. Further,

$$\operatorname{dist}(z^*, G^\perp) \ge \frac{1}{2} \inf\{\|z^* - w^*\| : w^* \in G^\perp, \|w^*\| = \|z^*\|\}.$$

(ii) Suppose G and G_0 are subspaces of Z, so that for every $z \in G\backslash\{0\}$ there exists $z_0 \in G_0$ so that $\|z_0\| = \|z\|$ and $\|z - z_0\| < \varepsilon\|z\|$. Then any $z^* \in G_0^\perp$ satisfies $\operatorname{dist}(z^*, G^\perp) < \varepsilon\|z^*\|$.

Below, we apply this lemma for finite dimensional G and G_0. In this case, the statement of (2) can be strengthened slightly: if for every $z \in G$ there exists $z_0 \in G_0$ so that $\|z_0\| = \|z\|$ and $\|z - z_0\| \le \varepsilon \|z\|$, then any $z^* \in G_0^\perp$ satisfies $\mathrm{dist}(z^*, G^\perp) \le \varepsilon \|z^*\|$.

Proof. (1) The equality

$$\mathrm{dist}(z^*, G^\perp) = \sup \left\{ |z^*(z)| : z \in G, \|z\| \le 1 \right\}$$

follows from the canonical identification between G^* and $Z^*/G^\perp$. To establish the "further" statement, if suffices to show that, if $\|z^*\| = 1$, and $\mathrm{dist}(z^*, G^\perp) < c$, then there exists a norm one $w^* \in G^\perp$ with $\|z^* - w^*\| < 2c$. To this end, find $u^* \in G^\perp$ so that $\|z^* - u^*\| < c$. By the triangle inequality, $\big| \|u^*\| - 1 \big| < c$. Let $w^* = u^*/\|u^*\|$, so $u^* = \|u^*\| w^*$, and therefore, $\|u^* - w^*\| = \big| \|u^*\| - 1 \big| < c$. Consequently,

$$\|z^* - w^*\| \le \|z^* - u^*\| + \|u^* - w^*\| < 2c.$$

(2) Pick a norm one $z^* \in G_0^\perp$. By (1), $\mathrm{dist}(z^*, G^\perp) = \sup \left\{ |z^*(z)| : z \in G, \|z\| = 1 \right\}$. For any z as in the right hand side, find $z_0 \in B_{G_0}$ so that $\|z - z_0\| < \varepsilon$. Then $|z^*(z)| \le |z^*(z_0)| + \|z - z_0\| < \varepsilon$. □

Proposition 6.10.11. *Suppose $\infty \ge u > \max\{v, p\} \ge v \ge 1$, $E = (\sum_i E_i)_u$ ($E_1, E_2, \ldots$ are finite dimensional; for $u = \infty$, consider the c_0-sum), and F^* contains a copy of $\ell_{v'}$, with $1/v + 1/v' = 1$. Then E^* p-dominates F^*.*

Proof. Assume F^* contains a normalized basic sequence, K-equivalent to the canonical basis of $\ell_{v'}$. Fix $C > 0$. Let $u' = u/(u-1)$ (so $1/u + 1/u' = 1$). In the course of a (C, p)-game, Player 1 can arrange $(y_i^*)_{i=1}^N$ to be $2K$-equivalent to the unit vector basis of $\ell_{v'}^N$, and force Player 2 to make $(x_i^*)_{i=1}^N$ to be 2-equivalent to the unit vector basis of $\ell_{u'}^N$ (this follows from a "gliding hump" argument, permitted by Lemma 6.10.10(2)). Then

$$\frac{1}{2K} \|(y_i^*)\|_{p,\mathrm{weak}} \le \|id : \ell_q^N \to \ell_{v'}^N\| = \begin{cases} N^{1/p - 1/v} & v > p \\ 1 & v \le p \end{cases}$$

(here id stands for the formal identity and $\frac{1}{p} + \frac{1}{q} = 1$). Similarly, $2\|(x_i^*)\|_{p,\mathrm{weak}} \ge N^{1/p - 1/u}$ (since $u > p$). Thus, $\|(x_i^*)\|_{p,\mathrm{weak}} > C \|(y_i^*)\|_{p,\mathrm{weak}}$, for N large enough. □

Above, we defined p-domination, and established examples when it occurs. Next, we use it to show that certain free Banach lattices cannot be lattice isomorphic.

Proposition 6.10.12. *Suppose $1 \leq p < \infty$, E and F are Banach spaces, and E^* p-dominates F^*. Then $\mathrm{FBL}^{(p)}[F]$ is not lattice isomorphic to a lattice quotient of $\mathrm{FBL}^{(p)}[E]$.*

The proof requires an auxiliary result:

Lemma 6.10.13. *Let $\frac{1}{p} + \frac{1}{q} = 1$. For any Banach space Z, and any $z_1^*, \ldots, z_n^* \in Z^*$, we have*

$$\|(z_i^*)\|_{p,\mathrm{weak}} = \sup\big\{\|\sum_{i=1}^n \gamma_i \widehat{z_i^*}\|_{\mathrm{FBL}^{(p)}[Z]^*} : \sum_{i=1}^n |\gamma_i|^q \leq 1\big\}.$$

Proof. Let $T : Z \to \ell_p^n$ be given by $z \mapsto (z_i^*(z))_{i=1}^n$ and consider its canonical extension $\widehat{T} : \mathrm{FBL}^{(p)}[Z] \to \ell_p^n$. Note that $(\widehat{T})^* : \ell_q^n \to \mathrm{FBL}^{(p)}[Z]^*$ maps the unit vector basis to $\widehat{z_i^*}$. Hence,

$$\begin{aligned}\|(z_i^*)\|_{p,\mathrm{weak}} &= \|T\| = \|\widehat{T}\| = \|\widehat{T}^*\| \\ &= \sup\big\{\|\sum_{i=1}^n \gamma_i \widehat{z_i^*}\|_{\mathrm{FBL}^{(p)}[Z]^*} : \sum_{i=1}^n |\gamma_i|^q \leq 1\big\}.\end{aligned}$$

□

Proof of Proposition 6.10.12. Henceforth, suppose $T : \mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$ is a surjective lattice homomorphism (by scaling, we can assume it is contractive). There exists $c > 0$ so that for any $g \in \mathrm{FBL}^{(p)}[F]$ there exists $f \in \mathrm{FBL}^{(p)}[E]$ so that $Tf = g$, $\|f\| \leq c^{-1}\|g\|$. We keep the earlier notation Φ_T. By Proposition 6.10.2, the inequality $c\|y^*\| \leq \|\Phi_T y^*\| \leq \|y^*\|$ holds for any $y^* \in F^*$.

Fix $\varepsilon \in (0, 1/4)$ and $C > 1$. Find $K > (C + \varepsilon)c^{-1}$. Now let us start a (K, p)-game.

Suppose $n-1$ rounds have been played; we have $E_1 \subseteq \ldots \subseteq E_{n-1} \subseteq E$, $F_1 \subseteq \ldots \subseteq F_{n-1} \subseteq F$; norm one $y_i^* \in F_i^\perp$ and $x_i^* \in E_i^\perp$, for $1 \leq i < n$, so that $\|t_i x_i^* - \Phi_T y_i^*\| < 4^{-i}\varepsilon$, for some $t_i \in [c, 1]$; these have been chosen in such a way that Player 1 can still win the (K, p)-game if they keep playing.

On the first step of the n-th round, Player 1 picks a finite dimensional $E_n \subseteq E$ which contains E_{n-1}, and permits winning. Then Player 2 chooses $F_n \subseteq F$, $F_n \supseteq F_{n-1}$ so that, for any norm one $y^* \in F_n^\perp$, and any $x \in E_n$, we have $|[\Phi_T y^*](x)| \leq 4^{-1-n}\varepsilon\|x\|$ (this is possible, by Corollary 6.10.8).

On the second step, Player 1 selects a norm one $y_n^* \in F_n^\perp$ consistent with victory. By Lemma 6.10.10(1), we have that

$$\inf\big\{\|\Phi_T y_n^* - w^*\| : w^* \in E_n^\perp, \|w^*\| = \|\Phi_T y_n^*\|\big\} \le 2\mathrm{dist}(\Phi_T y_n^*, E_n^\perp) < 4^{-n}\varepsilon.$$

Hence, Player 2 can find $x_n^* \in E_n^\perp$ with $\|x_n^*\| = 1$, for which there exists $t_n \in [c,1]$ so that $\|\Phi_T y_n^* - t_n x_n^*\| < 4^{-n}\varepsilon$.

Continue until we obtain $(y_i^*)_{i=1}^N$ and $(x_i^*)_{i=1}^N$ witnessing the victory of Player 1. That is, we can find $\alpha_1, \dots, \alpha_N \ge 0$ so that

$$\|(\alpha_i x_i^*)\|_{p,\mathrm{weak}} > K\|(\alpha_i y_i^*)\|_{p,\mathrm{weak}}.$$

By scaling, we can assume $\max_i \alpha_i = 1$. Denote $\|(\alpha_i y_i^*)\|_{p,\mathrm{weak}}$ by M. Then clearly $M \ge 1$. By convexity,

$$\|(\alpha_i t_i x_i^*)\|_{p,\mathrm{weak}} \ge c\|(\alpha_i x_i^*)\|_{p,\mathrm{weak}} > KcM.$$

Then

$$\begin{aligned}\big\|\big(\alpha_i \Phi_T y_i^*\big)\big\|_{p,\mathrm{weak}} &\ge \big\|\big(\alpha_i t_i x_i^*\big)\big\|_{p,\mathrm{weak}} - \sum_i \alpha_i \|\Phi_T y_i^* - t_i x_i^*\| \\ &> KcM - \sum_i 4^{-i}\varepsilon > (Kc - \varepsilon)M > CM.\end{aligned}$$

By Lemma 6.10.13,

$$M = \sup\big\{\|\sum_i \gamma_i \alpha_i \widehat{y_i^*}\|_{\mathrm{FBL}^{(p)}[F]^*} : \sum_i \gamma_i^q \le 1\big\}$$

and

$$\big\|\big(\alpha_i \Phi_T y_i^*\big)\big\|_{p,\mathrm{weak}} = \sup\big\{\|\sum_i \gamma_i \alpha_i T^* \widehat{y_i^*}\|_{\mathrm{FBL}^{(p)}[E]^*} : \sum_i \gamma_i^q \le 1\big\}.$$

Thus, $\|T^*\| > C$. This contradicts the assumption that $\|T\| \le 1$. $\square$

We also have a "local" criterion for free lattices being "different".

Proposition 6.10.14. *Fix $u, v \in [2,\infty]$, $p \in [1,\infty]$, and $u < \min\{v, p'\}$, where $1/p + 1/p' = 1$. Suppose E^* has cotype u, and F^* does not have cotype less than v. Then $\mathrm{FBL}^{(p)}[F]$ is not lattice isomorphic to a lattice quotient of $\mathrm{FBL}^{(p)}[E]$.*

Proof. Find $q \in (u, \min\{v, p'\})$. By [97, Chapter 14], there exists $C > 0$ such that for any n we can find $y_1^*, \ldots, y_n^* \in F^*$ with the property that, for any (α_i), we have

$$\max_i |\alpha_i| \leq \big\| \sum_i \alpha_i y_i^* \big\| \leq C \big(\sum_i |\alpha_i|^q \big)^{1/q}.$$

Consequently, $\min_i \|y_i^*\| \geq 1$, and $\|(y_i^*)\|_{q',\mathrm{weak}} \leq C$.

Suppose, for the sake of contradiction, that $T : \mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$ is a surjective lattice homomorphism (without loss of generality, T is contractive). Then, by Proposition 6.10.1, for (y_i^*) as above we have $\|(\Phi_T y_i^*)\|_{q',\mathrm{weak}} \leq C$.

