SHARP HADAMARD LOCAL WELL-POSEDNESS, ENHANCED UNIQUENESS AND POINTWISE CONTINUATION CRITERION FOR THE INCOMPRESSIBLE FREE BOUNDARY EULER EQUATIONS

MIHAELA IFRIM, BEN PINEAU, DANIEL TATARU, AND MITCHELL A. TAYLOR

Abstract. We provide a complete local well-posedness theory in H^s based Sobolev spaces for the free boundary incompressible Euler equations with zero surface tension on a connected fluid domain. Our well-posedness theory includes: (i) Local well-posedness in the Hadamard sense, i.e., local existence, uniqueness, and the first proof of continuous dependence on the data, all in low regularity Sobolev spaces; (ii) Enhanced uniqueness: Our uniqueness result holds at the level of the Lipschitz norm of the velocity and the $C^{1,\frac{1}{2}}$ regularity of the free surface; (iii) Stability bounds: We construct a nonlinear functional which measures, in a suitable sense, the distance between two solutions (even when defined on different domains) and we show that this distance is propagated by the flow; (iv) Energy estimates: We prove refined, essentially scale invariant energy estimates for solutions, relying on a newly constructed family of elliptic estimates; (v) Continuation criterion: We give the first proof of a sharp continuation criterion in the physically relevant pointwise norms, at the level of scaling. In essence, we show that solutions can be continued as long as the velocity is in $L^1_T W^{1,\infty}$ and the free surface is in $L^1_T C^{1,\frac{1}{2}}$, which is at the same level as the Beale-Kato-Majda criterion for the boundaryless case; (vi) A novel proof of the construction of regular solutions.

Our entire approach is in the Eulerian framework and can be adapted to work in more general fluid domains.

Contents

1. Introduction

In this article, we study the dynamics of an inviscid fluid droplet in the absence of surface tension. At time t, our fluid occupies a compact, connected, but not necessarily simply connected region $\overline{\Omega}_t \subseteq \mathbb{R}^d$, and its

2020 *Mathematics Subject Classification.* 76B03; 76B15; 35Q31.

Key words and phrases. Euler equations, free boundary, water waves, local well-posedness, continuation criterion.

motion is governed by the incompressible Euler equations

$$\begin{cases} \partial_t v + v \cdot \nabla v = -\nabla p - g e_d, \\ \nabla \cdot v = 0. \end{cases} \tag{1.1}$$

Here, v is the fluid velocity, p is the pressure, $g \geq 0$ is the gravitational constant, and e_d is the standard vertical basis vector. In the local theory of the droplet problem, the gravity can be freely neglected. However, it becomes important in the case of an unbounded fluid domain and in the case of a domain with a rigid bottom, so we retain it in (1.1) for completeness.

An essential role in the analysis of the droplet problem is played by the vector field

$$D_t := \partial_t + v \cdot \nabla,$$

which is called the *material derivative* and describes the particle trajectories. On the free boundary, we require the kinematic boundary condition

$$D_t \ \text{ is tangent to } \bigcup_t \{t\} \times \partial\Omega_t \subseteq \mathbb{R}^{d+1}, \tag{1.2}$$

which says that the domain Ω_t is transported along the material derivative (or equivalently, the particle trajectories), and that the normal velocity of $\Gamma_t := \partial\Omega_t$ is given by $v \cdot n_{\Gamma_t}$. Additionally, we require the dynamic boundary condition

$$p|_{\Gamma_t} = 0, \tag{1.3}$$

which represents the balance of forces at the fluid interface in the absence of surface tension. Using the above boundary conditions, it is easy to see that the energy

$$E := \int_{\Omega_t} \left(\frac{|v|^2}{2} + g x \cdot e_d \right) dx$$

is formally conserved. Throughout the article, we will refer to the system (1.1)-(1.3) as the *free boundary (incompressible) Euler equations.*

As is the case with all Euler flows, an important role in the above evolution is played by the *vorticity*, ω, defined by

$$\omega_{ij} = \partial_i v_j - \partial_j v_i.$$

By taking the curl of (1.1), the vorticity is easily seen to solve the following transport equation along the flow:

$$D_t \omega = -(\nabla v)^* \omega - \omega \nabla v. \tag{1.4}$$

If initially $\omega = 0$, then (1.4) guarantees that this condition is propagated dynamically. Such velocity fields are called *irrotational*, and the corresponding solutions to the free boundary incompressible Euler equations are called *water waves.*

By taking the divergence of (1.1), we obtain the following Laplace equation for the pressure:

$$\begin{cases} \Delta p = -\text{tr}(\nabla v)^2 \ \text{ in } \Omega_t, \\ p = 0 \ \text{ on } \Gamma_t. \end{cases} \tag{1.5}$$

For regular enough v on sufficiently regular Ω_t, the equation (1.5) uniquely determines the pressure from the velocity and domain. A key role in the study of the free boundary Euler equations is played by the *Taylor coefficient*, a, which is defined on the boundary Γ_t by

$$a := -\nabla p \cdot n_{\Gamma_t}. \tag{1.6}$$

Indeed, a classical result of Ebin [21] asserts that the free boundary Euler equations are ill-posed unless $a \geq 0$. For this reason, we will always assume that the initial data for the free boundary Euler equations verifies the following:

Taylor sign condition. There is a $c_0 > 0$ such that $a_0 := -\nabla p_0 \cdot n_{\Gamma_0} > c_0$ on Γ_0.

For irrotational data on compact simply connected domains, the Taylor sign condition is automatic by the strong maximum principle [33]. See also [26, 49] for similar results on unbounded domains when $g > 0$. Geometrically, enforcing $a_0 > 0$ ensures that the initial pressure p_0 is a non-degenerate defining function for the initial boundary hypersurface Γ_0, and thus can be used to describe the regularity of the boundary. As part of our well-posedness theorem below, we prove that the Taylor sign condition is propagated by the flow on some non-trivial time interval.

Another important role in this paper is played by the *material derivative of the Taylor coefficient*, $D_t a$, which turns out to be closely related to (a derivative of) the normal component of the velocity $v \cdot n_{\Gamma_t}$. We will elaborate further on this relation shortly when we discuss our choice of control parameters and good variables.

1.1. **The Cauchy problem: scaling, Sobolev spaces and control parameters.** A state for the free boundary Euler equations consists of a domain Ω and a velocity field v on Ω. A bounded connected domain Ω can be equally described by its boundary Γ. Hence, in the sequel, by a state we mean a pair (v, Γ).

Describing the time evolution of (v, Γ) along the free boundary incompressible Euler flow is most naturally done in a functional setting described via appropriate Sobolev norms. To understand the proper setting, it is very helpful to consider the scaling properties of our problem. The boundaryless incompressible Euler flow admits a two parameter scaling group. However, when considering the free boundary flow there is an additional constraint; namely, that the pointwise property $a \approx 1$ rests unchanged. At a technical level, this is reflected in the fact that the Taylor coefficient appears as a weight in the Sobolev norms which are used on Γ. Imposing this constraint leaves us with a one parameter family of scaling laws, which have the form

$$\begin{aligned} v_\lambda(t,x) &= \lambda^{-\frac{1}{2}} v\left(\lambda^{\frac{1}{2}} t, \lambda x\right), \\ p_\lambda(t,x) &= \lambda^{-1} p\left(\lambda^{\frac{1}{2}} t, \lambda x\right), \\ (\Gamma_\lambda)_t &= \{\lambda^{-1} x : x \in \Gamma_{\lambda^{\frac{1}{2}} t}\}. \end{aligned}$$

As noted earlier, the above transformations have the property that the Taylor coefficient has the dimensionless scaling,

$$a_\lambda(t,x) = a\left(\lambda^{\frac{1}{2}} t, \lambda x\right).$$

A first benefit we derive from the scaling law is to understand what are the matched Sobolev regularities for v and Γ. This leads us to the following definition.

Definition 1.1 (State space)**.** The *state space* $\mathbf{H}^s$ is the set of all pairs (v, Γ) such that Γ is the boundary of a bounded, connected domain Ω and such that the following properties are satisfied:

(i) (Regularity). $v \in H^s_{div}(\Omega)$ and $\Gamma \in H^s$, where $H^s_{div}(\Omega)$ denotes the space of divergence free vector fields in $H^s(\Omega)$.

(ii) (Taylor sign condition). $a := -\nabla p \cdot n_\Gamma > c_0 > 0$, where c_0 may depend on the choice of (v, Γ), and the pressure p is obtained from (v, Γ) by solving the elliptic equation (1.5) associated to (1.1) and (1.3).

For states (v, Γ) as above, we define their size by

$$\|(v, \Gamma)\|^2_{\mathbf{H}^s} := \|\Gamma\|^2_{H^s} + \|v\|^2_{H^s(\Omega)}.$$

Note, however, that $\mathbf{H}^s$ is not a linear space, so $\| \cdot \|_{\mathbf{H}^s}$ does not induce a norm topology in the usual sense. Heuristically, the state space $\mathbf{H}^s$ may be thought of as an infinite dimensional manifold, though a precise interpretation of this is beyond the scope of this paper. For our purposes, it suffices to define a consistent notion of topology on $\mathbf{H}^s$. Although we will not describe the precise topology in the introduction, this topology will allow us to define the space $C([0,T]; \mathbf{H}^s)$ of continuous functions with values in $\mathbf{H}^s$, as well as an appropriate notion of $\mathbf{H}^s$ continuity of the data-to-solution map $(v_0, \Gamma_0) \mapsto (v(t), \Gamma_t)$. Armed with these notions, it makes sense to talk about the Cauchy problem.

Problem 1.2 (Cauchy problem for the free boundary Euler equations)**.** Given an initial state $(v_0, \Gamma_0) \in \mathbf{H}^s$, find the unique solution $(v, \Gamma) \in C([0,T]; \mathbf{H}^s)$ in some time interval $[0, T]$.

A natural question to ask is what are the exponents s for which the Cauchy problem is well-posed in $\mathbf{H}^s$. Our first clue in this direction comes from scaling, which leads us to the critical exponent

$$s_c = \frac{d+1}{2},$$

and implicitly the lower bound $s \geq s_c$. However, this does not tell the entire story, as even in the boundaryless case a result of Bourgain-Li [11] shows that well-posedness holds only in the more restricted range

$$s > \frac{d}{2} + 1,$$

which is heuristically connected to another scaling law of the boundaryless problem; namely,

$$v(t, x) \mapsto \lambda^{-1} v(t, \lambda x).$$

This latter exponent range $s > \frac{d}{2} + 1$ is exactly what we consider in our work. Specifically, in this article we solve the Cauchy problem for the free boundary incompressible Euler equations at the same regularity level as the incompressible Euler equations on a fixed domain.

The reader who is more familiar with the boundaryless case may ask at this point why we confine ourselves to L^2 based Sobolev spaces, instead of using the full range of indices L^p as in the boundaryless case. The reason for this is precisely the boundary, where a portion of the dynamics is concentrated. In particular, as a subset of our problem we have the irrotational case $\omega = 0$, when the flow may be fully interpreted as the flow of the free boundary. This case, commonly identified as water waves, yields a dispersive flow, where L^p based Sobolev spaces are disallowed if $p \neq 2$. This is not to say that exponents $p \neq 2$ do not play a central role in our analysis. Instead, we use them, particularly the case $p = \infty$, in the definition of our *control parameters*, which control the size and growth of our energy functionals. Precisely, our analysis involves two such control parameters, which ideally should be appropriately scale invariant, as follows:

(i) An "elliptic" control parameter $A^\sharp$, used to control implicit constants in fixed time elliptic estimates, given by

$$A^\sharp = \|v\|_{\dot{C}^{\frac{1}{2}}(\Omega)} + \|\Gamma\|_{Lip}, \tag{1.7}$$

which is exactly invariant under scaling.

(ii) A "dynamical" control parameter $B^\sharp$, used to control the growth of energy in time, given by

$$B^\sharp = \|v\|_{Lip(\Omega)} + \|\Gamma\|_{\dot{C}^{1,\frac{1}{2}}}. \tag{1.8}$$

This latter control parameter is 1/2 derivatives above scaling, and instead the scale invariant quantity is $\|B^\sharp\|_{L^1_t}$, which is what will actually appear in our continuation criterion later on.

With these control parameters in hand, we would like to have energy estimates in the scale invariant form

$$\frac{d}{dt}E^k(v,\Gamma) \lesssim_{A^\sharp} B^\sharp E^k(v,\Gamma), \tag{1.9}$$

where E^k denotes a suitable energy at the $\mathbf{H}^k$ regularity. As noted earlier, these are our ideal choices, but for our results we need to make some small adjustments and relax them a bit, as follows:

a) Working with $A^\sharp$ would require edge case elliptic estimates in Lipschitz domains, bringing forth a broad host of issues which are less central to our problem, if even possible to overcome. So, instead, we will simply add ϵ derivatives to the norms in $A^\sharp$.

b) In the case of $B^\sharp$, we do not want to lose the sharp scaling, which is exactly as in the Beale-Kato-Majda criteria in the boundaryless case. Therefore, we do not want to add extra derivatives as we did with $A^\sharp$. However, as we shall soon see, the quantity $\|D_t a\|_{L^\infty(\Gamma)}$ appears as a control parameter in the L^2 estimate for the linearized equation. As it turns out, in order to propagate our low regularity difference bounds, control of $\|D_t a\|_{L^\infty(\Gamma)}$ will be needed. However, for the energy estimates, a careful analysis will show that the control parameter $B^\sharp$ is sufficient, if we slightly modify the form of the estimate (1.9). In both cases, maintaining the sharp top order control parameter is non-trivial. In the difference estimates, it requires a careful analysis on intersections of domains (and hence, in particular, performing elliptic theory on Lipschitz domains) and in the energy estimates it requires (amongst several other things) finding a way to appropriately absorb the logarithmic divergences occurring in the endpoint elliptic estimates when attempting to control $\|D_t a\|_{L^\infty(\Gamma)}$ by $B^\sharp$. To deal with this latter issue, we will take some inspiration from the proof of Beale-Kato-Majda [8].

The issues mentioned above have well-known counterparts in the boundaryless Euler flow. In fact, *strong ill-posedness* of the boundaryless Euler equations has been recently proven in the "ideal" pointwise spaces C^1 and *Lip* [12, 22].

1.2. **Historical comments.** The local well-posedness problem for the free boundary Euler equations has a long history. For irrotational flows, the first rigorous local existence result in Sobolev spaces was obtained by Wu [49, 50], in the late 1990s. Since then, various methods have been introduced to shorten the proofs, lower the regularity threshold and allow for more complicated geometries. For a small sample of such results we cite Beyer and Günther in [9], Lannes in [32], Alazard, Burq and Zuily in [4, 5], Hunter, Ifrim and Tataru in [26], Ai in [1, 2] and Ai, Ifrim and Tataru in [3]. Although physically restrictive, the irrotationality assumption allows one to reduce the dynamics to a system of equations on the free boundary. Depending on the choices made, this typically culminates in either the Zakharov-Craig-Sulem formulation of the water waves problem used in [1, 2, 4, 5, 32], or the holomorphic coordinates formulation used in [3, 26]. In either case, the reduction to a system of equations on $\mathbb{R}^{d-1}$ greatly simplifies the analysis.

For the free boundary Euler equations with non-trivial vorticity, certain generalized systems based on the above irrotational reductions have been proposed [13, 51]. However, historically, the most successful approach has been to use Lagrangian coordinates to fix the domain. For an execution of this approach to proving local existence, the reader may consult the papers of Christodoulou and Lindblad [14], Coutand and Shkoller [15] and Lindblad [33]. One may also compare with the article [31] of Kukavica and Tuffaha, which uses the so-called *arbitrary Lagrangian-Eulerian change of variables*, as well as the more recent advances in the Lagrangian analysis presented in [7, 19].

In contrast to the above articles, we will utilize a *fully Eulerian* strategy to prove the local well-posedness of the free boundary Euler equations. In other words, we will work directly with the physical equations (1.1)-(1.3), and avoid the use of any non-trivial coordinates changes. On time-independent domains, both the Lagrangian and Eulerian approaches have been widely successful in analyzing fluid equations. However, for free boundary problems, the Eulerian approach has seen relatively little attention, due to the obvious difficulty in having the domain of the fluid itself serve as a time-dependent unknown. Our aim in this article is to directly confront this issue. Corollaries of our newly obtained insights include:

(i) The first proof of the continuity of the data-to-solution map for this problem.
(ii) An enhanced uniqueness result, requiring only pointwise norms of very limited regularity.
(iii) Refined low regularity energy estimates with geometrically natural pointwise control parameters.
(iv) A new, direct proof of existence for regular solutions.
(v) A method to obtain rough solutions as unique limits of regular solutions at a Sobolev regularity that matches the optimal result for the Euler equations on $\mathbb{R}^d$.
(vi) An essentially scale invariant continuation criterion akin to that of Beale-Kato-Majda for the incompressible Euler equations on the whole space.

We will elaborate further on the ideas for obtaining the above results in Section 1.3. For now, it is important to note that we are not the first to utilize an Eulerian approach to analyze the well-posedness of fluid equations in the free boundary setting. The pioneering work in this regard is the remarkable series of papers by Shatah and Zeng [41, 42, 43]. However, Shatah and Zeng primarily consider the free boundary Euler equations with surface tension. While they are able to produce a solution to the pure gravity problem in the zero surface tension limit, it seems that their construction at least requires bounded curvature, which corresponds to greater regularity assumptions on the data than we need here. For this reason, the overlap between their analysis and ours tends to be on a more philosophical level, which we will elaborate on further in Section 1.3. A more direct comparison is with the memoir [47] of Wang, Zhang, Zhao and Zheng. In [47], the authors construct solutions to the free boundary Euler equations in an unbounded graph domain at the same *Sobolev* regularity that we achieve here. That is, they prove existence and uniqueness of solutions in H^s for $s > \frac{d}{2} + 1$. The approach in [47] is in the style of Alazard, Burq and Zuily [4, 5], though the addition of vorticity makes the execution much more technical. Our approach is completely different to the one that they follow and works well in more complicated fluid domains. Additionally, we prove properties (i)-(vi) above. We also remark that all other fully Eulerian approaches (see, e.g., [37, 38, 39]) follow Shatah and Zeng, and hence require the regularizing effect of surface tension and higher regularity. The one step towards a fully Eulerian proof without surface tension is the work [17] of de Poyferré, who proves energy estimates for the pure gravity shoreline problem. However, the energy estimates in [17] have H^s based control norms and no well-posedness proof is presented.

The goal of our paper is twofold. First, we intend to present a comprehensive, Hadamard style well-posedness theory, with an aim towards proving sharp results. At the same time, we provide a novel, geometric analysis, which we argue is more direct and streamlined than previous works. For instance, our proofs do not require paralinearization or Chemin-Lerner spaces as in [47]. Moreover, our existence scheme is new and direct - it does not use Nash-Moser, the approach in [47], or go through the zero surface tension limit as in [41, 42, 43]. For this reason, we believe that the techniques introduced in this paper will have a wide range of applicability.

Finally, we mention that the analysis we present here is for the case of a compact fluid domain. In the study of the free boundary Euler equations, it is also common to consider the case of an infinite ocean of either finite or infinite depth. The choice of compact fluid domain emphasizes the geometric nature of our problem, and removes the temptation to flatten the domain into a strip or a half-space. Although some changes need to be made, as with the analysis of the capillary problem [41, 42, 43] by Shatah and Zeng, the general strategy we use here can be adapted to all three geometries. That being said, to streamline the exposition, we do allow some of our estimates to depend on the domain volume, which is a conserved quantity for the droplet problem.

1.3. An overview of the main results.

In a nutshell, our main result asserts that the free boundary incompressible Euler equations are well-posed in $\mathbf{H}^s$ for $s > \frac{d}{2}+1$. However, simply stating this fails to convey the full strength of both the result and of its various aspects and consequences. Instead, it is more revealing to divide the result in a modular way into four independently interesting parts; namely, (a) uniqueness and stability, (b) well-posedness, (c) energy estimates and (d) the continuation criteria.

To set the stage for our results, let Ω_* be a bounded, connected domain with smooth boundary Γ_*. Given $\epsilon, \delta > 0$, consider the collar neighborhood $\Lambda_* := \Lambda(\Gamma_*, \epsilon, \delta)$ consisting of all hypersurfaces Γ which are δ-close to Γ_* in the $C^{1,\epsilon}$ topology. As long as $\delta > 0$ is small enough, hypersurfaces in Λ_* can be written as graphs over Γ_*. This permits us to define Sobolev and Hölder norms on these hypersurfaces in a consistent fashion. To state our results, we will assume that a collar neighborhood Λ_* has been fixed, and consider solutions with initial data (v_0, Γ_0) having $\Gamma_0 \in \Lambda_*$. A more precise description of the functional setting will be given later, in Section 3. For now, we remark that, while the collar neighborhood is very useful in order to uniformly define the $\mathbf{H}^s$ norms, it is not needed at all for the definition of our control parameters.

1.3.1. *Uniqueness and stability.*

We start by stating our uniqueness result, which requires the least in terms of notations and preliminaries. Here, of crucial importance are the control parameters

$$A := A_\epsilon := \|v\|_{C_x^{\frac{1}{2}+\epsilon}(\Omega_t)} + \|\Gamma_t\|_{C_x^{1,\epsilon}}, \ \epsilon > 0, \tag{1.10}$$

and

$$B_{\mathrm{diff}} := \|v\|_{W_x^{1,\infty}(\Omega_t)} + \|D_t p\|_{W_x^{1,\infty}(\Omega_t)} + \|\Gamma_t\|_{C_x^{1,\frac{1}{2}}}, \tag{1.11}$$

which represent slight adjustments of the ideal control parameters $A^\sharp$ and $B^\sharp$, as discussed earlier. Using these control parameters, our main uniqueness result is as follows:

Theorem 1.3 (Uniqueness). *Let $\epsilon, T > 0$ and let Ω_0 be a domain with boundary Γ_0 of $C^{1,\frac{1}{2}}$ regularity. Then for every divergence free initial data $v_0 \in W^{1,\infty}(\Omega_0)$, the free boundary Euler equations with the Taylor sign* *condition admit at most one solution* (v, Γ_t) *with* $\Gamma_t \in \Lambda_*$ *and*

$$\sup_{0\leq t\leq T} A_\epsilon(t) + \int_0^T B_{\mathrm{diff}}(t)\,dt < \infty.$$

To the best of our knowledge, Theorem 1.3 is the first uniqueness result for the free boundary Euler equations which involves only low regularity pointwise norms. Indeed, as far as we are aware, all other papers on this subject are content to prove uniqueness in the same class of H^s spaces for which they prove existence.

While uniqueness is a fundamental property in its own right, in our work it can be seen as a corollary of a far more useful stability result, which we now explain. Let (v, Γ_t) and $(v_h, \Gamma_{t,h})$ be two solutions to the free boundary Euler equations with corresponding domains Ω_t and $\Omega_{t,h}$. An obvious objective is to show that if (v, Γ_t) and $(v_h, \Gamma_{t,h})$ are "close" at time zero, then they remain close on a suitable timescale. However, since the domains Ω_t and $\Omega_{t,h}$ are evolving in time, we cannot compare the solutions (v, Γ_t) and $(v_h, \Gamma_{t,h})$ in a linear way. To resolve this issue, we construct a nonlinear functional which quantifies the distance between solutions and is propagated by the flow.

To avoid comparing solutions whose corresponding domains are very different, we harmlessly restrict ourselves to solutions (v, Γ_t) and $(v_h, \Gamma_{t,h})$ evolving in the same collar neighborhood Λ_*. For such solutions we define the nonlinear distance functional

$$D((v,\Gamma),(v_h,\Gamma_h)) := \frac{1}{2}\int_{\tilde{\Omega}_t} |v-v_h|^2 dx + \frac{1}{2}\int_{\tilde{\Gamma}_t} b|p-p_h|^2\,dS. \tag{1.12}$$

Here, p and p_h are the pressures, $\tilde{\Gamma}_t$ is the boundary of $\tilde{\Omega}_t := \Omega_t \cap \Omega_{t,h}$ and b is a suitable weight function. Morally speaking, the first term on the right-hand side of (1.12) measures the L^2 distance between v and v_h. On the other hand, by the Taylor sign condition, p and p_h are non-degenerate defining functions for Γ_t and $\Gamma_{t,h}$, so the second term on the right-hand side of (1.12) gives a measure of the distance between Γ_t and $\Gamma_{t,h}$. In Section 4, we prove that (1.12) does indeed act as a proper measure of distance between solutions. More crucially, we prove that this distance is propagated by the flow, in the sense that

$$\frac{d}{dt}D((v,\Gamma),(v_h,\Gamma_h)) \lesssim_{A,A_h} (B_{\mathrm{diff}} + B_{\mathrm{diff},h})D((v,\Gamma),(v_h,\Gamma_h)). \tag{1.13}$$

Here, A_h and $B_{\mathrm{diff},h}$ are the control parameters (1.10) and (1.11) corresponding to the solution $(v_h, \Gamma_{t,h})$. An immediate corollary of the stability estimate (1.13) is the aforementioned Theorem 1.3. However, (1.13) will also prove to be useful in various other scenarios. For example, we will use it in our proof of the continuity of the data-to-solution map, as well as in the construction of rough solutions as unique limits of regular solutions.

1.3.2. *Well-posedness.* Our second main result is concerned with the well-posedness problem. To fix the notations, we start with a collar neighborhood Λ_* and $s > \frac{d}{2}+1$. We then consider initial data $(v_0, \Gamma_0) \in \mathbf{H}^s$ with $\Gamma_0 \in \Lambda_*$. Viewing Γ_0 as a graph over Γ_*, we may unambiguously define its H^s norm. With this setup, we may state our well-posedness theorem as follows:

Theorem 1.4 (Hadamard local well-posedness)**.** *Fix* $s > \frac{d}{2}+1$ *and a collar* Λ_**. For any* (v_0, Γ_0) *in* $\mathbf{H}^s$ *with* $\Gamma_0 \in \Lambda_*$ *there exists a time* $T > 0$*, depending only on* $\|(v_0, \Gamma_0)\|_{\mathbf{H}^s}$ *and the lower bound in the Taylor sign condition, for which there exists a unique solution* $(v(t), \Gamma_t) \in C([0,T]; \mathbf{H}^s)$ *to the free boundary Euler equations satisfying a proportional uniform lower bound in the Taylor sign condition. Moreover, the data-to-solution map is continuous with respect to the* $\mathbf{H}^s$ *topology.*

The regularity of the velocity in Theorem 1.4 matches the optimal Sobolev regularity for the Euler equations on $\mathbb{R}^d$. Indeed, as shown by Bourgain and Li [11], the Euler equations are ill-posed in $H^s(\mathbb{R}^d)$ when $s = \frac{d}{2}+1$.

We note crucially that our article is not the first to reach the $s > \frac{d}{2} + 1$ Sobolev threshold for the free boundary Euler equations. Indeed, this threshold was achieved for the first time in the recent memoir [47], in the case of an unbounded fluid domain with graph geometry. However, it is important to note that the approach in [47] is very different from ours, as it passes through a paralinearization and utilizes properties of strip-like domains and Chemin-Lerner spaces. In particular, the approach in [47] cannot be easily modified to the droplet problem, whereas our approach applies equally well in unbounded domains. Moreover, there is no mention of the continuity of the data-to-solution map in [47]. To the best of our knowledge, Theorem 1.4 gives the *first* proof of this important property for the free boundary Euler equations. In addition, our approach significantly refines the well-posedness theory by adding properties (ii)-(vi) above as well as introduces an entirely new set of techniques that we believe will have broad applications.

When it comes to free boundary problems, the continuity of the data-to-solution map − if justified − is usually proven by reformulating the problem on a fixed domain and then working with the standard notion of continuous dependence on fixed domains. As far as we are aware, the only exception to this appears in the work [41, 42, 43] of Shatah and Zeng, where continuous dependence is proven for the free boundary Euler equations with surface tension directly in the Eulerian setting. The drawback of Shatah and Zeng's proof, however, is that it relies crucially on the regularizing effect of surface tension, so is not applicable to the pure gravity problem. In particular, Shatah and Zeng do not construct a distance functional, as we do here. For this reason, our robust proof which simultaneously avoids domain flattenings and works on a quasilinear problem without regularizing effects can be seen as one of the main novelties of our paper.