On the other hand, T^* is bounded below by some $c > 0$, hence by Proposition 6.10.2 the inequality $\|\Phi_T y^*\| \geq c\|y^*\|$ holds for any $y^* \in F^*$. By cotype u, $\max_\pm \| \sum_i \pm \Phi_T y_i^* \| \geq Kcn^{1/u}$ (K is the cotype constant), so

$$\|(\Phi_T y_i^*)\|_{q',\mathrm{weak}} \geq \max_\pm \big\| \sum_i \pm n^{-1/q} \Phi_T y_i^* \big\| \geq Kcn^{1/u - 1/q};$$

the latter exceeds C for large n. This is the desired contradiction. □

Corollary 6.10.15. *Suppose $r \in [1, 2)$, and $s \in (r, \infty]$. Then* $\mathrm{FBL}[L_r]$ *is not a lattice quotient of* $\mathrm{FBL}[L_s]$.

This corollary generalizes the classical result that, for r and s as above, L_r is not a quotient of L_s.

Proof. Following the usual convention, we assume $1/r + 1/r' = 1 = 1/s + 1/s'$. Let $E = L_s$, $F = L_r$, and note that E^* has cotype $\max\{2, s'\}$, while F^* has cotype $r' > \max\{2, s'\}$, but no smaller. Apply Proposition 6.10.14 with E, F as above, and $p = 1$. □

The above results leads one to ask:

Question 6.10.16. Suppose $\mathrm{FBL}^{(p)}[E]$ is lattice isomorphic to $\mathrm{FBL}^{(p)}[F]$. What properties do the spaces E and F necessarily share?

The results of Section 6.9 provide positive answers for certain properties (such as containing a complemented copy of ℓ_1, or ℓ_1^n, see Theorem 6.9.20, respectively Corollary 6.9.25). Some other properties are covered by the following partial result.

Proposition 6.10.17. *Suppose* $\mathrm{FBL}[E]$ *is lattice isomorphic to* $\mathrm{FBL}[F]$*, and* E *is a separable space which has* c_0 *as a quotient. Then:*

(i) If F *is reflexive, it cannot be* K*-convex.*

(ii) F *is not super-reflexive.*

Proof. (1) Suppose F is reflexive, and $T : \mathrm{FBL}[F] \to \mathrm{FBL}[E]$ is a lattice isomorphism. Let $\Phi_T : E^* \to F^*$ be the corresponding map given by Proposition 6.10.1. By the proof of [230, Proposition 2.e.9], E^* contains a weak* null sequence (e_i^*), equivalent to the ℓ_1 basis. The sequence $(\Phi_T e_i^*)$ is semi-normalized, and weakly null in F^*, hence, by [103], we can find $i_1 < i_2 < \dots$ so that $(\Phi_T e_{i_k}^*)$ is Schreier unconditional. We have

$$\max_{\pm} \| \sum_k \pm \alpha_k \Phi_T e_{i_k}^* \| \sim \max_{\pm} \| \sum_k \pm \alpha_k e_{i_k}^* \| \sim \sum_k |\alpha_k|,$$

hence for any n, and any choice of signs $\pm$,

$$\| \sum_{k=n+1}^{2n} \pm \alpha_k \Phi_T e_{i_k}^* \| \sim \max_{\pm} \| \sum_{k=n+1}^{2n} \pm \alpha_k \Phi_T e_{i_k}^* \| \sim \sum_{k=n+1}^{2n} |\alpha_k|,$$

which shows that F^* contains copies of ℓ_1^n uniformly. This is equivalent to the lack of K-convexity for F^*, hence also for F [97, Chapter 13].

(2) is a consequence of (1). Indeed, if F is super-reflexive, then it is necessarily reflexive. Also, it cannot contains copies of ℓ_1^n uniformly, which implies K-convexity. □

We do not know whether, under the hypotheses of Proposition 6.10.17, F necessarily has a c_0 quotient. One major obstacle is that a weakly null sequence may not have an unconditional subsequence [241] (see also [192]).

Note that Question 6.10.16 can be interpreted as inquiring which properties of Banach spaces are preserved under positively homogeneous bijections which are weak* to weak* continuous on bounded sets in both directions. For the discussion on Banach space properties preserved by other types of non-linear isomorphisms, see e.g. [9, Chapter 14]. For instance, there it is shown that Lipschitz isomorphisms preserve super-reflexivity (Proposition 6.10.17 above can be viewed as a weaker version of that).

Isometries between $\mathrm{FBL}^{(p)}$'s for finite p

To examine the existence of lattice isometries between lattices of the form $\mathrm{FBL}^{(p)}[E]$ and $\mathrm{FBL}^{(p)}[F]$, recall that a Banach space Z is called *smooth* if, for every point z on its unit sphere, there exists a unique support functional, which we call f_z (that is $f_z(z) = \|f_z\| = 1$). For more information on smoothness, and on the related topic of strict convexity, we refer to [98, Ch. 2].

Recall that if E and F are linearly isometric, then $\mathrm{FBL}^{(p)}[E]$ and $\mathrm{FBL}^{(p)}[F]$ are lattice isometric. A converse to this is the main result of this section, which can be considered as a Banach-Stone type theorem for free Banach lattices:

Theorem 6.10.18. *Suppose $1 \leq p < \infty$, and E, F are Banach spaces so that E^*, F^* are smooth. Then $T : \mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$ is a surjective lattice isometry if and only if $T = \overline{U}$, for some surjective isometry $U : E \to F$. Consequently, E and F are isometric if and only if $\mathrm{FBL}^{(p)}[E]$ is lattice isometric to $\mathrm{FBL}^{(p)}[F]$.*

It is known that Z is *strictly convex* (that is, the equality $\|z_1 + z_2\| = 2$ holds for $z_1, z_2 \in S_Z$ if and only if $z_1 = z_2$) whenever Z^* is smooth. For reflexive spaces, the converse implication holds as well.

Before proving Theorem 6.10.18, we recall some facts related to the geometry of the norm of a Banach space, and use them to describe the behavior of $\|(x, ty)\|_{p,\mathrm{weak}}$ for $t \approx 0$.

Suppose x is a point on the unit sphere of a Banach space Z. Denote by $\mathcal{F}(x)$ the set of support functionals for x – that is, of functionals x^* for which $\|x^*\| = 1 = x^*(x)$ (note that this set is weak* closed, hence weak* compact). Now suppose $y \in Z$, $\|y\| = 1$, and $\lambda \in \mathbb{R}$. It is known (see e.g. [191, Section 6]) that there exists $x^* \in \mathcal{F}(x)$ so that $x^*(y) = \lambda$ if and only if

$$\lim_{t\to 0^-} \frac{\|x + ty\| - 1}{t} \leq \lambda \leq \lim_{t\to 0^+} \frac{\|x + ty\| - 1}{t}. \tag{6.10.2}$$

In particular, if $\mathcal{F}(x) = \{x^*\}$ (in this case, $x^* = f_x$), then

$$\lim_{t\to 0} \frac{\|x + ty\| - 1}{t} = x^*(y).$$

We begin the proof of Theorem 6.10.18 with a lemma.

Lemma 6.10.19. *Suppose x, y are elements of the unit sphere of Z, and $1 \le p < \infty$. Let $\kappa = \sup_{x^* \in \mathcal{F}(x)} |x^*(y)|$. Then, for $t \to 0$,*

$$\|(x, ty)\|_{p,weak} = 1 + \frac{\kappa^p}{p}|t|^p + o(|t|^p).$$

Note that, in the definition of κ, sup can be replaced by max.

Proof. Replacing y by $-y$ if necessary, we assume (see (6.10.2)) that

$$\kappa = \lim_{t \to 0+} \frac{\|x + ty\| - 1}{t},$$

hence

$$\|x + ty\| = 1 + \kappa t + o(t) \text{ for } t \to 0^+.$$

Further, set

$$\kappa' = \lim_{t \to 0^-} \frac{\|x + ty\| - 1}{t},$$

hence $\|x - ty\| = 1 - \kappa' t + o(t)$ for $t \to 0^+$. By our assumption, $|\kappa'| \le \kappa$, hence

$$\max_{\pm} \|x \pm ty\| = 1 + \kappa|t| + o(t). \tag{6.10.3}$$

To complete the proof for $p = 1$, recall that $\big\|(x, ty)\big\|_{1,\mathrm{weak}} = \max_{\pm} \|x \pm ty\|$.

Now consider $p \in (1, \infty)$. To estimate $\|(x, ty)\|_{p,\mathrm{weak}}$ from below, find $x^* \in \mathcal{F}(x)$ so that $x^*(y) = \kappa$ (this is possible, due to the weak* compactness of $\mathcal{F}(x)$). Then

$$\|(x, ty)\|_{p,\mathrm{weak}} \ge \big(|x^*(x)|^p + |tx^*(y)|^p\big)^{1/p} = \big(1 + \kappa^p |t|^p\big)^{1/p}.$$

Taylor expansion gives

$$\big(1 + \kappa^p |t|^p\big)^{1/p} = 1 + \frac{\kappa^p}{p}|t|^p + o(|t|^p).$$

The rest of the proof is devoted to estimating

$$\|(x, ty)\|_{p,\mathrm{weak}} = \max\big\{\|\alpha x + \beta t y\| : |\alpha|^q + |\beta|^q \le 1\big\}$$

from above (here $q = p/(p-1)$, so $1/p + 1/q = 1$). First, we show that, for any $\varepsilon > 0$, there exists $t_0 > 0$ so that

$$\|\alpha x + \beta t y\| \le 1 + \frac{\kappa^p}{p}|t|^p + \varepsilon |t|^p \tag{6.10.4}$$

whenever $\alpha \geq 0$, $\alpha^q + |\beta|^q = 1$, and $|t| \leq t_0$.

The case of $\alpha \leq 1/2$ is easy: for $|t| \leq 1/2$, $\|\alpha x + \beta t y\| \leq \alpha + |t| \leq 1$. The remainder of the proof deals with $\alpha > 1/2$. Then $\alpha = (1 - |\beta|^q)^{1/q}$; by (6.10.3),

$$\begin{aligned}\|\alpha x + \beta t y\| &= \alpha \left\| x + \frac{\beta}{\alpha} t y \right\| \\ &\leq (1 - |\beta|^q)^{1/q} + |\beta||t|\kappa + \alpha\phi\Big(\frac{\beta}{\alpha} t\Big),\end{aligned} \tag{6.10.5}$$

where $\phi(s) = o(s)$ near 0. To analyze the supremum of the above expression, we show the existence of $B > 0$ (depending solely on p) so that

$$(1 - |\beta|^q)^{1/q} \leq 1 - 2|\beta||t| \;\text{ for }\; |\beta| \geq B|t|^{1/(q-1)}. \tag{6.10.6}$$

Taking this inequality for granted, combine (6.10.5) with (6.10.6): for $|\beta| \geq B|t|^{1/(q-1)}$ and $\alpha \geq 1/2$,

$$\|\alpha x + \beta t y\| \leq 1 - |\beta t| + \alpha\phi\Big(\frac{\beta}{\alpha} t\Big).$$

Find s_0 so that $|\phi(s)| \leq |s|/2$ whenever $|s| \leq s_0$. Then for $|t| \leq s_0/2$, we have

$$\Big|\phi\Big(\frac{\beta}{\alpha} t\Big)\Big| \leq \frac{1}{2}\Big|\frac{\beta}{\alpha} t\Big| \leq |\beta||t|,$$

and therefore, for such t,

$$\max\big\{\|\alpha x + \beta t y\| : |\alpha|^q + |\beta|^q \leq 1, |\beta| \geq B|t|^{1/(q-1)}\big\} \leq 1. \tag{6.10.7}$$