1.3.3. *Energy estimates.* Controlling the growth of solutions to our boundary value problem is essential for both local well-posedness and understanding potential blowup. This control is achieved via energy estimates. Due to the complex geometry of our problem, the first challenge is to construct good energy functionals.

Fix an integer $k \geq 0$. In light of Theorem 1.3 and the stability estimate (1.13), it is natural to try to construct an energy functional $E^k = E^k(v, \Gamma)$ satisfying $E^k(v, \Gamma) \approx_A \|(v, \Gamma)\|^2_{\mathbf{H}^k}$ and the estimate

$$\frac{d}{dt}E^k(v, \Gamma) \lesssim_A B_{\text{diff}} E^k(v, \Gamma).$$

Indeed, by Grönwall's inequality, this would yield the bound

$$\|(v, \Gamma)(t)\|^2_{\mathbf{H}^k} \lesssim \exp\left(\int_0^t C_A B_{\text{diff}}(s)\, ds\right) \|(v, \Gamma)(0)\|^2_{\mathbf{H}^k},$$

for some constant C_A depending only on A, the collar, and the verification of the Taylor sign condition. Morally speaking, such an estimate would then allow one to conclude that solutions to the free boundary Euler equations with the Taylor sign condition can be continued as long A remains bounded and $B_{\text{diff}} \in L^1_t$.

However, there is one issue with the above estimates. Note that the control parameter A in (1.10) depends only on the Hölder norms of our main variables (the surface and the velocity) at (nearly) the correct scale. However, the control parameter B_{diff} in (1.11) depends also on the auxiliary variable $D_t p$. From the point of view of the analysis of the free boundary Euler equations, this is completely natural. Indeed, even at the level of the linearized equation, one sees that the uniform norm of $\nabla D_t p$ (or more specifically the uniform norm of $D_t a$, but these are essentially equivalent) appears as a control parameter for the L^2 energy estimates in Proposition 2.2. On the other hand, for the purpose of providing a clear and physical description of how solutions to the free boundary Euler equations break down, we would ultimately like to use the control parameter $B := B^\sharp$ defined in (1.8), which depends only on the Hölder norms of Γ and v. To achieve this, our key observation is that, as long as $k > \frac{d}{2} + 1$, we can use a log of the energy to absorb endpoint losses, and hence prove an estimate of the form

$$\|D_t p\|_{W^{1,\infty}_x(\Omega_t)} \lesssim_A \log(1 + E^k) B. \tag{1.14}$$

An estimate akin to (1.14) is not to be expected in the difference estimates, as the distance functional is too low of regularity to absorb the logarithmic divergences inevitably arising from C^1 and $W^{1,\infty}$ elliptic estimates. With the above discussion in mind, the actual energy estimates we prove can be essentially stated as follows.

Theorem 1.5 (Energy estimates)**.** *Fix a collar neighborhood Λ_*, let $s \in \mathbb{R}$ with $s > \frac{d}{2} + 1$ and let $k > \frac{d}{2} + 1$ be an integer. Then for Γ restricted to Λ_* there exists an energy functional $\mathbf{H}^k \ni (v, \Gamma) \mapsto E^k(v, \Gamma)$ such that*

(i) (Energy coercivity).

$$E^k(v, \Gamma) \approx_A \|(v, \Gamma)\|^2_{\mathbf{H}^k}. \tag{1.15}$$

(ii) (Energy propagation). If, in addition to the above, $(v, \Gamma) = (v(t), \Gamma_t)$ is a solution to the free boundary incompressible Euler equations, then $E^k(t) := E^k(v(t), \Gamma_t)$ satisfies

$$\frac{d}{dt} E^k \lesssim_A B \log(1 + \|(v, \Gamma)\|_{\mathbf{H}^s}) E^k. \tag{1.16}$$

Here, A is as in (1.10) *and $B = B^\sharp$.*

By Grönwall's inequality, (1.15) and (1.16) yield the following single and double exponential bounds of the type

$$\begin{aligned} &\|(v(t), \Gamma_t)\|^2_{\mathbf{H}^k} \lesssim_A \exp\left(\int_0^t C_A B \log(1 + \|(v, \Gamma)\|_{\mathbf{H}^s}) ds\right) \|(v_0, \Gamma_0)\|^2_{\mathbf{H}^k}, \\ &\|(v(t), \Gamma_t)\|^2_{\mathbf{H}^k} \lesssim_A \exp\left(\log(1 + C_A \|(v_0, \Gamma_0)\|^2_{\mathbf{H}^k}) \exp \int_0^t C_A B \, ds\right), \end{aligned} \tag{1.17}$$

for all integers $k > \frac{d}{2} + 1$. We do not directly prove the analogue of Theorem 1.5 for noninteger exponents k. Nevertheless, as a consequence of our analysis in the last section of the paper, we do obtain the bounds (1.17) also for noninteger k. This is achieved by using frequency envelopes in order to combine the distance functional and the energy estimates akin to a nonlinear Littlewood-Paley type theory. It is also worth noting that a similar double exponential growth rate for the $L^1_T L^\infty_x$ norm of the vorticity appears in the classical Beale-Kato-Majda [8] criteria as a consequence of trying to weaken the natural control parameters of the problem.

In order to understand the form of the energy functionals used in Theorem 1.5, a key step is to identify Alinhac style *good variables* for the problem, which are as follows:

(i) The vorticity ω, which is measured in $H^{k-1}(\Omega)$.
(ii) The Taylor coefficient a, which is measured in $H^{k-1}(\Gamma)$.

(iii) The material derivative $D_t a$ of the Taylor coefficient, which is measured in $H^{k-\frac{3}{2}}(\Gamma)$.

Our energy functionals are constructed as certain combinations of well-chosen norms of the above good variables. The general strategy for constructing these norms is to apply appropriate vector fields and elliptic operators to ω, a and $D_t a$ at the $\mathbf{H}^k$ regularity in such a way that the resulting variables solve the linearized equation to leading order. After this, the nonlinear energy E^k may be essentially defined as the linear energy evaluated at these good variables. As it turns out, after completing this process, we arrived at essentially the same energy as [17], which was derived by different means. However, as can be immediately inferred from our control norms, the way we treat the energy is very different from [17]. Indeed, without going into details, we mention that the proof of Theorem 1.5 requires not only a delicate analysis of the fine structure and cancellations present in the free boundary Euler equations, but also the use of a new family of refined elliptic estimates. Although we refrain from stating them here in the introduction, these elliptic estimates serve as an important part of the paper. Moreover, since they are quite general, we believe that they will prove to be useful in other problems as well.

1.3.4. *Low regularity continuation criterion.* A very natural objective in the study of the Euler equations is to find a geometric characterization of how solutions break down. For the Euler equations without free boundary, this direction traces back to the famous paper of Beale, Kato and Majda [8]. In recent years, interest in sharp blow up criterion for the free boundary Euler equations has risen, and progress has been made by de Poyferré [16], Ginsberg [25], Wang and Zhang [46] and Wang, Zhang, Zhao and Zheng [47]. Here, we explain our rather definitive answer to this question, which is essentially a consequence of our local well-posedness result in Theorem 1.4 and the energy estimates in Theorem 1.5. However, to avoid topological issues, we must first introduce a notion of thickness for the fluid domain.

Definition 1.6. The fluid domain Ω has thickness at least $R > 0$ if for each $x \in \Gamma$, $B(x, R) \cap \Gamma$ is the graph of a $C^{1,\epsilon}$ function which separates $B(x, R)$ into two connected components.

With this notion in hand, our continuation criterion reads as follows:

Theorem 1.7 (Continuation criterion). *A solution $(v, \Gamma) \in C(\mathbf{H}^s)$, $s > \frac{d}{2} + 1$, of the free boundary incompressible Euler equations with the Taylor sign condition can be continued for as long as the following properties hold:*

a) (Uniform bound from below for the Taylor coefficient). There is a $c > 0$ such that

$$a \geq c > 0.$$

b) (Uniform thickness). There is an $R > 0$ such that Ω_t has thickness at least R.

c) (Control parameter bounds). The control parameters satisfy

$$A \in L^\infty_t, \qquad B \in L^1_t.$$

One may compare our continuation criteria for the free boundary problem with the classical Beale-Kato-Majda criteria for the boundaryless problem and note that they are essentially at the same level, with the natural addition of the $C^{1,\frac{1}{2}}$ boundary regularity bound. Another minor difference is that we use the Lipschitz bound on the velocity v rather than the uniform bound on the vorticity ω. One may ask whether it is possible to further relax our criterion in order to use only the vorticity bound. The major obstruction is that while in fixed domains the vorticity uniquely determines the velocity, in our case an appropriate boundary condition is also needed, which is best described via the $D_t a$ good variable. So, a potential conjecture might be that in order to use only the vorticity bound in the interior, one might have to compensate by adding a uniform bound on $D_t a$, as seen in the linear control parameter B_{lin} and in the difference estimates. That being said, in this paper we have opted for a continuation criteria involving only the natural variables v and Γ and no auxiliary pressure related terms.

As mentioned above, several recent articles [16, 25, 46, 47] have focused on obtaining improved continuation criterion for the free boundary Euler equations. The most significant of these contributions is the memoir [47], which proves that H^k solutions to the free boundary Euler equations with the Taylor sign condition can be continued after $t = T$ as long as properties a) and b) in Theorem 1.7 hold and

$$\sup_{t\in[0,T]} \left(\|\kappa(t)\|_{(L^p\cap L^2)(\Gamma_t)} + \|v(t)\|_{W^{1,\infty}(\Omega_t)}\right) < \infty \text{ for some } p > 2d-2. \tag{1.18}$$

Here, κ denotes the mean curvature of the surface. To motivate their result, [47] recalls a question of Craig and Wayne [30], which asks one to find (in the context of the irrotational water waves problem) the lowest Hölder regularity of the surface and velocity potential whose boundedness on $[0, T]$ implies that one can continue the solution past $t = T$. Although (1.18) makes significant progress on this question, it fails to achieve purely pointwise norms and is far from scale invariant. Moreover, the criterion (1.18) only applies to solutions which a priori live in integer based Sobolev spaces H^k. This limits the applicability of (1.18) to solutions with at least a half derivative of excess regularity. In contrast, Theorem 1.7 replaces the criterion $v \in L^\infty_T W^{1,\infty}_x$ by the sharp and scale invariant criterion $v \in L^1_T W^{1,\infty}_x$, and only requires control of Hölder norms of the free surface at the correct scale. In particular, Theorem 1.7 gives a rather definitive answer to Craig and Wayne's question for the full free boundary Euler equations. For the state-of-the-art result for the two-dimensional irrotational water waves problem, see [3]. Also, note that Theorem 1.7 applies to solutions in all Sobolev spaces $\mathbf{H}^s$ with $s > \frac{d}{2} + 1$, not just to those in integer spaces. This improvement is by no means trivial; rather, it follows from a careful usage of our distance functional.

1.4. **Outline of the paper.**

The article has a modular structure, where, for the essential part, only the main results of each section are used later.

1.4.1. *The linearized equations.* The starting point for our analysis, in Section 2, is to derive the linearization of our problem in Eulerian coordinates. The linearized system will serve as a guide to several of the choices made in our nonlinear analysis. In particular, it will suggest the correct variables to use, as well as the form of our distance functional. Moreover, when proving energy estimates, the Alinhac style good variables we construct will be shown to solve the linearized equations to leading order. This is also where the control parameters A and B_{lin} (an enhanced version of B) make their first appearance.

1.4.2. *Function spaces and the geometry of moving domains.* Section 3 describes the appropriate functional setting for our analysis. We begin by setting up a basic framework for our problem, including introducing low regularity control neighborhoods which will allow us to establish uniform control over constants in Sobolev and elliptic estimates in certain topologies for an appropriate family of domains. After defining the function spaces and norms that we will be using, we define the state space $\mathbf{H}^s$ where we will seek solutions to the free boundary Euler equations. Unlike in problems on fixed domains, the state space $\mathbf{H}^s$ will not be linear. However, it will be equipped with an appropriate notion of convergence, allowing us to define continuity of functions with values in $\mathbf{H}^s$ as well continuity of the data-to-solution map.

1.4.3. *Stability estimates and uniqueness.* The aim of Section 4 is to construct a nonlinear distance functional which will allow us to track the distance between two solutions at very low regularity. The general scheme is akin to the difference bounds in a weaker topology which are common in the study of quasilinear problems on fixed domains. However, here there are fundamental difficulties to overcome, as we are seeking to not only compare functions on different domains, but also track the evolution in time of this distance. These difficulties are embedded into the nonlinear character of our distance functional; both careful choices and delicate estimates are required to propagate this distance forward in time. To the best of our knowledge, this is only the second time difference estimates have been successfully proven in the free boundary setting. The other successful execution, which conceptually inspired the present approach, was in the case of a compressible gas [18, 27], which is very different from the incompressible liquid we consider here. In particular, unlike in the gas case, the boundary of our fluid contains non-trivial energy, requiring interesting geometric insights to understand.

As a consequence of our stability estimates, we deduce uniqueness of solutions at very low regularity. Also, as we shall see in later sections, the low regularity distance bounds we prove will serve both as an essential building block in our construction of rough solutions as unique limits of regular solutions as well as in the proof of the continuity of the data-to-solution map.

1.4.4. *Elliptic theory.* The main goal of Section 5 is to introduce a new family of refined elliptic estimates which will be crucial for obtaining the sharp pointwise control norms in the higher energy bounds. The secondary objective of Section 5 is to define a relevant Littlewood-Paley theory, collect various "balanced" product, Moser and Sobolev type estimates, and note several identities for operators and functions defined on moving domains. For the most part, the material in Section 5 does not rely on any specific structure of the Euler equations, so should be applicable to other free boundary problems as well. In Section 6, we construct the regularization operators which we will need for our existence scheme and the frequency envelopes for states $(v, \Gamma) \in \mathbf{H}^s$ that we will use to establish the refined properties of the data-to-solution map.

1.4.5. *Energy estimates.* In Section 7 we establish energy estimates within the $\mathbf{H}^k$ scale of spaces. As a first step, we construct a coercive energy functional $(v, \Gamma) \mapsto E^k(v, \Gamma)$ associated to each integer $k > \frac{d}{2} + 1$. The scheme here is to identify Alinhac style "good variables" (w_k, s_k) which solve the linearized equation modulo perturbative source terms. We then define our energy as the sum of the rotational energy and the linearized energy evaluated at these good variables. To prove the energy estimates, we split the argument in a modular fashion into two parts. First, we prove the coercivity of our energy functional; that is, we show that $E^k(v, \Gamma) \approx \|(v, \Gamma)\|^2_{\mathbf{H}^k}$. After this, we track the time evolution of the energy, establishing control of $E^k(v, \Gamma)$ in terms of the initial data, with growth dictated by the pointwise control parameters A and B. Both steps of this argument are delicate. In particular, the former makes extensive use of the refined elliptic estimates from Section 5, and the latter requires us to identify and exploit various structural properties and fine cancellations present in the Euler equations.

1.4.6. *Construction of regular solutions.* Section 8 is devoted to the construction of regular solutions to the free boundary Euler equations. The overarching scheme we utilize is similar to [27], which analyzed the case of a compressible gas. However, we stress that the main difficulties in the incompressible liquid case are quite different than for the gas, especially near the free boundary, as the surface of a liquid carries a non-trivial energy. As a general overview, the scheme we utilize is constructive, employing a time discretization via an Euler type method together with a separate transport step to produce good approximate solutions.

However, a naïve implementation of Euler's method loses derivatives. To overcome this, we ameliorate the derivative loss by an initial regularization of each iterate in our discretization. To ensure that the uniform energy bounds survive, such a regularization needs to be chosen carefully. For this, we employ a modular approach and try to decouple this process into two steps, where we regularize individually the domain and the velocity. We believe that this modular approach will serve as a recipe for a new and relatively simple method for constructing solutions to various free boundary problems. That being said, the execution of this scheme is still quite subtle, requiring several novel ideas in addition to those coming from [27].

1.4.7. *Rough solutions and continuous dependence.* The last section of the paper aims to construct rough solutions as strong limits of smooth solutions. This is achieved by considering a family of dyadic regularizations of the initial data, which generate corresponding smooth solutions. For these smooth solutions we control on one hand higher Sobolev norms $\mathbf{H}^k$, using our energy estimates, and on the other hand the L^2 type distance between consecutive ones, from our difference estimates. Combining the high and the low regularity bounds directly yields rapid convergence in all $\mathbf{H}^l$ spaces for $l < k$. To gain strong convergence in $\mathbf{H}^k$, we use frequency envelopes to more accurately control both the low and the high Sobolev norms above. This allows us to bound differences in the strong $\mathbf{H}^k$ topology. Interpolation and a similar argument yields local existence in fractional Sobolev spaces as well as continuous dependence of the solutions in terms of the initial data in the strong topology. Finally, our main continuation result in Theorem 1.7 follows along similar lines, given the careful treatment of our control norms in the energy and difference estimates.

For problems on $\mathbb{R}^d$, the scheme outlined above for obtaining rough solutions from smooth solutions, good energy estimates and difference estimates is more classical; see the expository article [28]. However, as we shall see, the fact that solutions are all defined on different domains leads to some new subtleties in our free boundary setting.

1.5. **Acknowledgements.** The first author was supported by the NSF CAREER grant DMS-1845037, by the Miller Foundation and by a Simons Fellowship. The other three authors were supported by the NSF grant DMS-2054975 as well as by a Simons Investigator grant from the Simons Foundation. Some of this work was carried out while all four authors were in residence at the Simons Laufer Mathematical Sciences Institute (formerly MSRI) in Berkeley, California, during the summer of 2023, participating in the program "Mathematical problems in fluid dynamics, Part II", which was supported by the National Science Foundation under Grant No. DMS-1928930.

2. The linearized equation

The first goal of this section is to formally derive the linearization of our problem, working entirely in Eulerian coordinates; this is the system of equations (2.6). Then, we prove Theorem 2.1, which asserts that the linearized system is well-posed in L^2, with energy bounds determined by our sharp control parameters. The key elements here are the linearized energy (2.9) and the basic energy estimate (2.10).

Conceptually, the linearized system is an essential piece of the puzzle. On a practical level, however, it is not immediately useful in proving well-posedness, as it is not clear that C^1 one parameter families of solutions exist in the first place. It is only a posteriori, after well-posedness is established, that the linearized energy estimates may be used to derive bounds for differences of solutions. Instead, we will use our understanding of the linearized system to guide us in our choice of distance functional in Section 4 and later in our choice of energy functionals in Section 7.

To derive the linearized system, we take a one parameter family of solutions (v_h, p_h) defined on domains $\Omega_{t,h}$, with $(v_0, p_0) := (v, p)$ and $\Omega_{t,0} := \Omega_t$. We define $w = \partial_h v_h|_{h=0}$ and $q = \partial_h p_h|_{h=0}$.

In Ω_t, the linearized equation is rather standard:

$$\begin{cases} \partial_t w + w \cdot \nabla v + v \cdot \nabla w = -\nabla q, \\ \nabla \cdot w = 0. \end{cases}$$

However, we also need to linearize the kinematic and dynamic boundary conditions on the surface Γ_t. For this, let us denote by $\Gamma_{t,h}$ the free surface at time t for the solution (v_h, p_h), so $\Gamma_{t,0} := \Gamma_t$. Fix a one parameter family of diffeomorphisms $\phi_h(t) : \Gamma_t \to \Gamma_{t,h}$, with $\phi_0(t) = Id_{\Gamma_t}$. The dynamic boundary condition (1.3) asserts that for every point $x \in \Gamma_t$,

$$p_h(t, \phi_h(t)(x)) = 0.$$

Differentiating in h and evaluating at $h = 0$ gives

$$q|_{\Gamma_t} = -\nabla p|_{\Gamma_t} \cdot \psi(t),$$

where $\psi(t) := \frac{\partial}{\partial h}\phi_h(t)|_{h=0}$. Using that $\nabla p|_{\Gamma_t}$ is normal to Γ_t we deduce that

$$q|_{\Gamma_t} = -\nabla p|_{\Gamma_t} \cdot n_{\Gamma_t} \psi(t) \cdot n_{\Gamma_t} =: as. \tag{2.1}$$

Here, we define $s := \psi(t) \cdot n_{\Gamma_t}$ which we loosely interpret as the normal velocity in the parameter h of the family $\Gamma_{t,h}$ at $h = 0$. We will use this as one of our linearized variables. Note that since $a > 0$, s does not depend on the choice of diffeomorphisms $\phi_h(t)$.

Next, we linearize the kinematic boundary condition. Analogously to $v \cdot n_{\Gamma_t}$ describing the normal velocity of the free surface, we expect $w \cdot n_{\Gamma_t}$ to describe the "normal velocity" of our linearized variable s. Therefore, up to a perturbative error, $D_t s$ should agree with $w \cdot n_{\Gamma_t}$. In fact, we obtain the relation

$$D_t s - w \cdot n_{\Gamma_t} = s(n_{\Gamma_t} \cdot \nabla v) \cdot n_{\Gamma_t}. \tag{2.2}$$

To derive (2.2), we note that (1.2) and (1.3) imply that

$$D_t p = 0 \quad \text{on } \Gamma_t. \tag{2.3}$$

This is the equation that we will linearize to obtain (2.2). As before, let $\phi_h(t) : \Gamma_t \to \Gamma_{t,h}$ be a diffeomorphism. We then have for $x \in \Gamma_t$,

$$[(\partial_t + v_h \cdot \nabla)p_h](t, \phi_h(t)(x)) = 0.$$

Taking h derivative and evaluating at $h = 0$ yields,

$$w \cdot \nabla p + D_t q + \nabla D_t p \cdot \psi = 0 \ \text{ on } \Gamma_t. \tag{2.4}$$

Using (2.1), and that $\nabla D_t p$ is normal to Γ_t by (2.3), we deduce (2.2) from (2.4) after some simple algebraic manipulation. Indeed, we have $\nabla p|_{\Gamma_t} = -a n_{\Gamma_t}$. Then using the relation $q_{|\Gamma_t} = as$, we compute $D_t q = aD_t s + sD_t a$. This reduces (2.4) to

$$-aw \cdot n_{\Gamma_t} + aD_t s + sD_t a + s\nabla D_t p \cdot n_{\Gamma_t} = 0. \tag{2.5}$$

After division by a, the first two terms in (2.5) evidently align with the left-hand side of (2.2). The right-hand side of (2.2) appears by commuting the gradient with the material derivative in the last term of (2.5), and by using the fact that $\nabla p \cdot D_t n_{\Gamma_t} = 0$ to rewrite $sD_t a = -sD_t(\nabla p \cdot n_{\Gamma_t}) = -sD_t \nabla p \cdot n_{\Gamma_t}$.

Putting everything together, the linearized system takes the form:

$$\begin{cases} D_t w + \nabla q = -w \cdot \nabla v \quad \text{in } \Omega_t, \\ \nabla \cdot w = 0 \quad \text{in } \Omega_t, \\ D_t s - w \cdot n_{\Gamma_t} = s(n_{\Gamma_t} \cdot \nabla v) \cdot n_{\Gamma_t} \quad \text{on } \Gamma_t, \\ q = as \quad \text{on } \Gamma_t, \end{cases} \tag{2.6}$$

where the terms on the right-hand side can be viewed as perturbative source terms.

In order to study the well-posedness of the linearized system (2.6), we introduce an enhanced version B_{lin} of the control parameter $B^\sharp$:

$$B_{lin}(t) := \|a^{-1} D_t a\|_{L^\infty(\Gamma_t)} + \|\nabla v\|_{L^\infty(\Omega_t)}. \tag{2.7}$$

Using this, we may state our main linearized well-posedness result as follows.

Theorem 2.1. *Let (v, Γ) be a solution to the free boundary incompressible Euler equations in a time interval $[0, T]$ so that $a > 0$, $A^\sharp$ stays uniformly bounded and $B_{lin} \in L^1_T$. Then the linearized system (2.6) for (w, s) is well-posed in $L^2(\Omega) \times L^2(\Gamma)$ in $[0, T]$.*

Here we recall that Ω and Γ are time dependent. The rest of this section is devoted to the proof of this very simple theorem. The basic strategy is to construct a suitable energy functional and prove corresponding energy estimates. Once this is done, well-posedness follows via a standard duality argument, which is left for the reader. To execute this argument, one simply notes that the adjoint system is essentially identical to the direct system (2.6), modulo perturbative terms, and that the energy estimates are time reversible.

Below, we will work with a slightly more general system, since this is what will appear in the higher order energy bounds later on. We define the *generalized linearized system* as follows:

$$\begin{cases} D_t w + \nabla q = f \quad \text{in } \Omega_t, \\ \nabla \cdot w = 0 \quad \text{in } \Omega_t, \\ D_t s - w \cdot n_{\Gamma_t} = g \quad \text{on } \Gamma_t, \\ q = as \quad \text{on } \Gamma_t, \end{cases} \tag{2.8}$$

where we allow for arbitrary source terms f and g on the right-hand side of the first and third equation.

It remains to prove a suitable energy estimate for the system (2.8). The natural energy associated to this system is

$$E_{lin}(w, s)(t) = \frac{1}{2} \int_{\Omega_t} |w|^2 \, dx + \frac{1}{2} \int_{\Gamma_t} as^2 \, dS. \tag{2.9}$$

Using (2.9), the main energy estimate for the generalized linear system is as follows:

Proposition 2.2. *Suppose $a > 0$. Then the system (2.8) satisfies the energy estimate*

$$\frac{d}{dt}E_{lin}(w,s)(t) \leq B_{lin}E_{lin}(w,s)(t) + \langle as, g\rangle_{L^2(\Gamma_t)} + \langle w, f\rangle_{L^2(\Omega_t)}. \tag{2.10}$$

We note that the energy functional (2.9) is also the energy functional for the linearized system (2.6), and that this proposition yields energy estimates for (2.6), thereby concluding the proof of Theorem 2.1.

Proof. We will make use of the following standard Leibniz type formulas (see; for example, [20, Appendix A]).

Proposition 2.3. *(i) Assume that the time-dependent domain Ω_t flows with Lipschitz velocity v. Then the time derivative of the time-dependent volume integral is given by*

$$\frac{d}{dt}\int_{\Omega_t} f(t,x)\,dx = \int_{\Omega_t} D_t f + f\nabla\cdot v\,dx.$$

(ii) Assume that the time-dependent hypersurface Γ_t flows with divergence free velocity v. Then the time derivative of the time-dependent surface integral is given by

$$\frac{d}{dt}\int_{\Gamma_t} f(t,x)\,dS = \int_{\Gamma_t} D_t f - f(n_{\Gamma_t}\cdot\nabla v)\cdot n_{\Gamma_t}\,dS.$$

Now, to prove the energy estimate (2.10), we apply Proposition 2.3 to obtain

$$\begin{aligned}\frac{d}{dt}E_{lin}(w,s)(t) &= \int_{\Omega_t} D_t w\cdot w\,dx + \int_{\Gamma_t} asD_t s\,dS + \frac{1}{2}\int_{\Gamma_t} D_t as^2\,dS - \frac{1}{2}\int_{\Gamma_t}[n_{\Gamma_t}\cdot\nabla v\cdot n_{\Gamma_t}]as^2\,dS\\ &\leq \int_{\Omega_t} D_t w\cdot w\,dx + \int_{\Gamma_t} asD_t s\,dS + B_{lin}E_{lin}(w,s)(t).\end{aligned} \tag{2.11}$$

Integrating by parts, we obtain

$$\begin{aligned}\int_{\Omega_t} D_t w\cdot w\,dx + \int_{\Gamma_t} asD_t s\,dS &= \int_{\Omega_t} w\cdot f\,dx + \int_{\Gamma_t} asD_t s\,dS - \int_{\Gamma_t} qw\cdot n_{\Gamma_t}\,dS\\ &= \langle as, g\rangle_{L^2(\Gamma_t)} + \langle w, f\rangle_{L^2(\Omega_t)}.\end{aligned}$$

Combining this with (2.11) completes the proof. □

3. Analysis on moving domains

One difficulty when working directly on moving domains is that many of the standard Sobolev and elliptic estimates have domain dependent constants. It is therefore necessary to work in a framework which allows for uniform control of these constants in certain topologies. This section is devoted to dealing with this issue. Our approach in this regard is somewhat analogous to that of Shatah and Zeng [41, 42, 43] and de Poyferré [17, Section 3], but with the key difference being that our control neighborhoods will only be uniform in the pointwise C^1 or $C^{1,\epsilon}$ topologies as opposed to the stronger L^2 based topologies considered in those papers. This will be essential for establishing the pointwise continuation criterion for solutions.