Finally, for $\varepsilon > 0$, find $s_1 > 0$ so that $|\phi(s)| \leq B^{-1}\varepsilon|s|$ for $|s| \leq s_1$. Then, for $|t| \leq s_1/2$ and $|\beta| \leq B|t|^{1/(q-1)}$,

$$\alpha\Big|\phi\Big(\frac{\beta}{\alpha} t\Big)\Big| \leq \alpha B^{-1}\varepsilon\Big|\frac{\beta}{\alpha} t\Big| \leq B^{-1}\varepsilon\beta|t| \leq \varepsilon|t|^p$$

(since $p = 1 + 1/(q-1)$). Therefore, for such t,

$$\begin{aligned}&\max\big\{\|\alpha x + \beta t y\| : |\alpha|^q + |\beta|^q \leq 1, |\beta| \leq B|t|^{1/(q-1)}\big\} \\ &\leq \max\big\{(1 - |\beta|^q)^{1/q} + \beta|t|\kappa : \beta \in [-1, 1]\big\} + \varepsilon|t|^p.\end{aligned}$$

By Hölder's Inequality,

$$\begin{aligned}&\max\big\{(1 - |\beta|^q)^{1/q} + \beta|t|\kappa : \beta \in [-1, 1]\big\} \\ &= \max\big\{\alpha \cdot 1 + \beta \cdot \kappa|t| : |\alpha|^q + |\beta|^q \leq 1\big\} \\ &= \|(1, \kappa|t|)\|_p = \big(1 + (\kappa|t|)^p\big)^{1/p} \leq 1 + \frac{\kappa^p|t|^p}{p},\end{aligned}$$

hence

$$
\begin{aligned}
&\max\left\{\|\alpha x+\beta ty\| : |\alpha|^q+|\beta|^q\le 1, |\beta|\le B|t|^{1/(q-1)}\right\} \\
&\le 1+\frac{\kappa^p}{p}|t|^p+\varepsilon|t|^p.
\end{aligned}
\tag{6.10.8}
$$

Together, (6.10.7) and (6.10.8) establish (6.10.4), with $t_0=\min\{s_0,s_1\}/2$.

It remains to establish (6.10.6). For convenience we shall only deal with non-negative values of β and t. That is, we have to show that

$$(1-2\beta t)^q\ge 1-\beta^q$$

for $\beta\ge Bt^{1/(q-1)}$. By Bernoulli's Inequality,

$$(1-2\beta t)^q\ge 1-2q\beta t,$$

hence it suffices to select B to guarantee that $1-2q\beta t\ge 1-\beta^q$ holds for $\beta\ge Bt^{1/(q-1)}$. Clearly $B=(2q)^{1/(q-1)}$ works. □

Proof of Theorem 6.10.18. Following [178], we define the following semi-inner product on E^*: for $x^*,y^*\in E^*$,

$$[y^*,x^*]=\begin{cases}0 & \text{if } x^*=0,\\ f_{x^*}(y^*) & \text{if } x^*\ne 0,\end{cases}$$

where $f_{x^*}\in E^{**}$ is the unique support functional at x^* – that is, $\|f_{x^*}\|=\|x^*\|=\sqrt{f_{x^*}(x^*)}$. A semi-inner product on F^* is defined in a similar fashion.

By Proposition 6.10.1, T is implemented by a surjective positively homogeneous map $\Phi_T:F^*\to E^*$, weak* to weak* continuous on bounded sets, which preserves the (p,weak)-norms of tuples; Φ_T^{-1} also has all these properties. By Lemma 6.10.19, Φ_T preserves absolute value of the semi-inner product defined above. By [178], there exist a linear surjective isometry $V:F^*\to E^*$ and a function $\sigma:F^*\to\{-1,1\}$ so that $\Phi_T f^*=\sigma(f^*)Vf^*$ for any $f^*\in F^*$. Due to the positive homogeneity of Φ_T, σ is constant on rays – that is, $\sigma(tf^*)=\sigma(f^*)$ for any $f^*\ne 0$, and $t>0$.

We claim that σ is a constant on the sphere of F^*. Indeed, otherwise, up to a sign change, we can assume that there exists a sequence (f_k^*) on the unit sphere of F^*, converging to f^* in norm, so that $\sigma(f_k^*)=1$ for any k, and $\sigma(f^*)=-1$ (we make use of the connectedness of the unit sphere). Then $(\Phi_T f_k^*)$ converges in norm, and hence also weak*, to $V f^*$. On the other hand, $\Phi_T f_k^* \to \Phi_T f^* = -V f^*$ weak*, which is a contradiction.

By changing sign if necessary, we can assume $\sigma = 1$ everywhere, hence $\Phi_T f^* = V f^*$ for any $f^* \in F^*$. The linear isometry V, and its inverse, are weak* to weak* continuous on bounded sets. It remains to show that V is an adjoint operator – that is, $V = U^*$, with some $U \in B(E, F)$ (such a U is automatically a surjective isometry). To this end, consider $V^* : E^{**} \to F^{**}$. By [111, Corollary 4.46], $e^{**} \in E^{**}$ comes from $\kappa_E(E)$ (where κ_E denotes the canonical embedding into the bidual) if and only if $\ker e^{**} \cap B_{E^*}$ is weak* closed; the same is true regarding $f^{**} \in F^{**}$. As V is a surjective isometry, we have

$$\ker(V^* e^{**}) \cap B_{F^*} = V^{-1}(\ker e^{**} \cap B_{E^*}).$$

Since V is weak* to weak* continuous on bounded sets, it follows that $\ker(V^* e^{**}) \cap B_{F^*}$ is weak* closed whenever $\ker e^{**} \cap B_{E^*}$ is. In other words, V^* maps $\kappa_E(E)$ into $\kappa_F(F)$. Consequently, $V = U^*$, where $U = \kappa_F^{-1} V^* \kappa_E \in B(E, F)$. □

The smoothness assumption is essential for the preceding proof. Without smoothness, we can obtain some partial results.

Proposition 6.10.20. *Suppose* $1 \le p < \infty$*, and* $\mathrm{FBL}^{(p)}[E]$ *is lattice isometric to* $\mathrm{FBL}^{(p)}[F]$*.*

(i) If E^* *is strictly convex, then so is* F^**.*

(ii) If both E *and* F *are reflexive, and* E^* *is smooth, then* F^* *is smooth as well.*

For the proof, we need a particular case of Lemma 6.10.19:

Corollary 6.10.21. *Suppose* $p \in [1, \infty)$*, and* $z, y \in Z$ *with* $\|z\| = 1 = \|y\|$*. Then* $\max_\pm \|z \pm y\| = 2$ *if and only if*

$$\lim_{t \to 0} p t^{-p} \Big(\big\| (z, ty) \big\|_{p,weak} - 1 \Big) = 1. \tag{6.10.9}$$

Proof. If $\|z + y\| = 2$ or $\|z - y\| = 2$, find $z^* \in \mathcal{F}(z)$ so that $|z^*(y)| = 1$. Apply Lemma 6.10.19. Conversely, if (6.10.9) holds, then there exists $z^* \in \mathcal{F}(z)$ with $|z^*(y)| = 1$. Then $\max_\pm \|z \pm y\| = 2$. □

Proof of Proposition 6.10.20(1). Suppose, for the sake of contradiction, that E^* is strictly convex, but F^* is not. Find norm one $y_0^*, y_1^* \in F^*$ so that $y_0^* \neq y_1^*$, and $\|y_1^* + y_0^*\| = 2$. For

$s \in (0,1)$ let $y_s^* = (1-s)y_0^* + sy_1^*$. It is easy to see that $\|y_s^*\| = 1$, and $\|y_0^* + y_s^*\| = 2$, for any $s \in (0,1)$. Consequently, by Corollary 6.10.21,

$$\lim_{t\to 0} pt^{-p}\Big(\big\|(y_0^*, ty_s^*)\big\|_{p,\mathrm{weak}} - 1\Big) = 1.$$

Find a positively homogeneous map $\Phi : F^* \to E^*$, weak* to weak* continuous on bounded sets, which implements a surjective lattice isometry $\mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$. Let $x_s^* = \Phi y_s^*$. Then for any $s \in (0,1)$, $\|x_s^*\| = 1$. Moreover, Φ preserves (p, weak)-norms of tuples, so

$$\lim_{t\to 0} pt^{-p}\Big(\big\|(x_0^*, tx_s^*)\big\|_{p,\mathrm{weak}} - 1\Big) = 1.$$

Consequently, by Corollary 6.10.21, $\max_\pm \|x_0^* \pm x_s^*\| = 2$. By the strict convexity of E^*, $x_s^* \in \{x_0^*, -x_0^*\}$ for any s. However, all x_s^*'s must be distinct, which gives a contradiction. □

The following topological result is likely known to experts.

Lemma 6.10.22. *Consider a Banach space Z, equipped with the weak topology. Let G be a closed subspace of Z. Then $Z\backslash G$ is path connected if* $\dim Z/G \geq 2$*, and is disconnected if* $\dim Z/G = 1$.

Proof. If $\dim Z/G = 1$, find $z^* \in Z^*$ so that $G = \ker z^*$. Then $Z\backslash G$ is a union of two open sets – $\{z \in Z : z^*(z) > 0\}$ and $\{z \in Z : z^*(z) < 0\}$ – hence disconnected.

Now suppose $\dim Z/G > 1$. For $z_0, z_1 \in Z\backslash G$, we need to find a path connecting these two points. By replacing Z by its subspace, we can and do assume that $\dim Z/G = 2$. Represent Z as $G \oplus H$, with $\dim H = 2$. Represent $z_i = g_i + h_i$, with $g_i \in G$ and $h_i \in H\backslash\{0\}$. Find norm-continuous functions $g : [0,1] \to G$ and $h : [0,1] \to H\backslash\{0\}$ so that $g(0) = g_0$, $g(1) = g_1$, $h(0) = h_0$, and $h(1) = h_1$. Then $t \mapsto g(t) + h(t)$ is the desired path. □

Proof of Proposition 6.10.20(2). Suppose, for the sake of contradiction, that E^* is smooth, but F^* is not. Find a positively homogeneous map $\Phi : F^* \to E^*$, weak* to weak* continuous on bounded sets, which implements a surjective lattice isometry $\mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$. Find a non-smooth point f^* on the unit sphere of F^*. Let $e^* = \Phi f^*$. Let $e = f_{e^*}$. It follows that $x^* \in E^*$ satisfies

$$\lim_{t\to 0} \frac{1}{t^p}\big(\|(e^*, tx^*)\|_{p,\mathrm{weak}} - 1\big) = 0$$

if and only if $x^* \in \ker e =: A$.

Now let B be the set of all $y^* \in F^*$ for which

$$\lim_{t\to 0} \frac{1}{t^p}\big(\|(f^*, ty^*)\|_{p,\text{weak}} - 1\big) = 0.$$

Then $B = \cap\big\{\ker f : f \in \mathcal{F}(f^*)\big\}$. Further, $A = \Phi(B)$, and $E^*\backslash A = \Phi(F^*\backslash B)$. By Lemma 6.10.22, $F^*\backslash B$ is connected (since $\dim F^*/B \geq 2$), and $E^*\backslash A$ is not (since $\dim E^*/A = 1$). However, a disconnected set cannot be a continuous image of a connected set. $\square$

Isomorphism between $\mathrm{FBL}^{(\infty)}$ lattices

It turns out that $\mathrm{FBL}^{(\infty)}[X]$ and $\mathrm{FBL}^{(\infty)}[Y]$ may be lattice isomorphic, or even isometric, even when X and Y are non-isomorphic. For motivation, we recall a result from [185]:

Proposition 6.10.23. *Let E be a Banach space. Then $C(B_{E^*})$ is the free unital AM-space over E. More specifically, for any compact Hausdorff space K and any norm one operator $T : E \to C(K)$ there exists a unique unital lattice homomorphism $\widehat{T} : C(B_{E^*}) \to C(K)$ such that $\widehat{T} \circ \phi_E = T$. Moreover, $\|\widehat{T}\| = 1$.*

From this result we deduce that the free *unital* AM-spaces over E and F are (isometrically) lattice isomorphic if and only if (B_{E^*}, w^*) and (B_{F^*}, w^*) are homeomorphic. In particular, since the dual ball of any separable infinite dimensional Banach space is weak* homeomorphic to the Hilbert cube $[0,1]^\omega$ by Keller's Theorem (see [110, Section 12.3]), the free unital AM-space cannot distinguish between separable Banach spaces. We now prove a similar result for $\mathrm{FBL}^{(\infty)}$. This has the added difficulty that one needs to build positive homogeneity into the homeomorphism.

We say that (Z_i) is a *finite dimensional decomposition* (*FDD* for short) in a Banach space Z if the spaces $Z_i \subseteq Z$ are finite dimensional, $Z = \overline{\operatorname{span}}[Z_i : i \in \mathbb{N}]$, and there exist projections P_i from Z onto Z_i so that $P_iP_j = 0$ whenever $i \neq j$, and $\sup_n \|\overline{P}_n\| < \infty$ (here $\overline{P}_n = P_1 + \ldots + P_n$), and, for any $z \in Z$, $\overline{P}_n z \to z$ in norm (equivalently, weakly [243]). Then $\overline{P}_n^*$ converges to I_{Z^*} in the point-weak* topology. We say that an FDD is *monotone* if $\overline{P}_n$ is contractive for every n. A classical renorming procedure makes an FDD monotone.

Theorem 6.10.24. *Suppose a Banach space X has a monotone FDD. Then $\mathrm{FBL}^{(\infty)}[X]$ is lattice isometric to $\mathrm{FBL}^{(\infty)}[c_0]$.*

Remark 6.10.25. This result contrasts sharply with those of Section 6.10. For instance, whereas the above yields that $\mathrm{FBL}^{(\infty)}[c_0]$ and $\mathrm{FBL}^{(\infty)}[\ell_1]$ are lattice isometric, combining

Proposition 6.10.11 with Proposition 6.10.12, we conclude that $\mathrm{FBL}^{(p)}[c_0]$ and $\mathrm{FBL}^{(p)}[\ell_1]$ are not lattice isomorphic for any $p \in [1, \infty)$. In fact, we know no examples of non-isomorphic Banach spaces E, F for which $\mathrm{FBL}^{(p)}[E]$ and $\mathrm{FBL}^{(p)}[F]$ $(1 \leq p < \infty)$ are lattice isomorphic.

The following easy observation will be used throughout the proof of Theorem 6.10.24.

Lemma 6.10.26. *Suppose X is a Banach space with an FDD implemented by projections (P_n). Then a norm bounded net $(x_\alpha^*) \subseteq X^*$ weak* converges if and only if the net $(\overline{P}_n^* x_\alpha^*)_\alpha$ is norm convergent for every n. Further, (x_α^*) weak* converges to $x^* \in X^*$ if and only if $\overline{P}_n^* x_\alpha^* \to \overline{P}_n^* x^*$ in norm for any n.*

Proof of Theorem 6.10.24. Let (P_i) be the FDD projections in X, and let $E_i = \mathrm{Range}\, P_i^* \hookrightarrow X^*$. Then $\overline{P}_n^* = P_1^* + \ldots + P_n^*$ is a projection onto $\overline{E}_n = E_1 \oplus \ldots \oplus E_n$. Let $F_i = \ell_1^{n_i}$, with $n_i = \dim E_i$. Find a bi-continuous bijection $\rho_i : E_i \to F_i$, so that $\|\rho_i y^*\| = \|y^*\|$ and $\rho_i(ty^*) = t\rho_i(y^*)$ for any y^* and $t \in \mathbb{R}$. Finally, we identify c_0 with $Y = (\sum_i F_i^*)_{c_0}$. Denote the coordinate projections from Y onto F_i^* by Q_i, and let $\overline{Q}_n = Q_1 + \ldots + Q_n$. Then $\overline{Q}_n^*$ is a projection from Y^* onto $\overline{F}_n = F_1 \oplus \ldots \oplus F_n$. Note also that $Y^* = (\sum_i F_i)_1 \sim \ell_1$.

We recursively define continuous norm-preserving positively homogeneous bijections $\Psi_n : \overline{E}_n \to \overline{F}_n$ (henceforth we say that a map Ψ is *norm-preserving* if $\|\Psi z\| = \|z\|$ on the domain of Ψ).

To begin, let $\Psi_1 x^* = \rho_1 x^*$. Now suppose Ψ_{n-1} with the desired properties has been defined; let us describe Ψ_n. Any $x^* \in \overline{E}_n$ can be written, in a unique way, as $x^* = x_0^* + x_n^*$, where $x_0^* \in \overline{E}_{n-1}$, and $x_n^* \in E_n$. If $x_0^* = 0$, let $\Psi_n x^* = \rho_n x_n^*$, while if $x_n^* = 0$, let $\Psi_n x^* = \Psi_{n-1} x^*$. Otherwise, let

$$\Psi_n(x_0^* + x_n^*) = \kappa_n\Big(\frac{\|x_n^*\|}{\|x_0^* + x_n^*\|}\Big)\Psi_{n-1}(x_0^*) + t\rho_n x_n^*, \tag{6.10.10}$$

where $\kappa_n(s) = 1 - 4^{-n}\big(1 - e^{-s/2}\big)$, and

$$t = t(x_0^*, x_n^*) = \frac{\|x_0^* + x_n^*\|}{\|x_n^*\|} - \kappa_n\Big(\frac{\|x_n^*\|}{\|x_0^* + x_n^*\|}\Big)\frac{\|x_0^*\|}{\|x_n^*\|}. \tag{6.10.11}$$

Note that $\|x_0^* + x_n^*\| \geq \|x_0^*\|$ (due to $\overline{P}_{n-1}$ being contractive), hence $t > 0$. In fact, t was selected to guarantee that

$$\|\Psi_n(x_0^* + x_n^*)\| = \kappa_n\Big(\frac{\|x_n^*\|}{\|x_0^* + x_n^*\|}\Big)\|x_0^*\| + t\|x_n^*\| = \|x_0^* + x_n^*\|,$$

that is, to make Ψ_n norm-preserving. Finally, observe that t is a continuous function of (x_0^*, x_n^*), provided $x_n^* \neq 0$.

We next note that $1 \geq \kappa_n(s) > 1 - 4^{-n}$ for any $s \geq 0$, and so,

$$\|\overline{Q}_{n-1}^*\big(\Psi_{n-1}x_0^* - \Psi_n(x_0^* + x_n^*)\big)\| \leq 4^{-n}\|x_0^*\|. \tag{6.10.12}$$

Clearly, Ψ_n maps $\overline{E}_{n-1} + E_n$ to $\overline{F}_{n-1} + F_n$; also, it is positively homogeneous. We next establish that Ψ_n is continuous. Continuity at (x_0^*, x_n^*) when both x_0^* and x_n^* are different from 0 is straightforward, and that at $(0,0)$ follows from Ψ_n being norm-preserving.

Now suppose $x_n^* = 0$, while $\|x_0^*\| \neq 0$. The map

$$(z_0^*, z_n^*) \mapsto \overline{Q}_{n-1}^* \Psi_n(z_0^* + z_n^*) = \kappa_n\Big(\frac{\|z_n^*\|}{\|z_0^* + z_n^*\|}\Big)\Psi_{n-1}(z_0^*)$$

is clearly continuous at $(x_0^*, 0)$. In addition, (6.10.11) implies that

$$t(z_0^*, z_n^*)\|z_n^*\| = \|z_0^* + z_n^*\| - \kappa_n\Big(\frac{\|z_n^*\|}{\|z_0^* + z_n^*\|}\Big)\|z_0^*\| \underset{z_n^* \to 0}{\longrightarrow} 0$$

uniformly over $\{z_0^* \in \overline{E}_{n-1} : \|z_0^*\| > c\}$ (for any $c > 0$), or equivalently,

$$\big\|Q_n^* \Psi_n(z_0^* + z_n^*)\big\| = \|t(z_0^*, z_n^*)\rho_n z_n^*\| = \|t(z_0^*, z_n^*) z_n^*\| \to 0$$

uniformly over the same set. Thus, the continuity of Ψ_n at $(x_0^*, 0)$ is verified.

Conversely, suppose $x_0^* = 0$, and $x_n^* \neq 0$. By the continuity of Ψ_{n-1} at 0, the map

$$(z_0^*, z_n^*) \mapsto \overline{Q}_{n-1}^* \Psi_n(z_0^* + z_n^*) = \kappa_n\Big(\frac{\|z_n^*\|}{\|z_0^* + z_n^*\|}\Big)\Psi_{n-1}(z_0^*)$$

is continuous at $(0, x_n^*)$. Also, the continuity of t and ρ_n implies that of

$$(z_0^*, z_n^*) \mapsto Q_n^* \Psi_n(z_0^* + z_n^*) = t(z_0^*, z_n^*)\rho_n z_n^*.$$

The continuity of Ψ_n at $(0, x_n^*)$ follows.

To establish injectivity, we suppose that $\Psi_n(x_0^* + x_n^*) = \Psi_n(z_0^* + z_n^*)$ for some $x_0^*, z_0^* \in \overline{E}_{n-1}$ and $x_n^*, z_n^* \in E_n$, and show that $x_0^* = z_0^*$, $x_n^* = z_n^*$. As Ψ_n is norm-preserving, we can assume, by scaling, that $\|x_0^* + x_n^*\| = 1 = \|z_0^* + z_n^*\|$.

Applying $\overline{Q}_{n-1}^*$, we see that $\kappa_n(\|x_n^*\|)\Psi_{n-1}(x_0^*) = \kappa_n(\|z_n^*\|)\Psi_{n-1}(z_0^*)$. By the induction hypothesis, Ψ_{n-1} is continuous and positively homogeneous, hence there exists $\alpha > 0$ so that $z_0^* = \alpha x_0^*$. In this case, $\kappa_n(\|z_n^*\|) = \alpha^{-1}\kappa_n(\|x_n^*\|)$.

Likewise, applying Q_n^*, we conclude that there exists $\beta > 0$ so that $z_n^* = \beta x_n^*$. Therefore, we have $\alpha\kappa_n(\beta\|x_n^*\|) = \kappa_n(\|x_n^*\|)$.

If $\alpha = 1$, then, due to κ_n being strictly decreasing, we conclude that $\beta = 1$ as well, hence $x_0^* + x_n^* = z_0^* + z_n^*$. To establish the injectivity of Ψ_n, we therefore should rule out the possibility of $\alpha \neq 1$. By relabeling, it suffices to show that α cannot be less than 1. In fact, we shall prove the following fact:

$$\text{if } \alpha \in (0,1),\ s > 0,\ \alpha\kappa_n(\beta s) = \kappa_n(s),\ \text{ then } \beta < \alpha. \tag{6.10.13}$$

Once this is shown, we have that

$$\|\alpha x_0^* + \beta x_n^*\| = \alpha\big\|x_0^* + \frac{\beta}{\alpha}x_n^*\big\| < \big\|x_0^* + \frac{\beta}{\alpha}x_n^*\big\|. \tag{6.10.14}$$

Further, as $\overline{P}_{n-1}^*$ is contractive, we have $\|x_0^*\| \leq \|x_0^* + x_n^*\| = 1$, and therefore, by the convexity of the norm,

$$\|x_0^* + \frac{\beta}{\alpha}x_n^*\| \leq \frac{\beta}{\alpha}\|x_0^* + x_n^*\| + \Big(1 - \frac{\beta}{\alpha}\Big)\|x_0^*\| \leq 1.$$

Together with (6.10.14), this gives $\|z_0^* + z_n^*\| = \|\alpha x_0^* + \beta x_n^*\| < 1$, producing the desired contradiction.