3.1. Function spaces.

To begin, we precisely define the function spaces and norms that we will be using. Throughout, $\Omega \subseteq \mathbb{R}^d$ will denote a bounded, connected domain. We define $H^s(\Omega)$, $s \geq 0$, as the set of all $f \in L^2(\Omega)$ such that

$$\|f\|_{H^s(\Omega)} := \inf\left\{\|F\|_{H^s(\mathbb{R}^d)} : F \in H^s(\mathbb{R}^d),\ F|_\Omega = f\right\} \tag{3.1}$$

is finite. Here, $\|\cdot\|_{H^s(\mathbb{R}^d)}$ is defined in the standard way, via the Fourier transform. We let $H^s_0(\Omega)$ denote the closure of $C^\infty_0(\Omega)$ in $H^s(\Omega)$ and identify $H^{-s}(\Omega)$ isometrically with the dual space $(H^s_0(\Omega))^*$. Importantly, with this definition of the H^s norm, the constants in Sobolev embedding theorems (either $H^s \to L^p$ or $H^s \to C^\alpha$) are independent of Ω. For regular enough domains and integer s, the norm defined in (3.1) is equivalent to the standard one. We will precisely quantify this equivalence later.

We next define the regularity of the boundary of a connected domain Ω, which is characterized in terms of the regularity of local coordinate parameterizations of $\partial\Omega$. Indeed, in general, an m-dimensional manifold $\mathcal{M} \subseteq \mathbb{R}^d$ is said to be of class $C^{k,\alpha}$ or H^s, $s > \frac{d}{2}$, if, locally in linear frames, $\mathcal{M}$ can be represented by graphs with the same regularity.

If $s > \frac{d+1}{2}$, then given Ω as above with boundary of class H^s, we can define what it means to be an H^r function on $\partial\Omega$ for $s \geq r \geq -s$. Indeed, these are simply the functions whose coordinate representatives are locally in $H^r(\mathbb{R}^{d-1})$. It is easy to see that the space of H^r functions on $\partial\Omega$, $s \geq r \geq -s$, can be made into a Banach space. Indeed, a norm can be chosen by taking a covering of $\partial\Omega$ by a finite number of coordinate patches and an adapted partition of unity. However, there is one problem with this approach. Although such a norm is well-defined up to equivalence, the precise value of the norm is dependent on the choice of local coordinates. Since we will be dealing with a family of domains, we need to make sure that we define norms on their boundaries in a consistent and uniform way.

3.2. **Collar coordinates.** As a first step towards resolving the above issue, we fix a bounded, connected reference domain Ω_* with smooth boundary $\Gamma_* := \partial\Omega_*$. We define H^s and $C^{k,\alpha}$ based norms on Γ_* by making an appropriate choice of local parameterizations of Γ_*. Letting $\delta > 0$ be a small positive constant, we define $N(\Gamma_*, \delta)$ to be the collection of all C^1 hypersurfaces Γ such that there exists a C^1 diffeomorphism $\Phi_\Gamma : \Gamma_* \to \Gamma$ with

$$\|\Phi_\Gamma - id_{\Gamma_*}\|_{C^1(\Gamma_*)} < \delta.$$

If $\delta > 0$ is small enough, we can represent hypersurfaces $\Gamma \in N(\Gamma_*, \delta)$ as graphs over Γ_*. Indeed, we denote the outward unit normal to Γ_* by n_{Γ_*}. Following [43, Section 2.1], if we have a smooth unit vector field $\nu : \Gamma_* \to \mathbb{S}^{d-1}$ which is suitably transversal to Γ_* (that is, $\nu \cdot n_{\Gamma_*} > 1 - c$ for some small $c > 0$), it follows from the implicit function theorem that there exists a $\delta > 0$, determined by Γ_* and ν, such that the map

$$\varphi : \Gamma_* \times [-\delta, \delta] \to \mathbb{R}^d, \quad \varphi(x, \mu) = x + \mu\nu(x)$$

is a C^1 diffeomorphism from its domain to a collar neighborhood of Γ_*. If $\delta > 0$ is small enough, the above coordinate system associates each hypersurface $\Gamma \in N(\Gamma_*, \delta)$ with a unique function $\eta_\Gamma : \Gamma_* \to \mathbb{R}$ such that

$$\Phi_\Gamma(x) := \varphi(x, \eta_\Gamma(x)) = x + \eta_\Gamma(x)\nu(x) \tag{3.2}$$

is a diffeomorphism in $C^1(\Gamma_*, \Gamma \subseteq \mathbb{R}^d)$. We can think of the map Φ_Γ as a way to represent Γ as a (global) graph over Γ_*. With this notation in hand, we can now define what it means to be a H^s hypersurface which is close to Γ_*.

Definition 3.1. For $\delta > 0$ small enough and $\alpha \in [0, 1)$, define the control neighborhood $\Lambda(\Gamma_*, \alpha, \delta)$ as the collection of all hypersurfaces $\Gamma \in N(\Gamma_*, \delta)$ such that the associated map $\eta_\Gamma : \Gamma_* \to \mathbb{R}$ satisfies

$$\|\eta_\Gamma\|_{C^{1,\alpha}(\Gamma_*)} < \delta.$$

Definition 3.2. Suppose $s \geq 0$, $\Gamma \in N(\Gamma_*, \delta)$ for $\delta > 0$ small enough, and the associated map $\eta_\Gamma : \Gamma_* \to \mathbb{R}$ satisfies $\eta_\Gamma \in H^s(\Gamma_*)$. We then define the H^s norm of Γ by

$$\|\Gamma\|_{H^s} := \|\eta_\Gamma\|_{H^s(\Gamma_*)}.$$

In the above definitions, $\|\eta_\Gamma\|_{C^{1,\alpha}(\Gamma_*)}$ and $\|\eta_\Gamma\|_{H^s(\Gamma_*)}$ are computed with respect to fixed, independent of Γ, local coordinates on Γ_*. In an analogous way, we define for $\gamma \in [0,1)$ and integers $k \geq 0$, the $C^{k,\gamma}$ norm, $\|\Gamma\|_{C^{k,\gamma}}$. As was essentially noted in [43, Section 2.1], when $0 < \delta \ll 1$, each $\Gamma \in \Lambda(\Gamma_*, \alpha, \delta)$ is associated to a well-defined domain Ω.

Remark 3.3. One key point in Definition 3.1 is that we only require Γ be close to Γ_* in the $C^{1,\alpha}$ topology, as opposed to the stronger L^2 based topologies used in [17, 41, 42, 43]. In practice, we will want the control topology to be as weak as possible. For our purposes, we will typically take $\alpha = \epsilon > 0$ for some arbitrarily small (but fixed) constant $\epsilon > 0$.

Remark 3.4. A second key point in Definition 3.1 concerns the choice of the small parameter δ. This will not be arbitrarily small, but instead its size may also be chosen to depend on weaker topologies; namely, (i) the $C^{1,\epsilon}$ norm of Γ_* and (ii) the thickness (see Definition 1.6) of the domain Ω. This will serve two purposes:

- To allow us to place any rough H^s boundary Γ within a suitable control neighborhood $\Lambda(\Gamma_*, \epsilon, \delta)$.
- To allow us to obtain the robust continuation result in Theorem 1.7, which does not require any reference to control neighborhoods.

Following the discussion in the above two remarks, throughout the article we will often abbreviate $\Lambda(\Gamma_*, \epsilon, \delta)$ by Λ_*, where the suppressed parameters $\varepsilon > 0$ and $\delta > 0$ are understood to be small but fixed universal parameters, which depend only on s and on the thickness of Ω.

3.3. **State space.** Fix a collar neighborhood Λ_* and $s > \frac{d}{2} + 1$. We define $\mathbf{H}^s$ as the set of all pairs (v, Γ) such that $\Gamma \in \Lambda_*$ is the boundary of a bounded, connected domain Ω and such that the following properties are satisfied:

(i) (Regularity). $v \in H^s_{div}(\Omega)$ and $\Gamma \in H^s$, where $H^s_{div}(\Omega)$ denotes the space of divergence free vector fields in $H^s(\Omega)$.

(ii) (Taylor sign condition). $a := -\nabla p \cdot n_\Gamma > c_0 > 0$, where c_0 may depend on the choice of (v, Γ), and the pressure p is obtained from (v, Γ) by solving the standard elliptic equation (1.5) associated to (1.1) and (1.3).

Given initial data (v_0, Γ_0) in the state space $\mathbf{H}^s$, our eventual goal will be to construct local solutions $(v(t), \Gamma_t)$ that evolve continuously in $\mathbf{H}^s$. To accomplish this, we must define a suitable notion of topology on our state space. This will enable us to establish two key properties of our flow; namely,

(i) Continuity of solutions with values in $\mathbf{H}^s$.

(ii) Continuous dependence of solutions $(v(t), \Gamma_t)$ as functions of the initial data (v_0, Γ_0).

Note that since $\mathbf{H}^s$ is not a linear space, the above two continuity properties require some explanation. To measure the size of individual states $(v, \Gamma) \in \mathbf{H}^s$, we define $\|(v, \Gamma)\|^2_{\mathbf{H}^s} := \|\Gamma\|^2_{H^s} + \|v\|^2_{H^s(\Omega)}$. However, since $\mathbf{H}^s$ is not a linear space, $\|\cdot\|_{\mathbf{H}^s}$ does not induce a norm topology in the usual sense. Hence, we still need an appropriate way of comparing different states. Motivated by [18, 27], we define convergence in $\mathbf{H}^s$ as follows.

Definition 3.5. We say that a sequence $(v_n, \Gamma_n) \in \mathbf{H}^s$ converges to $(v, \Gamma) \in \mathbf{H}^s$ if

(i) (Uniform Taylor sign condition). For some $c_0 > 0$ independent of n, we have

$$a_n, a > c_0 > 0.$$

(ii) (Domain convergence). $\Gamma_n \to \Gamma$ in H^s. That is, $\eta_{\Gamma_n} \to \eta_\Gamma$ in $H^s(\Gamma_*)$ where η_{Γ_n} and η_Γ correspond to the collar coordinate representations of Γ_n and Γ, respectively.

(iii) (Norm convergence). For every $\epsilon > 0$ there exists a smooth divergence free function $\tilde{v}$ defined on a neighborhood $\tilde{\Omega}$ of $\overline{\Omega}$ with $\|\tilde{v}\|_{H^s(\tilde{\Omega})} < \infty$ and satisfying

$$\|v - \tilde{v}\|_{H^s(\Omega)} \leq \epsilon$$

and

$$\limsup_{n\to\infty} \|v_n - \tilde{v}\|_{H^s(\Omega_n)} \leq \epsilon.$$

With the above notion of convergence, it makes sense to define $C([0,T];\mathbf{H}^s)$. We remark, however, that in [17, 41, 42, 43], $C([0,T];\mathbf{H}^s)$ is defined in a slightly different way, via the existence of an extension to a continuous function with values in $H^s(\mathbb{R}^d)$. In Section 5.3.4, we construct a family of extension operators which depend continuously in a suitable sense on the domain, making the above two notions of continuity essentially interchangeable.

4. Difference estimates and uniqueness

Comparing different solutions is key to any well-posedness result. Since our problem is quasilinear, such a comparison cannot be achieved uniformly in the leading $\mathbf{H}^s$ topology, but instead only in weaker topologies. The main result of this section provides a Lipschitz bound for the distance between two solutions in the L^2 topology, akin to our bounds for the linearized equation. Notably, our distance bounds propagate at the level of our control parameters, which require for instance a Lipschitz bound on the velocity but no higher regularity. This is what will allow us to establish uniqueness of solutions under very weak regularity assumptions. Moreover, as we shall see shortly, these low regularity distance bounds also serve as an essential building block in our construction of rough solutions as unique limits of smooth solutions, as well as in our proof of the continuity of the data-to-solution map.

The fundamental difficulty in achieving our distance bounds is the need to compare states which live on different domains. To overcome this difficulty, we construct a "distance functional" which *simultaneously* captures the distance between (functions on) different domains and admits a time evolution that we are able to track. To the best of our knowledge, no such low regularity difference bounds or even uniqueness results were previously known for any incompressible free boundary Euler model. Instead, we take our cue from the work [27] of the first and the third authors, which considers a similar free boundary problem but for a compressible Euler model. We note, however, that the similarity between the uniqueness argument here and its counterpart in [27] is only at the conceptual level, as the two flows have very different behaviors both inside the domain and near the free boundary.

4.1. **The distance functional.** Our first objective is to use the linearized energy as a guide to construct a distance functional which will be suitable for comparing nearby solutions. We begin by fixing a collar neighborhood $\Lambda(\Gamma_*, \epsilon, \delta)$, where $\epsilon > 0$ and $\delta > 0$ are small. We then suppose that we have two states (v, Γ), (v_h, Γ_h) with respective domains Ω, Ω_h. We let η_Γ and η_{Γ_h} be the corresponding representations of Γ and Γ_h as graphs over Γ_*. Following the linearized energy estimate, we aim to define analogues of the linearized variables w and s, which heuristically should measure the L^2 distance between v and v_h and the distance between Γ and Γ_h, respectively. One technical caveat is that v and v_h are not defined on the same domain. For this reason, we define $\widetilde{\Omega} = \Omega \cap \Omega_h$. We can represent the free boundary $\widetilde{\Gamma}$ for $\widetilde{\Omega}$ as a graph over Γ_* via the function $\eta_{\widetilde{\Gamma}} = \eta_\Gamma \wedge \eta_{\Gamma_h}$. Note that although the graph representation $\eta_{\widetilde{\Gamma}}$ is well-defined, $\widetilde{\Gamma}$ is only Lipschitz in general, so will not be in $\Lambda(\Gamma_*, \epsilon, \delta)$.