Keeping in mind that κ_n is strictly decreasing, we establish (6.10.13) by showing that the map $\alpha \mapsto \alpha\kappa_n(\alpha s)$ is strictly increasing on $(0,\infty)$. Multiplying by s, and using substitution $\alpha s = t$, we are reduced to showing that $t \mapsto t\kappa_n(t)$ is strictly increasing. Differentiating the function on the right, we obtain

$$\big[t\kappa_n(t)\big]' = \kappa_n(t) + t\kappa_n'(t) = \kappa_n(t) - 4^{-n}\frac{te^{-t/2}}{2} > 0,$$

since $\kappa_n(t) > 1 - 4^{-n}$. Thus, $t \mapsto t\kappa_n(t)$ is indeed increasing.

Finally we show that Ψ_n is surjective, that is, for any norm one $y_0^* \in \overline{F}_{n-1}$, $y_n^* \in F_n$, and $a, b \in [0, \infty)$, $ay_0^* + by_n^*$ belongs to the range of Ψ_n. By scaling, we assume $a + b = 1$ (in other words, $\|ay_0^* + by_n^*\| = 1$). If either a or b equals 0, then $ay_0^* + by_n^*$ lies in the range of Ψ_n, due to the surjectivity of ρ_n, respectively Ψ_{n-1}. Henceforth, we consider non-zero a and b.

Find $x_0^* \in \overline{E}_{n-1}$ and $x_n^* \in E_n$ so that $\Psi_{n-1}x_0^* = y_0^*$ and $\rho_n x_n^* = y_n^*$ (then $\|x_0^*\| = 1 = \|x_n^*\|$). Our goal is to find $\alpha, \beta > 0$ so that

$$\Psi_n(\alpha x_0^* + \beta x_n^*) = ay_0^* + by_n^*. \tag{6.10.15}$$

To this end, note first that the function

$$\beta \mapsto \Big\| \frac{a}{\kappa_n(\beta)} x_0^* + \beta x_n^* \Big\|$$

is continuous on $[0, \infty)$, equals $a < 1$ when $\beta = 0$, and tends to ∞ as β does the same. Find β for which this function equals 1, and let $\alpha = a/\kappa_n(\beta)$. Then $\|\alpha x_0^* + \beta x_n^*\| = 1$, and $\alpha\kappa_n(\|\beta x_n^*\|/\|\alpha x_0^* + \beta x_n^*\|) = a$. Therefore, $\kappa_n(\|\beta x_n^*\|/\|\alpha x_0^* + \beta x_n^*\|)\Psi_{n-1}(\alpha x_0^*) = ay_0^*$. From the definition,

$$\Psi_n(\alpha x_0^* + \beta x_n^*) = \kappa_n\Big(\frac{\|\beta x_n^*\|}{\|\alpha x_0^* + \beta x_n^*\|}\Big)\Psi_{n-1}(\alpha x_0^*) + t\beta\rho_n x_n^*;$$

we know that $\rho_n x_n^* = y_n^*$, and, by norm preservation, $t\beta = b$. Thus, α and β have the required properties.

For $x^* \in X^*$, we define $\Psi x^* = \text{weak}^* - \lim_n \Psi_n \overline{P}_n^* x^*$. We shall show that Ψ is well defined (the limit above indeed exists, even in the norm topology), is positively homogeneous, norm preserving, and weak* continuous. To begin, note that $\|x^*\| = \lim_n \|\overline{P}_n^* x^*\|$ holds for any $x^* \in X^*$. Indeed, $\|x^*\| \geq \|\overline{P}_n^* x^*\|$ for any n, by monotonicity. On the other hand, $x^* = \text{weak}^* - \lim_n \overline{P}_n^* x^*$, hence $\|x^*\| \leq \liminf \|\overline{P}_n^* x^*\|$. Recall that, for $j \geq n$, we have $\overline{Q}_n^* \overline{Q}_j^* = \overline{Q}_n^*$. As $\overline{Q}_n^*$ is contractive, (6.10.12) implies that

$$\begin{aligned} &\|\overline{Q}_n^* \Psi_j \overline{P}_j^* x^* - \overline{Q}_n^* \Psi_{j+1} \overline{P}_{j+1}^* x^*\| \\ &\leq \|\Psi_j \overline{P}_j^* x^* - \overline{Q}_j^* \Psi_{j+1} \overline{P}_{j+1}^* x^*\| \leq 4^{-j}\|x^*\|. \end{aligned} \tag{6.10.16}$$

Consequently, for every n, the sequence $\big(\overline{Q}_n^* \Psi_j \overline{P}_j^* x^*\big)_j$ is Cauchy, hence convergent (in norm). Note that $\|\Psi_j \overline{P}_j^* x^*\| \nearrow \|x^*\|$, hence, in particular, the sequence $(\Psi_j \overline{P}_j^* x^*)_j$ is norm bounded. Therefore, Lemma 6.10.26 (applied with the sequence $(\Psi_j \overline{P}_j^* x^*)_j$ instead of the net $(x_\alpha^*)_\alpha$, and with projections Q_n instead of P_n) shows that the weak* limit of $(\Psi_j \overline{P}_j^* x^*)$ exists (and therefore, Ψ is well defined), with $\|\Psi x^*\| \leq \|x^*\|$.

On the other hand, for any n, $\|\Psi x^*\| \geq \limsup_m \|\overline{Q}_n^* \Psi_m \overline{P}_m^* x^*\|$. For $m > n$, we can use (6.10.16) to write a telescopic sum:

$$\begin{aligned}\|\Psi_n \overline{P}_n^* x^* - \overline{Q}_n^* \Psi_m \overline{P}_m^* x^*\| &\leq \sum_{j=n}^{m-1} \|\overline{Q}_n^* \Psi_j \overline{P}_j^* x^* - \overline{Q}_n^* \Psi_{j+1} \overline{P}_{j+1}^* x^*\| \\ &\leq \sum_{j=n}^{m-1} 4^{-j} \|x^*\| \leq 2 \cdot 4^{-n} \|x^*\|,\end{aligned} \tag{6.10.17}$$

hence $\|\overline{Q}_n^* \Psi_m \overline{P}_m^* x^*\| \geq \|\Psi_n \overline{P}_n^* x^*\| - 2 \cdot 4^{-n} \|x^*\|$, and therefore,

$$\begin{aligned}\|\Psi x^*\| &\geq \limsup_n \left(\|\Psi_n \overline{P}_n^* x^*\| - 2 \cdot 4^{-n} \|x^*\|\right) \\ &= \limsup_n \|\Psi_n \overline{P}_n^* x^*\| = \|x^*\|.\end{aligned}$$

Thus, Ψ is norm-preserving. As Ψ_n is positively homogeneous for any n, so is Ψ.

We observe that the sequence $(\Psi_n \overline{P}_n^* x^*)$ is not merely weak*-convergent, but also Cauchy, hence convergent in norm. To this end, consider $m > n$, and recall several relevant facts.

(i) (6.10.17) shows that $\|\Psi_n \overline{P}_n^* x^* - \overline{Q}_n^* \Psi_m \overline{P}_m^* x^*\| \leq 2 \cdot 4^{-n} \|x^*\|$.

(ii) There exists $t \in [0,1]$ so that $t \Psi_n \overline{P}_n^* x^* = \overline{Q}_n^* \Psi_m \overline{P}_m^* x^*$; consequently, $\|\Psi_n \overline{P}_n^* x^* - \overline{Q}_n^* \Psi_m \overline{P}_m^* x^*\| = \|\Psi_n \overline{P}_n^* x^*\| - \|\overline{Q}_n^* \Psi_m \overline{P}_m^* x^*\|$.

(iii) Further,

$$\begin{aligned}&\|(I - \overline{Q}_n^*) \Psi_m \overline{P}_m^* x^*\| = \|\Psi_m \overline{P}_m^* x^*\| - \|\overline{Q}_n^* \Psi_m \overline{P}_m^* x^*\| \\ &= \left(\|\Psi_m \overline{P}_m^* x^*\| - \|\Psi_n \overline{P}_n^* x^*\|\right) + \left(\|\Psi_n \overline{P}_n^* x^*\| - \|\overline{Q}_n^* \Psi_m \overline{P}_m^* x^*\|\right).\end{aligned}$$

(iv) Finally, recall that Ψ_n and Ψ_m are norm-preserving.

In light of the above,

$$\begin{aligned}
&\|\Psi_n \overline{P}_n^* x^* - \Psi_m \overline{P}_m^* x^*\| \\
&= \|\Psi_n \overline{P}_n^* x^* - \overline{Q}_n^* \Psi_m \overline{P}_m^* x^*\| + \|(I - \overline{Q}_n^*)\Psi_m \overline{P}_m^* x^*\| \\
&= 2\|\Psi_n \overline{P}_n^* x^* - \overline{Q}_n^* \Psi_m \overline{P}_m^* x^*\| + \big(\|\Psi_m \overline{P}_m^* x^*\| - \|\Psi_n \overline{P}_n^* x^*\|\big) \\
&\le 4 \cdot 4^{-n}\|x^*\| + \big(\|\overline{P}_m^* x^*\| - \|\overline{P}_n^* x^*\|\big).
\end{aligned}$$

As $\|\overline{P}_n^* x^*\| \nearrow \|x^*\|$, we conclude that $(\Psi_n \overline{P}_n^* x^*)$ is a Cauchy sequence.

Next we establish that Ψ is weak* continuous on bounded sets. Suppose a net $(x_\alpha^*) \subseteq B_{X^*}$ converges weak* to x^* (hence $x^* \in B_{X^*}$ as well). We shall show that (Ψx_α^*) weak* converges to Ψx^*. In light of Lemma 6.10.26, we have to prove that, for any $n \in \mathbb{N}$ and $\varepsilon > 0$, the inequality $\|\overline{Q}_n^* \Psi x_\alpha^* - \overline{Q}_n^* \Psi x^*\| < \varepsilon$ holds for α large enough.

Fix $m \ge n$ so that $4^{1-m} < \varepsilon/2$. As $\overline{Q}_n^* \Psi x^* = \lim_j \overline{Q}_n^* \Psi_j \overline{P}_j^* x^*$, (6.10.16) implies

$$\|\overline{Q}_n^* \Psi x^* - \overline{Q}_n^* \Psi_m \overline{P}_m^* x^*\| \le \sum_{j=m}^{\infty} \|\overline{Q}_n^* \Psi_j \overline{P}_j^* x^* - \overline{Q}_n^* \Psi_{j+1} \overline{P}_{j+1}^* x^*\| < 2 \cdot 4^{-m},$$

and likewise, $\|\overline{Q}_n^* \Psi x_\alpha^* - \overline{Q}_n^* \Psi_m \overline{P}_m^* x_\alpha^*\| < 2 \cdot 4^{-m}$ for any α.

For m as above, we have $\lim_\alpha \overline{P}_m^* x_\alpha^* = \overline{P}_m^* x^*$ (as $\overline{P}_m$ has finite rank, weak* and norm convergence coincide). By the continuity of Ψ_m, the equality $\lim_\alpha \Psi_m \overline{P}_m^* x_\alpha^* = \Psi_m \overline{P}_m^* x^*$ holds as well. In particular, $\|\Psi_m \overline{P}_m^* x_\alpha^* - \Psi_m \overline{P}_m^* x^*\| < \varepsilon/2$ for α large enough. For such α,

$$\begin{aligned}
&\|\overline{Q}_n^* \Psi x_\alpha^* - \overline{Q}_n^* \Psi x^*\| \le \|\Psi_m \overline{P}_m^* x_\alpha^* - \Psi_m \overline{P}_m^* x^*\| + \\
&\|\overline{Q}_n^* \Psi x^* - \overline{Q}_n^* \Psi_m \overline{P}_m^* x^*\| + \|\overline{Q}_n^* \Psi x_\alpha^* - \overline{Q}_n^* \Psi_m \overline{P}_m^* x_\alpha^*\| \\
&< \frac{\varepsilon}{2} + 2 \cdot 2 \cdot 4^{-m} < \varepsilon,
\end{aligned}$$

as desired.

Next we define a map Φ, and prove that it is the inverse of Ψ, possessing the desired properties. First let $\Phi_n = \Psi_n^{-1} : \overline{F}_n \to \overline{E}_n$. This map is clearly positively homogeneous and norm-preserving; it is also continuous, by Inverse Function Theorem. We shall show that, for any $y^* \in \ell_1$, the sequence $(\Phi_n \overline{Q}_n^* y^*)$ is weak* convergent. Once this is done, we let $\Phi y^* = \text{weak}^* - \lim \Phi_n \overline{Q}_n^* y^*$.

First fix n, $y_0^* \in \overline{F}_{n-1}$, and $y_n^* \in F_n$. Let $x_0^* = \overline{P}_{n-1}^* \Phi_n(y_0^* + y_n^*)$ and $x_n^* = P_n^* \Phi_n(y_0^* + y_n^*)$. By (6.10.10),

$$y_0^* = \overline{Q}_{n-1}^*(y_0^* + y_n^*) = \kappa_n\Big(\frac{\|x_n^*\|}{\|x_0^* + x_n^*\|}\Big)\Psi_{n-1}x_0^*.$$

Apply Φ_{n-1} to both sides to obtain

$$\begin{aligned}\Phi_{n-1}\overline{Q}_{n-1}^*(y_0^* + y_n^*) &= \kappa_n\Big(\frac{\|x_n^*\|}{\|x_0^* + x_n^*\|}\Big)x_0^* \\ &= \kappa_n\Big(\frac{\|x_n^*\|}{\|x_0^* + x_n^*\|}\Big)\overline{P}_{n-1}^*\Phi_n(y_0^* + y_n^*),\end{aligned}$$

and therefore,

$$\begin{aligned}&\big\|\Phi_{n-1}\overline{Q}_{n-1}^*(y_0^* + y_n^*) - \overline{P}_{n-1}^*\Phi_n(y_0^* + y_n^*)\big\| \\ &= \Big(1 - \kappa_n\Big(\frac{\|x_n^*\|}{\|x_0^* + x_n^*\|}\Big)\Big)\big\|\overline{P}_{n-1}^*\Phi_n(y_0^* + y_n^*)\big\| \\ &\le \Big(1 - \kappa_n\Big(\frac{\|x_n^*\|}{\|x_0^* + x_n^*\|}\Big)\Big)\|y_0^* + y_n^*\| \le 4^{-n}\|y_0^* + y_n^*\|.\end{aligned}$$

Consequently, for $m > n$ and $y^* \in \ell_1$,

$$\|\overline{P}_n^*\Phi_m\overline{Q}_m^*y^* - \Phi_n\overline{Q}_n^*y^*\| \le 2 \cdot 4^{-n}\|y^*\|. \tag{6.10.18}$$

For $n < m < k$ we therefore have:

$$\begin{aligned}&\|\overline{P}_n^*\Phi_k\overline{Q}_k^*y^* - \overline{P}_n^*\Phi_m\overline{Q}_m^*y^*\| = \|\overline{P}_n^*\overline{P}_m^*\Phi_k\overline{Q}_k^*y^* - \overline{P}_n^*\Phi_m\overline{Q}_m^*y^*\| \\ &\le \|\overline{P}_m^*\Phi_k\overline{Q}_k^*y^* - \Phi_m\overline{Q}_m^*y^*\| \le 2 \cdot 4^{-m}\|y^*\|.\end{aligned}$$

Thus, for any n, the sequence $(\overline{P}_n^*\Phi_m\overline{Q}_m^*y^*)_m$ is Cauchy, hence convergent in norm. Additionally, $\|\Phi_m\overline{Q}_m^*y^*\| = \|\overline{Q}_m^*y^*\| \le \|y^*\|$, hence, by Lemma 6.10.26, $(\Phi_m\overline{Q}_m^*y^*)$ has a weak* limit, say x^*. Note that $\|x^*\| = \|y^*\|$. Indeed, $\|x^*\| \le \liminf \|\Phi_m\overline{Q}_m^*y^*\| = \|y^*\|$. On the other hand, $\|x^*\| \ge \limsup_m \|\overline{P}_n^*\Phi_m\overline{Q}_m^*y^*\|$ for any n. Combining (6.10.18) with the fact that $\|y^*\| = \lim_n \|\Phi_n\overline{Q}_n^*y^*\|$, we obtain the opposite inequality.

Thus, the map Φ is well-defined, positively homogeneous, and norm preserving. The weak* continuity of Φ is established in the same manner as that of Ψ.

It remains to show that $\Phi\Psi = I_{X^*}$, and $\Psi\Phi = I_{Y^*}$. We shall only establish the first of these identities, as the second one can be treated similarly. Fix $x^* \in X^*$, and let $y^* = \Psi x^*$.

Further, for $n \in \mathbb{N}$ let $x_n^* = \overline{P}_n^* x^*$ and $y_n^* = \overline{Q}_n^* y^*$. As we observed before, $x^* = \text{weak}^* - \lim x_n^*$, and $y^* = \text{weak}^* - \lim y_n^*$. By the definition of Ψ, y_n^* is a scalar multiple of $\Psi_n x_n^*$, with $\|\Psi_n x_n^* - y_n^*\| \leq 4^{-n} \|x^*\|$. Consequently, $\lim_n \|\Phi_n y_n^* - x_n^*\| = 0$, and so,

$$\Phi y^* = \text{weak}^* - \lim \Phi_n y_n^* = \text{weak}^* - \lim x_n^* = x^*,$$

which gives $\Phi\Psi x^* = x^*$.

To summarize: we have defined an invertible map $\Psi : X^* \to Y^*$ so that Ψ itself, and its inverse, are positively homogeneous, norm preserving, and weak* continuous on bounded sets. Composition with $\Phi = \Psi^{-1}$ induces a lattice homomorphism $T : \mathrm{FBL}^{(\infty)}[X] \to \mathrm{FBL}^{(\infty)}[c_0]$ so that $\Phi = \Phi_T$; then $\Psi = \Phi_{T^{-1}}$. Consequently, $\mathrm{FBL}^{(\infty)}[X]$ and $\mathrm{FBL}^{(\infty)}[c_0]$ are lattice isometric. □

Remark 6.10.27. Note in particular, that the map Ψ given in the proof of Theorem 6.10.24 is positively homogeneous and provides a weak* homeomorphism between rB_{X^*} and rB_{ℓ_1} for every $r > 0$. This provides an improvement to [100] where homeomorphisms between rB_{ℓ_p} and rB_{ℓ_q} were constructed using some heavy machinery from topology.

For brevity, we shall use the notation $\mathcal{U} = \mathrm{FBL}^{(\infty)}[c_0]$. Above, we have shown that $\mathrm{FBL}^{(\infty)}[X]$ is lattice isometric to $\mathcal{U}$ whenever X has a monotone FDD. One can ask whether this lattice isomorphism (or even lattice isometry) holds for any separable X. While we cannot answer this question, below we list some partial results.

For future use, we describe a class of spaces possessing a monotone basis (and, consequently, a monotone FDD). The results below may be known to experts, but we have not been able to find them in the literature.

Proposition 6.10.28. *Any separable $\mathcal{L}_{\infty,1+}$ space has a monotone basis. Therefore, any separable AM-space has a monotone basis.*

Proof. (1) By [222], any separable $\mathcal{L}_{\infty,1+}$ space X can be written as $\overline{\cup_i E_i}$, where $E_1 \subseteq E_2 \subseteq \ldots$, and E_n is isometric to ℓ_∞^n. For each n, there exists a contractive projection R_n from E_n to E_{n-1} (for convenience set $R_1 = 0$). For each i, find a norm one x_i in $\ker R_i$; clearly (x_i) is a monotone basis.

(2) If X is an AM-space, then (see [231, Section 1.b]) X^{**} is an L_∞ space, which is a $\mathcal{L}_{\infty,1+}$ space. Then X is an $\mathcal{L}_{\infty,1+}$ space as well, by Local Reflexivity (see [191]). □

Corollary 6.10.29. *For every separable Banach space X we have:*

(i) If X has an FDD, then $\mathrm{FBL}^{(\infty)}[X]$ is lattice isomorphic to $\mathcal{U}$.

(ii) If X has the Bounded Approximation Property, then $\mathrm{FBL}^{(\infty)}[X]$ is lattice isomorphic to a lattice complemented sublattice of $\mathcal{U}$.

(iii) There exist a linear isometry $J : \mathrm{FBL}^{(\infty)}[X] \to \mathcal{U}$ and a contractive lattice homomorphism $P : \mathcal{U} \to \mathrm{FBL}^{(\infty)}[X]$ so that $PJ = id_{\mathrm{FBL}^{(\infty)}[X]}$.

Proof. (1) is obtained by combining Theorem 6.10.24 with the fact that any Banach space with an FDD can be renormed to make this FDD monotone. (2) follows from (1), since any separable Banach space with the BAP embeds complementably into a Banach space with a basis [230, Theorem 1.e.13].

(3): By Proposition 6.10.28, $Y = \mathrm{FBL}^{(\infty)}[X]$ has a monotone basis, hence there exists a lattice isometry $T : \mathrm{FBL}^{(\infty)}[Y] \to \mathcal{U}$. The formal identity $id : Y \to Y$ extends to a contractive surjective lattice homomorphism $\widehat{id} : \mathrm{FBL}^{(\infty)}[Y] \to Y$. Then $J = T\phi_Y$ and $P = \widehat{id}T^{-1}$ have the desired property. □

Remark 6.10.30. We do not know whether $\mathrm{FBL}^{(\infty)}[E]$ and $\mathrm{FBL}^{(\infty)}[F]$ must be lattice isometric whenever E and F are separable Banach spaces. However, Proposition 6.9.15 and Remark 6.9.16 use WCG arguments to show that there exists non-separable Banach spaces E and F of the same density character, for which $\mathrm{FBL}^{(\infty)}[E]$ and $\mathrm{FBL}^{(\infty)}[F]$ are not lattice isomorphic. A different approach to distinguishing $\mathrm{FBL}^{(\infty)}[E]$ from $\mathrm{FBL}^{(\infty)}[F]$ exploits topological properties of E^* and F^*. If $\mathrm{FBL}^{(\infty)}[E]$ and $\mathrm{FBL}^{(\infty)}[F]$ are lattice isomorphic, then there exists a positively homogeneous map $\Phi : F^* \to E^*$, so that both Φ and its inverse are weak* continuous on bounded sets, and there exists $C \geq 1$ so that $C^{-1}\|\Phi f^*\| \leq \|f^*\| \leq C\|\Phi f^*\|$ for any $f^* \in F^*$. Then B_{F^*} is weak* sequentially compact if and only if B_{E^*} is (see [99, Chapter XIII] for general facts about weak* sequential compactness). Now suppose κ is a cardinal, greater or equal than the continuum. Let $E = \ell_1(\kappa)$ and $F = \ell_2(\kappa)$. Both spaces have density character κ. Then B_{F^*} is weak* sequentially compact (see [99, Chapter XIII, Theorem 4] for a more general fact), while B_{E^*} is not (see [99, p. 226]).

Remark 6.10.31. To our knowledge, weak* continuous non-linear maps have not been deeply studied in the Banach space literature. One application has to do with extensions of operators into $C(K)$ spaces [105, p. 490]: if F is a subspace of E, then any operator $T : F \to C(K)$ admits an extension $\widetilde{T} : F \to C(K)$ with $\|\widetilde{T}\| \leq C\|T\|$ if and only if there exists a weak*-continuous map $\Phi : B_{F^*} \to C \cdot B_{E^*}$ so that $\Phi f^*|_F = f^*$ for any $f^* \in F^*$ (Φ implements a "Hahn-Banach extension"). In [332] such extensions are used to show that, if F is a subspace of c_0, then any operator $T : F \to C(K)$ has an extension to c_0.