To measure the (signed) distance between Γ and Γ_h, we define $s_h^*:\Gamma_*\to\mathbb{R}$ by

$$s_h^*(x)=\eta_{\Gamma_h}(x)-\eta_\Gamma(x). \tag{4.1}$$

As will become evident below, although s_h^* correctly measures the distance between the free hypersurfaces, it has the "wrong" domain. To fix this, we define the variable $s_h:\widetilde{\Gamma}\to\mathbb{R}$ by pushing s_h^* forward to the hypersurface $\widetilde{\Gamma}$. In other words, for $x\in\widetilde{\Gamma}$, we define $s_h(x)=s_h^*(\pi(x))$, where π denotes the canonical projection, mapping the image of $\Gamma_*\times[-\delta,\delta]$ under φ back to Γ_*. For convenience, we also extend ν to a vector field X defined on the image of φ via $X(x)=\nu(\pi(x))$. We will not actually use the displacement function s_h directly in the difference estimates below. In particular, it will not act as our desired analogue of the linearized variable s. This is because its dynamics are somewhat awkward to work with. Instead of using s_h, it is far more convenient (and geometrically natural) to use the the pressure difference $p-p_h$ (along with a suitable weight to be defined below) to measure the distance between Γ and Γ_h. To motivate this, recall that for solutions to the free boundary Euler equations, the Taylor sign condition implies that p and p_h are non-degenerate defining functions for Γ_t and $\Gamma_{t,h}$ within a suitable collar neighborhood. Therefore, on the boundary of $\tilde{\Omega}_t=\Omega_t\cap\Omega_{t,h}$, $p-p_h$ should be proportional to the displacement function s_h. The dynamics of $p-p_h$ turn out to be much easier to work with than those of s_h, as terms involving $p-p_h$ will appear naturally when we use the free boundary Euler equations to compare solutions.

With the above motivation in mind and using the linearized equation as a guide, we define our distance functional as follows:

$$D((v,\Gamma),(v_h,\Gamma_h)):=D(v,v_h):=\frac{1}{2}\int_{\widetilde{\Omega}}|v-v_h|^2\,dx+\frac{1}{2}\int_{\tilde{\Gamma}}b|p-p_h|^2\,dS, \tag{4.2}$$

where the weight function b is defined by

$$b:=a^{-1}1_{\tilde{\Gamma}\cap\Gamma}+a_h^{-1}1_{\tilde{\Gamma}\cap\Gamma_h}.$$

As $p-p_h$ vanishes on $\Gamma\cap\Gamma_h$, we may rewrite the distance functional in the slightly more convenient form

$$D(v,v_h)=\frac{1}{2}\int_{\tilde{\Omega}}|v-v_h|^2\,dx+\frac{1}{2}\int_{\mathcal{A}}a^{-1}|p-p_h|^2\,dS+\frac{1}{2}\int_{\mathcal{A}_h}a_h^{-1}|p-p_h|^2\,dS,$$

where $\mathcal{A}:=\tilde{\Gamma}\cap\Gamma-\Gamma\cap\Gamma_h$ and $\mathcal{A}_h:=\tilde{\Gamma}\cap\Gamma_h-\Gamma\cap\Gamma_h$.

Letting $\overline{F}$ denote the average of F along the flow φ between the free surfaces, the fundamental theorem of calculus implies that for $x\in\widetilde{\Gamma}$,

$$p_h(x)-p(x)=\begin{cases}-\overline{\nabla p_h\cdot X}s_h(x) & \text{if } x\in\mathcal{A},\\ -\overline{\nabla p\cdot X}s_h(x) & \text{if } x\in\mathcal{A}_h.\end{cases} \tag{4.3}$$

Therefore, thanks to the Taylor sign condition and assuming the regularity $p,p_h\in C^{1,\epsilon}$, we should have $|p-p_h|\approx|s_h|$ on $\widetilde{\Gamma}$ within a tight enough collar neighborhood. The precise manner in which we have this proportionality will be made clear shortly. Finally, note that, for solutions to the free boundary Euler equations, a simple computation yields the following equation for $v-v_h$ in $\tilde{\Omega}_t$:

$$\begin{cases}D_t(v-v_h)+\nabla(p-p_h)=(v_h-v)\cdot\nabla v_h,\\ \nabla\cdot(v-v_h)=0.\end{cases} \tag{4.4}$$

Remark 4.1. Although it is not particularly important for the difference estimates, we note that the distance functional (4.2) makes sense for general (not necessarily dynamical) states (v,Γ) and (v_h,Γ_h). Indeed, given suitable states (v,Γ) and (v_h,Γ_h), we can always associate pressures p and p_h by solving the standard elliptic equation associated to (1.1) and (1.3). As we will see in Section 7, it is very important that our energy functional for the $\mathbf{H}^k$ energy bounds be defined for general states $(v,\Gamma)\in\mathbf{H}^k$.

4.2. **Difference estimates.** We are now ready to propagate difference bounds for two solutions to the free boundary Euler equations.

Theorem 4.2 (Difference Bounds). *Let $0<\epsilon,\delta\ll 1$ and let $\Lambda_*=\Lambda(\Gamma_*,\epsilon,\delta)$ be a collar neighborhood. Suppose that (v,Γ_t) and $(v_h,\Gamma_{t,h})$ are solutions to the free boundary Euler equations that evolve in the collar in a time interval $[0,T]$ and satisfy a,$a_h>c_0>0$. Then we have the estimate*

$$\frac{d}{dt}D(v,v_h)\lesssim_{A,A_h}(B+B_h)D(v,v_h)$$

where

$$B:=\|v\|_{W^{1,\infty}(\Omega_t)}+\|\Gamma_t\|_{C^{1,\frac{1}{2}}}+\|D_tp\|_{W^{1,\infty}(\Omega_t)},\qquad A:=\|v\|_{C^{\frac{1}{2}+\epsilon}(\Omega_t)}+\|\Gamma_t\|_{C^{1,\epsilon}},$$

B_h and A_h are the analogous quantities corresponding to v_h, p_h, $D_t^hp_h$ and $\Gamma_{t,h}$ and we have implicitly assumed that our solutions have regularity $B,B_h\in L^1_T$ and $A,A_h\in L^\infty_T$.

Remark 4.3. It is worth remarking that all of the results in this section hold equally well if the control parameter B is replaced by

$$B_\epsilon=\|v\|_{C^{1,\epsilon}(\Omega_t)}+\|\Gamma_t\|_{C^{1,\frac{1}{2}}},$$

which depends solely on the regularity of v and Γ_t. This is because we will later prove an elliptic estimate of the form

$$\|D_tp\|_{W^{1,\infty}(\Omega_t)}\lesssim_A B_\epsilon.$$

See Lemma 7.9 and Remark 7.10 for details. We prefer, however, to work with the control parameter B defined above as its L^1_T norm is scale invariant.

Proof. For simplicity of notation, we drop the t subscript for domains below. We also use $\lesssim_A$ as a shorthand for $\lesssim_{A,A_h}$. To ensure that we can estimate expressions involving the pressure in terms of the control parameters A and B above, we need the bounds

$$\|p\|_{C^{1,\epsilon}(\Omega)}\lesssim_A 1,\qquad \|p\|_{C^{1,\frac{1}{2}}(\Omega)}\lesssim_A B, \tag{4.5}$$

as well as the analogous bounds for p_h. The proof that these bounds hold will be postponed until later when the requisite elliptic estimates are developed. See Lemma 7.5 and Lemma 7.9 for details. Now, to proceed with the difference estimate, we recall the identity

$$\frac{d}{dt}D(v,v_h)=\frac{1}{2}\frac{d}{dt}\int_{\tilde{\Omega}}|v-v_h|^2\,dx+\frac{1}{2}\frac{d}{dt}\int_{\mathcal{A}}a^{-1}|p-p_h|^2\,dS+\frac{1}{2}\frac{d}{dt}\int_{\mathcal{A}_h}a_h^{-1}|p-p_h|^2\,dS. \tag{4.6}$$

To compute the first term, we would like to use Reynolds' transport theorem, as in Proposition 2.3. However, here we do not have a good velocity field $\tilde{v}$ so that $\tilde{\Omega}$ flows with velocity $\tilde{v}$. Constructing such a field seems to be at the very least impractical, so we will instead allow for a correction term which is a boundary integral. For this purpose, suppose that $D(t)$ is a time-dependent domain for which we may define at almost every point of the boundary a normal velocity v_b for the boundary. Note that if $D(t)$ were flowing with velocity v, then $v_b=v\cdot n_{\partial D(t)}$, where $n_{\partial D(t)}$ is the outward unit normal. For more general velocity fields v on $D(t)$, we have the following proposition.

Proposition 4.4. *Given a velocity field v defined on a time-dependent domain $D(t)$ with Lipschitz boundary flowing with normal velocity v_b, we have*

$$\frac{d}{dt}\int_{D(t)} f\,dx = \int_{D(t)} D_t f + \nabla\cdot v f\,dx + \int_{\partial D(t)} f(v_b - v\cdot n_{\partial D(t)})\,dS.$$

The proof is a straightforward application of the divergence theorem.

In our setting, we need to make a vector field choice on $\tilde{\Omega}_t$; this will simply be the velocity v, though we could have equally chosen v_h. We remark that in the corresponding argument in [27] the average of the two was used, in order to better symmetrize the problem. However, the argument here is slightly more robust, and such a choice is not needed.

For this choice of v, we examine the boundary weight $v\cdot n_{\partial D(t)} - v_b$ appearing in the above formula. For this we use the disjoint boundary decomposition

$$\tilde{\Gamma} = \mathcal{A}\cup\mathcal{A}_h\cup(\Gamma\cap\Gamma_h),$$

where the normal $n_{\tilde{\Gamma}}$ is given a.e. by

$$n_{\tilde{\Gamma}} = \begin{cases} n_\Gamma & \text{in } \mathcal{A}\cup(\Gamma\cap\Gamma_h),\\ n_{\Gamma_h} & \text{in } \mathcal{A}_h\cup(\Gamma\cap\Gamma_h),\end{cases}$$

with the two normals agreeing a.e. on $\Gamma\cap\Gamma_h$. Correspondingly, for almost every point on $\tilde{\Gamma}$ we have $|v_b - v\cdot n_{\tilde{\Gamma}}| \leq |v - v_h|$, as can be seen by working with the collar parameterization $\eta_\Gamma\wedge\eta_{\Gamma_h}$ for $\tilde{\Gamma}$ and the kinematic boundary conditions for Γ and Γ_h.

We now use Proposition 4.4 and the incompressibility of v for each of the three terms in (4.6). We begin by studying the first term, where we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\tilde{\Omega}} |v-v_h|^2\,dx \leq \frac{1}{2}\int_{\tilde{\Omega}} D_t|v-v_h|^2\,dx + \frac{1}{2}\int_{\tilde{\Gamma}} |v-v_h|^3\,dS. \tag{4.7}$$

We note that, unlike in the case of the linearized equation, here we obtain a nonzero boundary term. However, this term has the redeeming feature that it is cubic in the difference $v - v_h$. To estimate it, we use a simple variant of the trace theorem. Indeed, as $\Gamma,\Gamma_h\in\Lambda_*$, we may find a smooth vector field X defined on $\mathbb{R}^d$ with C^k bounds uniform in Λ_* which is also uniformly transverse to $\tilde{\Gamma}$. By the divergence theorem, we then have

$$\begin{aligned}\frac{1}{2}\int_{\tilde{\Gamma}} |v-v_h|^3\,dS &\lesssim \int_{\tilde{\Gamma}} X\cdot n_{\tilde{\Gamma}}|v-v_h|^3\,dS \lesssim (B+B_h)\|v-v_h\|^2_{L^2(\tilde{\Omega})}\\ &\lesssim (B+B_h)D(v,v_h).\end{aligned} \tag{4.8}$$

Now, for the remaining term in (4.7), we use (4.4) and integrate by parts to obtain

$$\begin{aligned}\frac{1}{2}\int_{\tilde{\Omega}} D_t|v-v_h|^2\,dx &= \int_{\tilde{\Omega}}(v-v_h)D_t(v-v_h)\,dx\\ &= -\int_{\tilde{\Gamma}}(p-p_h)(v-v_h)\cdot n_{\tilde{\Gamma}}\,dS + \int_{\tilde{\Omega}}(v-v_h)\cdot[(v_h-v)\cdot\nabla v_h]\,dx\\ &\leq -\int_{\tilde{\Gamma}}(p-p_h)(v-v_h)\cdot n_{\tilde{\Gamma}}\,dS + (B+B_h)D(v,v_h).\end{aligned} \tag{4.9}$$

Using the decomposition $\tilde{\Gamma} = \mathcal{A} \cup \mathcal{A}_h \cup (\Gamma \cap \Gamma_h)$ and using that $p - p_h = 0$ on $\Gamma \cap \Gamma_h$ by the dynamic boundary condition (1.3), we can write

$$
\begin{aligned}
-\int_{\tilde{\Gamma}} (p - p_h)(v - v_h) \cdot n_{\tilde{\Gamma}} \, dS &= -\int_{\mathcal{A}} (p - p_h)(v - v_h) \cdot n_\Gamma \, dS - \int_{\mathcal{A}_h} (p - p_h)(v - v_h) \cdot n_{\Gamma_h} \, dS \\
&= \int_{\mathcal{A}} a^{-1} (p - p_h)(v - v_h) \cdot \nabla p \, dS + \int_{\mathcal{A}_h} a_h^{-1} (p - p_h)(v - v_h) \cdot \nabla p_h \, dS.
\end{aligned}
$$