Note that, by Bartle-Graves Theorem (see e.g. [245] for its generalizations), a norm continuous Φ like this exists for any $C > 1$; by [49, Corollary 7.4], we cannot in general make Φ uniformly continuous. On the other hand, we can ask for which pairs $F \hookrightarrow E$ there exists a positively homogeneous weak*-continuous map $\Phi : B_{F^*} \to C \cdot B_{E^*}$ so that $\Phi f^*|_F = f^*$ for any $f^* \in F^*$.

Another instance where non-linear weak* continuous maps between Banach spaces have been considered is in [164, 237]. In these works, among other things it is shown that if two dual Banach spaces E^* and F^* are uniformly homeomorphic with respect to the weak* topologies, then necessarily E and F must be linearly isomorphic.

Finally we investigate Banach space properties of $\mathrm{FBL}^{(\infty)}[E]$.

Proposition 6.10.32. *If X is a separable Banach space, then* $\mathrm{FBL}^{(\infty)}[X]$ *is isomorphic to* $C[0,1]$ *as a Banach space.*

Remark 6.10.33. The situation is different in the non-separable setting. For instance, [46] gives an example of a non-separable AM-space X, which is not isomorphic to a complemented subspace of any $C(K)$. Clearly X is complemented in $\mathrm{FBL}^{(\infty)}[X]$, hence $\mathrm{FBL}^{(\infty)}[X]$ cannot be isomorphic to a complemented subspace of a $C(K)$ space either.

Remark 6.10.34. As noted above, if E is an AM-space, then it is complemented in $\mathrm{FBL}^{(\infty)}[E]$. This need not be true if E is merely an ℓ_1 predual. Indeed, by [48], there exists a (necessarily separable) Banach space E, so that $E^* = \ell_1$ isometrically, and E is not isomorphic to a complemented subspace of any $C(K)$. By Proposition 6.10.32, $\mathrm{FBL}^{(\infty)}[E]$ is isomorphic to $C[0,1]$, hence E cannot be complemented in $\mathrm{FBL}^{(\infty)}[E]$.

To prove Proposition 6.10.32, we introduce some notation. Suppose K is a compact Hausdorff space, and $B \subseteq K$ is a closed subset. Let $C_B(K) = \{f \in C(K) : f|_B = 0\}$. The following lemma may be known to experts.

Lemma 6.10.35. *Suppose K is a compact metrizable space, and $B \subseteq K$ is a closed subset for which $K \backslash B$ is uncountable. Then $C_B(K)$ is linearly isomorphic to $C[0,1]$.*

Proof. Throughout the proof, we rely heavily on Milutin's Theorem (see e.g. [329, p. III.D.19]), which states that $C(S)$ is isomorphic to $C[0,1]$ whenever S is compact, metrizable, and uncountable. In particular, this is true for $S = K$.

Note first that $C_B(K)$ is a complemented subspace of $C(K) \sim C[0,1]$. Indeed, consider the restriction operator $v : C(K) \to C(B) : f \mapsto f|_B$. By [329, p. III.D.17], v has a right inverse u: specifically, u is a "linear extension" operator $u : C(B) \to C(K)$, which satisfies $vu = I_{C(B)}$. Then uv is a projection on $C(K)$, whose kernel is $C_B(K)$.

Next show that, conversely, $C[0,1]$ embeds complementably into $C_B(K)$. Once this is established, invoke the fact that $C[0,1]$ is isomorphic to $c_0(C[0,1])$ [329, II.B.24], and use Pełczyński decomposition (cf. [9, Theorem 2.2.3]) to complete the proof.

Pick $\delta > 0$ small enough so that the closed set $V = \{k \in K : \mathrm{dist}(k, B) \geq \delta\}$ is uncountable. By [329, p. III.D.16], there exists a linear extension operator $u : C(V) \to C_B(K)$. As before, denote by $v : C_B(K) \to C(V)$ the restriction operator; then vu is a projection from $C_B(K)$ onto $C(V)$. Therefore, $C(V) \sim C[0,1]$ embeds complementably into $C_B(K)$. □

Proof of Proposition 6.10.32. If X is finite dimensional, then $\mathrm{FBL}^{(\infty)}[X]$ is lattice isomorphic to $C(S_X)$ (S_X being the unit sphere of X), hence linearly isomorphic to $C[0,1]$, by Milutin's Theorem. To handle the case of separable infinite dimensional X, below we briefly review the construction from [47], recently re-examined in [263].

For brevity, denote the unit ball of X^*, equipped with its weak* topology, by A. Fix a dense sequence (h_k) in the unit ball of $\mathrm{FBL}^{(\infty)}[X]$, and let $h = \sum_{k=1}^{\infty} 2^{-k}|h_k|$. For $n \in \mathbb{N}$ let

$$A_n = \{x^* \in A : 2^{-n} \leq h(x^*) \leq 2^{1-n}\}$$

(as $h(0) = 0$, the set A_n is non-empty for n large enough). Let $\overline{A}$ be the one-point compactification of the "formal" disjoint union $\sqcup_n A_n$ (achieved by adding a point we call ∞). Define the map $T : \mathrm{FBL}^{(\infty)}[X] \to C(\overline{A})$ as follows: for $f \in \mathrm{FBL}^{(\infty)}[X]$,

$$[Tf](a) = \begin{cases} 2^{-n} f(a)/h(a) & \text{if } a \in A_n, \\ 0 & \text{if } a = \infty. \end{cases}$$

It is easy to check that T is a (non-surjective) lattice isomorphism.

Fix n for a moment. For $a, b \in A_n$, the equality $[Tf](a) = [Tf](b)$ holds for any $f \in \mathrm{FBL}^{(\infty)}[X]$ if and only if the ratio $f(a)/f(b)$ is independent of f. Hahn-Banach Theorem shows that this happens if and only if a is a positive scalar multiple of b. Denote this relation on A_n by $\sim$. In the notation of [47], let $K_n = A_n/\sim$. Then T gives rise to a lattice isomorphism from $\mathrm{FBL}^{(\infty)}[X]$ onto $Z \subseteq C(K)$, where $K = K_1 \sqcup K_2 \sqcup \ldots \sqcup \{\infty\}$ is the one-point compactification of $K_1 \sqcup K_2 \sqcup \ldots$. Note that any element of the unit sphere of X^* gives rise, via the evaluation map, to at most one point of K_n for each n, hence K is uncountable.

Now suppose $t \in K_m$, and $s \in K_n$. If there exists $\lambda \in \mathbb{R}$ so that $z(t) = \lambda z(s)$ for every $z \in Z$, then, as shown in [47], $n \neq m$, and $\lambda = 2^{n-m}$. For each m, denote by B_m the set of all $t \in K_m$ for which there is $s \in K_n$, $n < m$, so that $z(t) = 2^{n-m} z(s)$ holds for every $z \in Z$. Note that such an $s \in K_n$, if it exists, must be unique. As shown in [263], the sets B_m are closed.

Let $B = \overline{\bigcup_m B_m}$. By [47], Z (hence also $\mathrm{FBL}^{(\infty)}[X]$) is isomorphic to $C_B(K)$. To show that $K \backslash B$ is uncountable, pick the smallest m for which K_m is uncountable. As B_m is countable, $K_m \backslash B_m$ is uncountable. Note that $K_m \cap (K \backslash B) = K_m \backslash B_m$, hence $K \backslash B$ is uncountable. The result now follows from Lemma 6.10.35. $\square$

6.11 Nonlinear summing maps and applications

The norm for the free p-convex Banach lattice has an obvious similarity with the p-summing norm of a linear operator. A very substantial body of literature is devoted to the study of p-summing norms, their applications, and generalizations in the linear case. We refer to [97] for a comprehensive exposition of this theory. Our aim is to establish analogues of a few of these classical results in our setting. The material in this section is taken from [185] and will be explored more comprehensively in [220]

We begin by introducing a more general version of the free p-convex norm involving two indices $1 \leqslant p, q < \infty$ and investigating the fundamental properties of this new norm. For a

Banach space E and a function $f \in H[E]$, define

$$\|f\|_{p,q} = \sup\Big\{ \Big(\sum_{k=1}^{n} \big|f(x_k^*)\big|^p\Big)^{\frac{1}{p}} \,:\, n \in \mathbb{N},\, x_1^*, \dots, x_n^* \in E^*,\, \mu_q(x_1^*, \dots, x_n^*) \leqslant 1 \Big\} \tag{6.11.1}$$

and

$$H_{p,q}[E] = \big\{ f \in H[E] \,:\, \|f\|_{p,q} < \infty \big\}.$$

Here, $\mu_q(x_1^*, \dots, x_n^*) := \sup_{x\in B_E} \big(\sum_{k=1}^{n} |x_k^*(x)|^q\big)^{\frac{1}{q}}$ is the weak q-summing norm. Denote by $\|\cdot\|_\infty$ the supremum norm on B_{E^*}, and let $H_\infty[E]$ be the sublattice of $H[E]$ of all positively homogeneous functions which are bounded on B_{E^*}. Note that $\|f\|_\infty \leqslant \|f\|_{p,q}$ for every $1 \leqslant p, q < \infty$ and $f \in H[E]$, and consequently $H_{p,q}[E] \subseteq H_\infty[E]$. We abbreviate $\|f\|_p = \|f\|_{p,p}$ and note that $H_p[E] = H_{p,p}[E]$.

The following lemma is straightforward.

Lemma 6.11.1. *Given $1 \leqslant p, q < \infty$ and a Banach space E, the space $\big(H_{p,q}[E], \|\cdot\|_{p,q}\big)$ equipped with the pointwise vector lattice operations is a p-convex Banach lattice with p-convexity constant one.*

It is also easy to see that this space is of interest only for $p \geqslant q$.

Lemma 6.11.2. *Let $1 \leqslant p < q < \infty$. Then $H_{p,q}[E] = \{0\}$ for every Banach space E.*

Proof. Let $p, q \in [1, \infty)$, and suppose that $H_{p,q}[E]$ contains a nonzero function f. Choose $x^* \in E^*$ such that $f(x^*) \neq 0$. Then, for every $n \in \mathbb{N}$, we have

$$n^{\frac{1}{p}} \big|f(x^*)\big| = \Big(\sum_{k=1}^{n} \big|f(x^*)\big|^p\Big)^{\frac{1}{p}} \leqslant \|f\|_{p,q} \cdot \mu_q(\underbrace{x^*, \dots, x^*}_{n}) = \|f\|_{p,q}\, n^{\frac{1}{q}}\, \|x^*\|\,,$$

which implies that

$$n^{\frac{1}{p}-\frac{1}{q}} \leqslant \frac{\|x^*\|}{\big|f(x^*)\big|}\, \|f\|_{p,q}\,.$$

Since the right-hand side is independent of n, we conclude that $\frac{1}{p} - \frac{1}{q} \leqslant 0$, that is, $p \geqslant q$. $\square$

Our next result provides the general comparison among these norms. The argument follows the same approach as in the Inclusion Lemma [97, p. 2.8].

Proposition 6.11.3. *Let $1 \leqslant q_j \leqslant p_j < \infty$ for $j = 1, 2$, and suppose that $p_1 \leqslant p_2$, $q_1 \leqslant q_2$, and $\frac{1}{q_1} - \frac{1}{p_1} \leqslant \frac{1}{q_2} - \frac{1}{p_2}$. Then*

$$\|f\|_{p_2,q_2} \leqslant \|f\|_{p_1,q_1}$$

for every $f \in H[E]$. In particular, $H_{p_1,q_1}[E] \subseteq H_{p_2,q_2}[E]$.