Now, define

$$
J := \int_{\mathcal{A}} a^{-1} (p - p_h)(v - v_h) \cdot \nabla p \, dS + \frac{1}{2} \frac{d}{dt} \int_{\mathcal{A}} a^{-1} |p - p_h|^2 \, dS,
$$

and

$$
J_h := \int_{\mathcal{A}_h} a_h^{-1} (p - p_h)(v - v_h) \cdot \nabla p_h \, dS + \frac{1}{2} \frac{d}{dt} \int_{\mathcal{A}_h} a_h^{-1} |p - p_h|^2 \, dS.
$$

Combining (4.8) and (4.9), we obtain

$$
\frac{d}{dt} D(v, v_h) \lesssim (B + B_h) D(v, v_h) + J + J_h.
$$

It remains to show that

$$
J + J_h \lesssim_A (B + B_h) D(v, v_h).
$$

We show the details for J. The treatment of J_h will be virtually identical. We begin by using Proposition 2.3 to expand

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_{\mathcal{A}} a^{-1} |p - p_h|^2 \, dS = &-\frac{1}{2} \int_{\mathcal{A}} a^{-2} D_t a |p - p_h|^2 \, dS - \frac{1}{2} \int_{\mathcal{A}} a^{-1} |p - p_h|^2 [n_\Gamma \cdot \nabla v \cdot n_\Gamma] \, dS \\
&+ \int_{\mathcal{A}} a^{-1} (p - p_h) D_t (p - p_h) \, dS.
\end{aligned} \tag{4.10}
$$

The validity of the identity (4.10) is justified by noting that $|p - p_h|^2$ vanishes to second order on $\Gamma \cap \Gamma_h$, so one can extend by zero to write the integral on the left-hand side as an integral over Γ, apply standard identities there, and then return to an integral over $\mathcal{A}$. From (4.10) and adding the first term in the definition of J, we obtain (noting that by the kinematic and dynamic boundary conditions, we have $D_t p = 0$ on $\mathcal{A}$),

$$
J \lesssim_A -\int_{\mathcal{A}} a^{-1} (p - p_h) D_t^h p_h dS + \int_{\mathcal{A}} a^{-1} (p - p_h)(v - v_h) \cdot \nabla (p - p_h) dS + B D(v, v_h).
$$

In the above, we used the standard identity (5.35) to control $D_t a$. For the first term on the right-hand side we use that $D_t^h p_h$ vanishes on Γ_h, (4.3), the fundamental theorem of calculus, the Taylor sign condition and (4.5), to estimate

$$
|D_t^h p_h| \lesssim_A \|\nabla D_t^h p_h\|_{L^\infty} |s_h| \approx_A \|\nabla D_t^h p_h\|_{L^\infty} |p - p_h| \lesssim_A (B + B_h) |p - p_h|.
$$

Hence,

$$
\int_{\mathcal{A}} a^{-1} (p - p_h) D_t^h p_h dS \lesssim_A (B + B_h) D(v, v_h).
$$

It remains to estimate the cubic term, and show that

$$
\left| \int_{\mathcal{A}} a^{-1} (p - p_h)(v - v_h) \cdot \nabla (p - p_h) \, dS \right| \lesssim_A (B + B_h) D(v, v_h). \tag{4.11}
$$

We will need to perform a more careful analysis here, so that only the pointwise control terms appear in the estimate. Note that if we had instead settled for L^2 based control parameters, this cubic term could be handled relatively easily.

We recall that $\mathcal{A} \subseteq \Gamma$. Given a point $x \in \mathcal{A}$, its distance to Γ_h is proportional to $|(p-p_h)(x)|$. We consider a locally finite Vitali type covering of the set $\mathcal{A}$ with countably many balls $B_j = B(x_j, r_j)$ of radius r_j proportional to $|(p-p_h)(x_j)|$, so that in particular we have $B_j \subseteq \Omega_h$. We denote by D_j the energy of the difference in the region B_j, i.e., the integral in (4.2) restricted to B_j. Then

$$\sum_j D_j \lesssim D((v,\Gamma),(v_h,\Gamma_h)).$$

Hence, by the uniform bound on a^{-1}, it would suffice to show that

$$\int_{\mathcal{A}\cap B_j} |(p-p_h)(v-v_h)\cdot \nabla(p-p_h)|\, dS \lesssim_A (B+B_h)D_j. \tag{4.12}$$

We will indeed show that this bound holds for the bulk of the expression on the left. However, for the remaining part we will return to a global argument. For $\mathcal{A}$ we just use the uniform Lipschitz bound in this analysis. We first note that in $\tilde{\Omega}\cap B_j$ we have

$$|p-p_h| \approx_A r_j,$$

which after integration yields a good bound for r_j within B_j:

$$\int_{\mathcal{A}\cap B_j} |p-p_h|^2\, dS \approx_A r_j^{d+1} \lesssim_A D_j. \tag{4.13}$$

Next we consider $v - v_h$, for which we use the $C^{\frac{1}{2}}$ norm, which is part of our control norm A, in order to estimate the surface integral by the ball integral. This yields

$$\int_{\mathcal{A}\cap B_j} |v-v_h|^2\, dS \lesssim_A r_j^{-1}\int_{\tilde{\Omega}\cap B_j} |v-v_h|^2\, dx + r_j^d A^2 \lesssim_A r_j^{-1}D_j + r_j^d A^2 \lesssim_A r_j^{-1}D_j. \tag{4.14}$$

It remains to consider $\nabla(p-p_h)$. Our starting point is the global bound

$$\|\nabla p\|_{C^{\frac{1}{2}}(\Omega)} + \|\nabla p_h\|_{C^{\frac{1}{2}}(\Omega_h)} \lesssim_A B + B_h, \tag{4.15}$$

which is noted in (4.5). This allows us to replace $\nabla(p-p_h)$ with its average $\overline{\nabla(p-p_h)}_j$ in any smaller ball $\tilde{B}_j \subseteq \tilde{\Omega}\cap B_j$ of comparable size, because

$$\|\nabla(p-p_h) - \overline{\nabla(p-p_h)}_j\|_{L^\infty(\tilde{\Omega}\cap B_j)} \lesssim_A r_j^{\frac{1}{2}}(B+B_h).$$

Putting everything together we arrive at

$$\int_{\mathcal{A}\cap B_j} |(p-p_h)(v-v_h)\cdot(\nabla(p-p_h) - \overline{\nabla(p-p_h)}_j)|\, dS \lesssim_A (B+B_h)D_j,$$

which represents the bulk of (4.12).

It remains to estimate the contribution of the local average of $\nabla(p-p_h)$. Here we view $p-p_h$ as a solution to the following Laplace equation in $\tilde{\Omega}$:

$$\begin{cases} \Delta(p-p_h) = -\mathrm{tr}(\nabla v)^2 + \mathrm{tr}(\nabla v_h)^2, \\ p - p_{h|\tilde{\Gamma}} = \tilde{g} := p1_{\mathcal{A}_h} - p_h 1_{\mathcal{A}}. \end{cases}$$

We split the problem for $p-p_h$ into an inhomogeneous one with homogeneous boundary condition, and a homogeneous one with inhomogeneous boundary condition,

$$p - p_h = (p-p_h)_{inh} + (p-p_h)_{hom}.$$

For the inhomogeneous problem we can write the source term in divergence form to estimate

$$\|\mathrm{tr}(\nabla v)^2 - \mathrm{tr}(\nabla v_h)^2\|_{H^{-1}(\tilde{\Omega})} \lesssim (B + B_h) D^{\frac{1}{2}},$$

which by a simple energy estimate gives a global L^2 bound

$$\|\nabla(p - p_h)_{inh}\|_{L^2(\tilde{\Omega})} \lesssim_A (B + B_h) D^{\frac{1}{2}}.$$

This in turn yields a bound for the corresponding averages by Hölder's inequality,

$$\sum_j r_j^d |\overline{\nabla(p - p_h)_{inh,j}}|^2 \lesssim_A (B + B_h)^2 D.$$

The contribution of this into (4.11) is then estimated using (4.13) and (4.14) as follows:

$$\begin{aligned} J_{inh} := & \sum_j \int_{\mathcal{A} \cap B_j} |p - p_h||v - v_h||\overline{\nabla(p - p_h)_{inh,j}}| \, dS \\ \lesssim_A & \sum_j r_j^{\frac{d+1}{2}} \|v - v_h\|_{L^2(\mathcal{A} \cap B_j)} |\overline{\nabla(p - p_h)_{inh,j}}| \\ \lesssim_A & \sum_j D_j^{\frac{1}{2}} r_j^{\frac{d}{2}} |\overline{\nabla(p - p_h)_{inh,j}}| \\ \lesssim_A & (B + B_h) D, \end{aligned}$$

where in the last step we have used Cauchy-Schwarz with respect to j.

For the homogeneous term, on the other hand, we need to carefully examine the regularity of the Dirichlet data $\tilde{g}$. On one hand, by the definition of the distance D we have the L^2 bound

$$\|\tilde{g}\|^2_{L^2(\tilde{\Gamma})} \lesssim_A D. \tag{4.16}$$

On the other hand, by (4.15), on each of the two regions $\mathcal{A}_h$ respectively $\mathcal{A}$, we have formally

$$\|\tilde{g}\|_{C^{1,\frac{1}{2}}(\mathcal{A}_h)} + \|\tilde{g}\|_{C^{1,\frac{1}{2}}(\mathcal{A})} \lesssim_A B + B_h. \tag{4.17}$$

This bound has to be carefully interpreted, which we do within the proof of Lemma 4.5 below.

A formal interpolation between (4.16) and (4.17) would yield a $W^{1,6}(\tilde{\Gamma})$ bound for $\tilde{g}$. We make this bound rigorous in the following.

Lemma 4.5. *The function $\tilde{g}$ above satisfies the bound*

$$\|\tilde{g}\|_{W^{1,6}(\tilde{\Gamma})} \lesssim (B + B_h)^{\frac{2}{3}} D^{\frac{1}{6}}. \tag{4.18}$$

Proof. We begin by noting that the two components $g := p 1_{\mathcal{A}_h}$ and $g_h := -p_h 1_{\mathcal{A}}$ of $\tilde{g}$ are nonzero on disjoint sets $\mathcal{A}_h$ respectively $\mathcal{A}$, and vanish on the corresponding boundaries $\partial\mathcal{A}_h$, respectively $\partial\mathcal{A}$. Hence, we can prove the bound (4.18) separately for the two components. We consider g, which lives on $\mathcal{A}_h \subseteq \Gamma_h$. Here not only is Γ_h a Lipschitz surface, but it also has a $C^{1,\frac{1}{2}}$ bound of B_h (which is not the case for $\tilde{\Gamma}$).

Using a standard partition of unity we can reduce the problem to the case when Γ_h is a graph,

$$\Gamma_h = \{x_d = \phi(x')\},$$

where

$$\|\phi\|_{Lip} \lesssim_A 1, \qquad \|\phi\|_{C^{1,\frac{1}{2}}} \lesssim B_h. \tag{4.19}$$

We denote the Lipschitz projection of $\mathcal{A}_h$ by $\mathcal{P}\mathcal{A}_h \subseteq \mathbb{R}^{d-1}$. We can equivalently consider g as a function on $\mathcal{P}\mathcal{A}_h$, in which case the bound (4.18) becomes

$$\|\nabla g\|_{L^6(\mathcal{P}\mathcal{A}_h)} \lesssim_A (B+B_h)^{\frac{2}{3}} D^{\frac{1}{6}}. \tag{4.20}$$

We now summarize the information that we have on g as a function on $\mathcal{P}\mathcal{A}_h$:

(i) (L^2 control).

$$\|g\|^2_{L^2(\mathcal{P}\mathcal{A}_h)} \lesssim_A D,$$

which comes from (4.16).

(ii) (Hölder control).

$$\|\nabla g\|_{C^{\frac{1}{2}}(\mathcal{P}\mathcal{A}_h)} \lesssim_A B+B_h,$$

which is a consequence of (4.15), (4.19) and chain rule.

(iii) (Zero boundary data).

$$g = 0 \quad \text{on} \quad \partial\mathcal{P}\mathcal{A}_h.$$

We will prove that these three properties imply the desired bound (4.20). The difficulty here is that we do not know that $\nabla g = 0$ on $\partial\mathcal{P}\mathcal{A}_h$; else we could simply extend g by 0 outside $\mathcal{P}\mathcal{A}_h$ and this becomes a standard interpolation bound. Further, we do not a priori control the regularity of the boundary $\partial\mathcal{P}\mathcal{A}_h$.

Without any loss of generality we assume that $g > 0$ on $\mathcal{P}\mathcal{A}_h$; else we split this set into connected components where g has constant sign, modulo a set where $\nabla g = 0$ a.e. To prove the desired bound we will use a well-chosen Vitali covering of the set $S = \mathcal{P}\mathcal{A}_h \setminus \{\nabla g = 0\}$ with balls. This choice is as follows: For each $x \in S$ we consider a ball $B_x = B(x, r_x)$ with radius $r_x = c^2(B+B_h)^{-2}|\nabla g(x)|^2$ where $c > 0$ is a small universal constant, chosen so that $|\nabla g|$ is nearly constant on B_x, i.e.,

$$|\nabla g(y) - \nabla g(x)| \lesssim c|\nabla g(x)| \ll |\nabla g(x)|, \qquad y \in B_x.$$

The union of the balls B_x with $x \in S$ clearly covers S, so Vitali's lemma allows us to extract a countable disjoint subfamily of such balls $B_j = B_{x_j}$ so that

$$S \subseteq \bigcup 5B_j.$$

Since ∇g is almost constant on B_x and $g(x) > 0$, a key observation is that there must exist a nontrivial sector $C_x \subseteq B_x$ where

$$g > 0 \quad \text{in } C_x, \qquad |C_x| \approx |B_x|.$$

Since $g = 0$ on $\partial\mathcal{P}\mathcal{A}_h$, it follows that we must have $C_x \subseteq S$; this is what allows us to bypass the lack of geometric information on the set $\mathcal{P}\mathcal{A}_h$.