Proof. We begin by observing that the result follows easily for $q_1 = q_2$, and if $p_1 = p_2$, then the inequalities $q_1 \leqslant q_2$ and $\frac{1}{q_1} - \frac{1}{p_1} \leqslant \frac{1}{q_2} - \frac{1}{p_2}$ imply that $q_1 = q_2$. Thus, we may assume that $p_1 < p_2$ and $q_1 < q_2$, and then define

$$\frac{1}{p} = \frac{1}{p_1} - \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} - \frac{1}{q_2},$$

which satisfy $1 < p \leqslant q < \infty$ by the hypotheses.

Let $f \in H[E]$ and fix any $x_1^*, \ldots, x_n^* \in E^*$ with $\mu_{q_2}(x_1^*, \ldots, x_n^*) \leqslant 1$. For $1 \leqslant k \leq n$, define $\lambda_k = \left|f(x_k^*)\right|^{p_2/p}$. By the homogeneity of f, we have

$$\sum_{k=1}^{n} \left|f(x_k^*)\right|^{p_2} = \sum_{k=1}^{n} \left|f(\lambda_k x_k^*)\right|^{p_1} \leqslant \|f\|_{p_1,q_1}^{p_1} \; \mu_{q_1}(\lambda_1 x_1^*, \ldots, \lambda_n x_n^*)^{p_1}. \tag{6.11.2}$$

Hölder's inequality shows that

$$\Big(\sum_{k=1}^{n} \left|\lambda_k x_k^*(x)\right|^{q_1}\Big)^{\frac{1}{q_1}} \leqslant \Big(\sum_{k=1}^{n} \lambda_k^q\Big)^{\frac{1}{q}} \Big(\sum_{k=1}^{n} \left|x_k^*(x)\right|^{q_2}\Big)^{\frac{1}{q_2}} \leqslant \Big(\sum_{k=1}^{n} \lambda_k^q\Big)^{\frac{1}{q}} \leqslant \Big(\sum_{k=1}^{n} \lambda_k^p\Big)^{\frac{1}{p}}$$

for every $x \in B_E$ because $\mu_{q_2}(x_1^*, \ldots, x_n^*) \leqslant 1$ and $p \leqslant q$. Taking the supremum over $x \in B_E$ and using the definition of λ_k, we obtain

$$\mu_{q_1}(\lambda_1 x_1^*, \ldots, \lambda_n x_n^*) \leqslant \Big(\sum_{k=1}^{n} \left|f(x_k^*)\right|^{p_2}\Big)^{\frac{1}{p}}. \tag{6.11.3}$$

We now substitute (6.11.3) into (6.11.2) and rearrange the inequality to conclude that

$$\Big(\sum_{k=1}^{n} \left|f(x_k^*)\right|^{p_2}\Big)^{1-\frac{p_1}{p}} \leqslant \|f\|_{p_1,q_1}^{p_1}.$$

This completes the proof because $1 - \frac{p_1}{p} = \frac{p_1}{p_2}$. □

Proposition 6.11.4. *Let E be a Banach space whose dual has finite cotype $r \geqslant 2$, and suppose that $1 \leqslant q < p < \infty$ satisfy $\frac{1}{q} - \frac{1}{p} \geqslant 1 - \frac{1}{r}$. Then $H_{p,q}[E] = H_\infty[E]$ with equivalence of norms.*

Proof. By [97, Corollary 11.17], every weakly summable sequence in E^* is strongly r-summable, and there exists a constant $K > 0$ such that

$$\Big(\sum_{k=1}^{n} \|x_k^*\|^r\Big)^{\frac{1}{r}} \leqslant K\mu_1(x_1^*, \ldots, x_n^*)$$

for every finite sequence $(x_k^*)_{k=1}^n$ in E^*. Hence, for $f \in H_\infty[E]$, we have

$$\Big(\sum_{k=1}^{n} \big|f(x_k^*)\big|^r\Big)^{\frac{1}{r}} \leqslant \|f\|_\infty \Big(\sum_{k=1}^{n} \|x_k^*\|^r\Big)^{\frac{1}{r}} \leqslant K\ \|f\|_\infty\ \mu_1(x_1^*, \ldots, x_n^*).$$

Taking the supremum over all $n \in \mathbb{N}$ and $x_1^*, \ldots, x_n^* \in E^*$ with $\mu_1(x_1^*, \ldots, x_n^*) \leq 1$, we conclude that $\|f\|_{r,1} \leqslant K\,\|f\|_\infty$.

Since $1 \leqslant q < p < \infty$ satisfy $\frac{1}{q} - \frac{1}{p} \geqslant 1 - \frac{1}{r}$, we can apply Proposition 6.11.3 with $p_2 = p$, $q_2 = q$, $p_1 = r$, and $q_1 = 1$ to obtain $\|f\|_{p,q} \leqslant \|f\|_{r,1} \leqslant K\,\|f\|_\infty$, so that $f \in H_{p,q}[E]$ and the (p, q)- and supremum norms are equivalent. □

As in the classical setting, the Dvoretzky–Rogers Theorem can be used to show that in general these norms are different:

Proposition 6.11.5. *Let E be an infinite-dimensional Banach space, and suppose that $1 \leqslant q \leqslant p < \infty$ satisfy $\frac{1}{q} - \frac{1}{p} < \frac{1}{2}$. Then $H_{p,q}[E] \subsetneq H_\infty[E]$.*

Proof. By the Dvoretzky–Rogers Theorem [97, Theorem 10.5], there exists a weakly q-summable sequence $(x_k^*)_{k\in\mathbb{N}}$ in E^* which fails to be strongly p-summable. Now consider the function $f\colon E^* \to \mathbb{R}$ defined via $f(x^*) = \|x^*\|$. Clearly, $f \in H_\infty[E]$, and for every $n \in \mathbb{N}$, we have

$$\Big(\sum_{k=1}^{n} \|x_k^*\|^p\Big)^{\frac{1}{p}} \leqslant \|f\|_{p,q}\ \mu_q(x_1^*, \ldots, x_n^*).$$

Letting $n \to \infty$, we see that $\|f\|_{p,q} = \infty$. Thus $f \notin H_{p,q}[E]$. □

Pietsch's Domination Theorem (see, e.g., [97, Theorem 2.12]) is a cornerstone of the linear theory of p-summing operators with several important factorization results among its consequences. We conclude by providing analogues of [26, Propositions 2.12 and 2.13] for $1 \leqslant p < \infty$.

Given a Banach space E, equip the unit ball $B_{E^{**}}$ of its bidual with the relative weak* topology, and denote the set of regular Borel probability measures on $B_{E^{**}}$ by $\mathfrak{P}(B_{E^{**}})$. This is a convex, weak* compact subset of the dual space of $C(B_{E^{**}})$. Each measure $\mu \in \mathfrak{P}(B_{E^{**}})$ induces a function $f_\mu^p\colon E^* \to \mathbb{R}_+$ via the definition

$$f_\mu^p(x^*) = \left(\int_{B_{E^{**}}} \left|x^{**}(x^*)\right|^p \, d\mu(x^{**})\right)^{\frac{1}{p}}$$

for every $x^* \in E^*$. This provides a link between $H_p[E]_+$ and $\mathfrak{P}(B_{E^{**}})$, as we now explain.

Proposition 6.11.6. *Let $1 \leqslant p < \infty$ and $\mu \in \mathfrak{P}(B_{E^{**}})$. Then $f_\mu^p \in H_p[E]_+$ with $\left\|f_\mu^p\right\|_p \leqslant 1$.*

Proof. The function f_μ^p is clearly positive and positively homogeneous. For $n \in \mathbb{N}$ and $x_1^*, \dots, x_n^* \in E^*$, we have

$$\begin{aligned}\left(\sum_{k=1}^n \left|f_\mu^p(x_k^*)\right|^p\right)^{\frac{1}{p}} &= \left(\int_{B_{E^{**}}} \sum_{k=1}^n \left|x^{**}(x_k^*)\right|^p \, d\mu(x^{**})\right)^{\frac{1}{p}} \\ &\leqslant \sup_{x^{**} \in B_{E^{**}}} \left(\sum_{k=1}^n \left|x^{**}(x_k^*)\right|^p\right)^{\frac{1}{p}} = \mu_p(x_1^*, \dots, x_n^*),\end{aligned}$$

where the last equality follows from the weak* density of B_E in $B_{E^{**}}$. Hence $\left\|f_\mu^p\right\|_p \leqslant 1$. □

Proposition 6.11.7. *Let $1 \leqslant p < \infty$. For every $f \in H_p[E]_+$, there is a measure $\mu \in \mathfrak{P}(B_{E^{**}})$ such that $f(x^*) \leqslant \|f\|_p \, f_\mu^p(x^*)$ for every $x^* \in E^*$.*

Proof. This proof is based on the proof of Pietsch's Domination Theorem given in [97, p. 2.12]. For every nonempty finite subset M of E^*, define $g_M\colon B_{E^{**}} \to \mathbb{R}$ by

$$g_M(x^{**}) = \sum_{x^* \in M} \Big(f(x^*)^p - \|f\|_p^p \cdot \left|x^{**}(x^*)\right|^p\Big).$$

Then g_M is weak* continuous, and so the set Q of all such functions g_M is contained in $C(B_{E^{**}})$. Given nonempty finite subsets M_1 and M_2 of E^* and $0 < \lambda < 1$, the positive homogeneity of f implies that $\lambda \cdot g_{M_1} + (1-\lambda) \cdot g_{M_2} = g_{M_3}$, where

$$M_3 = \left\{\lambda^{1/p} x^* \ : \ x^* \in M_1\right\} \cup \left\{(1-\lambda)^{1/p} x^* \ : \ x^* \in M_2\right\}.$$

This shows that Q is a convex set.

The definition of the norm $\|\cdot\|_p$ implies that Q is disjoint from the strictly positive cone

$$P = \Big\{h \in C(B_{E^{**}}) \ : \ h(x^{**}) > 0 \text{ for every } x^{**} \in B_{E^{**}}\Big\}.$$

Since P is open and convex, the geometric version of the Hahn–Banach Theorem guarantees the existence of a functional $\mu \in C(B_{E^{**}})^*$ and a constant $c \in \mathbb{R}$ such that $\mu(g) \leq c < \mu(h)$ for every $g \in Q$ and $h \in P$.

Choosing $M = \{0\} \subseteq E^*$, we have $g_M = 0$. Therefore $0 \in Q$, and so $c \geqslant 0$. On the other hand, as every strictly positive constant function belongs to P, we must have $c \leqslant 0$. It follows that $c = 0$, which implies that $\mu(h) \geqslant 0$ for every $h \in C(B_{E^{**}})_+$. In other words, μ is a positive regular Borel measure such that

$$\int_{B_{E^{**}}} g \, d\mu \leqslant 0 < \int_{B_{E^{**}}} h \, d\mu$$

for every $g \in Q$ and $h \in P$. This inequality is unaffected by scaling of μ, so we may assume that $\mu \in \mathfrak{P}(B_{E^{**}})$. For every $x^* \in E^*$, the function $g_{\{x^*\}}$ belongs to Q, and therefore

$$0 \geqslant \int_{B_{E^{**}}} \Big(f(x^*)^p - \|f\|_p^p \cdot \big|x^{**}(x^*)\big|^p \Big) d\mu(x^{**}) = f(x^*)^p - \|f\|_p^p \; f_\mu^p(x^*)^p$$

because μ is a probability measure. □

We can summarize the conclusions of Propositions 6.11.6 and 6.11.7 as follows.

Corollary 6.11.8. *Let $1 \leqslant p < \infty$ and $f \in H[E]_+$. Then $f \in H_p[E]_+$ if and only if, for some constant $C > 0$, there is a measure $\mu \in \mathfrak{P}(B_{E^{**}})$ such that $f(x^*) \leqslant C \cdot f_\mu^p(x^*)$ for every $x^* \in E^*$. Furthermore, when $f \in H_p[E]$, its norm $\|f\|_p$ can be computed as the infimum of all constants C for which such a measure μ exists.*

Applications of the above type of result to FBL$[E]$ can be found in [26]; the p-convex case follows similar lines.

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