On C_x, the function g is almost linear with slope approximately $|\nabla g(x)|$. Therefore, we must have

$$\|g\|^2_{L^2(C_x)} \gtrsim r_x^{d+1}|\nabla g(x)|^2.$$

We will use this bound to estimate from above the L^6 norm of ∇g in each $5B_j$ as follows:

$$\begin{aligned}\|\nabla g\|^6_{L^6(5B_j)} &\lesssim r_{x_j}^{d-1}|\nabla g(x_j)|^6 \\ &\lesssim \|g\|^2_{L^2(C_j)} r_{x_j}^{-2}|\nabla g(x_j)|^4 \\ &\approx \|g\|^2_{L^2(C_j)}(B+B_h)^4.\end{aligned}$$

Now, we sum over j, using the disjointness of the balls B_j and thus of C_j. This gives

$$\sum_j \|\nabla g\|_{L^6(5B_j)}^6 \lesssim \|g\|_{L^2(S)}^2 (B+B_h)^4 \lesssim_A D(B+B_h)^4,$$

which concludes the proof of the lemma.

□

Now we use the bound in Lemma 4.5 to solve the homogeneous Dirichlet problem in $\tilde{\Omega}$ and to obtain the estimate

$$\|\nabla(p-p_h)_{hom}^*\|_{L^6(\tilde{\Gamma})} \lesssim (B+B_h)^{\frac{2}{3}} D^{\frac{1}{6}},$$

where $*$ stands for the nontangential maximal function. This bound is due to Verchota [45], but see also the further discussion by Jerison-Kenig [29, Theorem 5.6] as well as the case of C^1 boundaries considered earlier by Fabes-Jodeit-Rivière [23].

The exponent 6 is allowed above provided that the Lipschitz norm of the boundary is sufficiently small. Precisely, the upper limit of the allowed exponents goes to infinity as the corner size decreases to 0. The smallness of the intersection angle between Γ and Γ_h is a consequence of the $C^{1,\epsilon}$ common regularity bound together with the use of a sufficiently refined collar region.

To use the nontangential maximal function bound, within the ball $B_j = B(x_j, r_j)$ we consider a smaller ball

$$\tilde{B}_j = B(x_j - \frac{1}{2} r_j n_j, \frac{1}{4} r_j).$$

For $y \in \tilde{B}_j$ we have

$$|\nabla(p-p_h)_{hom}(y)| \lesssim |\nabla(p-p_h)_{hom}^*(z)|, \qquad z \in \tilde{\Gamma} \cap \frac{1}{4} B_j.$$

Taking averages on the left and integrating on the right, we arrive at

$$r_j^{d-1} |\overline{\nabla(p-p_h)_{hom,j}}|^6 \lesssim_A \|\nabla(p-p_h)_{hom}^*\|_{L^6(\tilde{\Gamma}\cap\frac{1}{4}B_j)}^6.$$

Since the balls B_j are disjoint, summation in j yields

$$\sum_j r_j^{d-1} |\overline{\nabla(p-p_h)_{hom,j}}|^6 \lesssim (B+B_h)^4 D. \tag{4.21}$$

On the other hand, for $v - v_h$ we use the interpolation bound (4.8), which gives

$$\|v-v_h\|_{L^3(\tilde{\Gamma})} \lesssim (B+B_h)^{\frac{1}{3}} D^{\frac{1}{3}}. \tag{4.22}$$

We are now ready to estimate the corresponding contribution to (4.11) using also (4.13) and (4.14) as follows:

$$\begin{aligned} J_{hom} &:= \sum_j \int_{\mathcal{A}\cap B_j} |p-p_h||v-v_h||\overline{\nabla(p-p_h)_{hom,j}}|\, dS \\ &\lesssim_A \sum_j r_j (r_j^{\frac{2(d-1)}{3}} \|v-v_h\|_{L^3(\mathcal{A}\cap B_j)}) |\overline{\nabla(p-p_h)_{hom,j}}| \\ &\lesssim_A \sum_j r_j^{\frac{d+1}{2}} \|v-v_h\|_{L^3(\mathcal{A}\cap B_j)} (r_j^{\frac{d-1}{6}} |\overline{\nabla(p-p_h)_{hom,j}}|) \\ &\lesssim_A (B+B_h)D. \end{aligned}$$

At the last step we have applied Hölder's inequality in j with exponents 2, 3 and 6, using (4.13), (4.22) and (4.21). This completes the proof of (4.12) and therefore the proof of Theorem 4.2.

□

One consequence of the difference bounds is the following uniqueness result.

Theorem 4.6 (Uniqueness). *Let $\epsilon > 0$ and let Ω_0 be a bounded domain with boundary $\Gamma_0 \in \Lambda(\Gamma_*, \epsilon, \delta)$. Then for $\Gamma_0 \in C^{1,\frac{1}{2}}$ and divergence free $v_0 \in W^{1,\infty}(\Omega_0)$ satisfying the Taylor sign condition, the free boundary Euler equations admit at most one solution (v, Γ_t) on a time interval $[0, T]$ with $\Gamma_t \in \Lambda(\Gamma_*, \epsilon, \delta)$ and*

$$\sup_{0\le t\le T} \|v\|_{C_x^{\frac{1}{2}+\epsilon}(\Omega_t)} + \int_0^T \|v\|_{W_x^{1,\infty}(\Omega_t)} + \|D_t p\|_{W_x^{1,\infty}(\Omega_t)} + \|\Gamma_t\|_{C_x^{1,\frac{1}{2}}} \, dt < \infty.$$

Proof. Suppose (v, Ω_t) and $(v_h, \Omega_{t,h})$ are a pair of solutions satisfying the conditions of the theorem with the same initial data. From the differences estimates, we immediately obtain $v = v_h$ on $\Omega_t \cap \Omega_{t,h}$. Next, we argue that the domain Ω_t coincides with $\Omega_{t,h}$. First, we note that the intersection is non-empty if $\delta > 0$ is small enough. We now show $\Omega_t \subseteq \Omega_{t,h}$. It suffices to show $\Omega_t \subseteq \overline{\Omega}_{t,h}$. If this is not true, then there is $x \in \Gamma_{t,h}$ such that $x \in \Omega_t$. Such a point must lie on $\partial(\Omega_t \cap \Omega_{t,h})$. Therefore, from the estimate for the distance functional, we have $p(x) = 0$. However, within a small enough collar neighborhood, the Taylor sign condition tells us that the level set $\{p = 0\}$ corresponds exactly to the free surface Γ_t. This is a contradiction to x being an interior point of Ω_t. Therefore $\Omega_t \subseteq \Omega_{t,h}$. The reverse inclusion follows by an identical argument. □

5. Balanced elliptic estimates

In this section, we prove a collection of refined elliptic estimates which will be crucial for obtaining the sharp pointwise control norms in the higher energy bounds. These estimates will turn out to be quite general and should be applicable to other free boundary problems. In a sense, they can be seen as significant refinements of the so-called *tame estimates* which have been fundamental in the analysis of many water waves problems (see the discussion in [6, 32]), but are not nearly sufficient for our purposes. Indeed, as we will soon see, our proofs of the higher energy bounds require estimates for various elliptic operators which more precisely balance the contributions of the input function and the domain regularity, simultaneously, in both pointwise and L^2 based norms. This simultaneous balance cannot be achieved with the known tame estimates, which often only seem to balance the contributions in L^2 based norms or involve domain dependent constants in pointwise norms which are significantly off scale. The technical utility of our balanced estimates will become readily apparent in Section 7, where they will be used to efficiently dispatch with expressions involving relatively complicated iterated applications of the Dirichlet-to-Neumann operator and various other elliptic operators.

In the following, we will always assume that Ω is a bounded domain with boundary $\Gamma \in \Lambda_* := \Lambda(\Gamma_*, \epsilon_0, \delta)$ for suitably small (but fixed) constants $\epsilon_0, \delta > 0$. Most of the bounds in this section do not make reference to a particular velocity function, and so, the implicit constants in many of the estimates will only depend on the surface component of the control parameter A; namely, $A_\Gamma := \|\Gamma\|_{C^{1,\epsilon_0}}$. Hence, for this section, by the relation $X \lesssim_A Y$, we mean $X \le C(A_\Gamma)Y$ for some constant C depending exclusively on A_Γ. The only exception to this rule (which we will make note of explicitly) will be in Section 5.6, where we will use the full control parameter A to establish estimates for commutators of various elliptic operators with D_t. We will also harmlessly let A depend on the domain volume throughout, as the volume of the domain will be conserved in the dynamic problem.

Throughout the section, by a slight abuse of notation, we will follow the convention that a parameter ϵ may vary from line to line by a fixed scalar factor. Generally speaking, we will take $\epsilon > 0$ to be any positive constant with $\epsilon \ll \epsilon_0$.

5.1. **Extension operators in Λ_* and product type estimates on Ω.** To establish the desired elliptic estimates, it will be convenient to have an extension operator which is bounded from $H^s(\Omega) \to H^s(\mathbb{R}^d)$ for $s \geq 0$, and $C^{k,\alpha}(\Omega) \to C^{k,\alpha}(\mathbb{R}^d)$ for a suitable range of k and α with bounds depending only on the implicit constant A. Among other things, this will enable us to recover many of the standard product type estimates which are well-known on $\mathbb{R}^d$. To this end, let $\varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant M. Let $\Omega = \{(x, y) \in \mathbb{R}^d : y > \varphi(x)\}$. Moreover, for $1 \leq p \leq \infty$ and an integer $k \geq 0$, let $W^{k,p}(\Omega)$ denote the usual Sobolev space consisting of distributions whose derivatives up to order k belong to $L^p(\Omega)$. It is a classical result of Stein [44, Theorem 5', p. 181] that there exists a linear operator $\mathcal{E}$ mapping functions on Ω to functions on $\mathbb{R}^d$ with the property that $\mathcal{E} : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^d)$ is well-defined and continuous for all $1 \leq p \leq \infty$ and integers k. Moreover, the norm of $\mathcal{E} : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^d)$ depends only on the dimension d, the order of differentiability k and the Lipschitz constant M. The operator $\mathcal{E}$ is called *Stein's extension operator*. As one can see directly from its definition [44, Equation (24), p. 182], $\mathcal{E}$ also maps $C^1(\Omega) \to C^1(\mathbb{R}^d)$.

As explained in Section 3.3 of [44], a partition of unity argument allows one to construct an extension operator $\mathcal{E} = \mathcal{E}_\Omega$ on all Lipschitz domains Ω, with constant depending only on d, k, p, the number and size of the balls needed to cover the boundary, and the Lipschitz constant of the defining function on each ball. Since for a tight enough collar Λ_* one can use the same balls to cover all elements of Λ_* with control of the Lipschitz constant on each ball, this shows that Stein's extension operator has norm bounds that are uniform for domains with boundary in Λ_*.

In the above discussion, the definition of the $W^{k,p}$ norm was the usual one, defined by requiring the first k weak-derivatives to be in L^p. However, as noted earlier, we also define the H^s norm of a function f as the infimum of the H^s norms of all possible extensions of f to $\mathbb{R}^d$. Clearly, $\| \cdot \|_{W^{k,2}} \lesssim \| \cdot \|_{H^k}$ with constant independent of the domain. However, by the above, for domains with boundary in Λ_*, the reverse inequality also holds, with implicit constant depending on A_Γ.

From [35, Theorem B.8] we know that for any non-empty open subset Ω of $\mathbb{R}^d$ and any $s_0, s_1 \in \mathbb{R}$ we have the identification

$$(H^{s_0}(\Omega), H^{s_1}(\Omega))_{\theta,2} = H^s(\Omega), \text{ where } s = (1-\theta)s_0 + \theta s_1 \text{ and } 0 < \theta < 1,$$

with equivalent norms uniform in the collar. Thus, by interpolation, we have the following result.

Proposition 5.1. *Let Ω be a bounded domain with boundary $\Gamma \in \Lambda_*$. Then for every $s \geq 0$ and $0 \leq \alpha \leq 1 + \epsilon_0$, Stein's extension operator $\mathcal{E}$ satisfies*

$$\|\mathcal{E}\|_{C^\alpha(\Omega)\to C^\alpha(\mathbb{R}^d)}, \qquad \|\mathcal{E}\|_{H^s(\Omega)\to H^s(\mathbb{R}^d)} \lesssim_A 1$$

uniformly in Λ_*.

Proof. The H^s case follows from interpolation between integer powers. For C^α, we first note from [35, Theorem A.1] (and higher order variants, c.f. [24, Lemma 6.37]) that there are extension operators with the above $C^\alpha \to C^\alpha$ bound. That Stein's operator has this property then follows by making use of such extensions and interpolating, similar to [34, p. 11-12]. □

Remark 5.2. As mentioned in [29, Proposition 2.17], by an interpolation argument, one can also prove that Stein's extension operator maps the Besov space $B^{p,q}_{\alpha}(\Omega)$ to $B^{p,q}_{\alpha}(\mathbb{R}^d)$ for all $\alpha > 0$, $1 \le p, q \le \infty$ and Lipschitz domains Ω. However, we will not require anything this precise.

5.2. **Littlewood-Paley decomposition and paraproducts on Ω.** Using the Stein extension operator, many of the standard paraproduct estimates on $\mathbb{R}^d$ pass over to Ω.

5.2.1. *Littlewood-Paley decomposition.* For a distribution u on $\mathbb{R}^d$, we will make use of the standard Littlewood-Paley decomposition

$$u = \sum_{k \ge 0} P_k u,$$

where for $k > 0$, P_k corresponds to a Fourier multiplier with smooth symbol supported in the dyadic frequency region $|\xi| \approx 2^k$ and P_0 corresponds to a multiplier localized to the unit ball. The notation $P_{<k}$, $P_{\le k}$, $P_{\ge k}$ and $P_{>k}$ will have the usual meaning. Using the Stein extension operator, we may also consider Littlewood-Paley projections when u is defined only on Ω. In this case, we abuse notation, and write $P_k u$ instead of $P_k \mathcal{E} u$, with corresponding definitions for $P_{<k}$, $P_{\le k}$, etc. We will also often write u_k, $u_{<k}$, etc. as shorthand for the above operators applied to u.

5.2.2. *Paraproducts on Ω.* The above decomposition allows us to make use of some of the standard tools of paradifferential calculus (see e.g. [10] and [36]) on $\mathbb{R}^d$ and apply them to functions defined on Ω. For bilinear expressions, we will make heavy use of the Littlewood-Paley trichotomy (now defined for functions on Ω with suitable regularity),

$$f \cdot g = T_f g + T_g f + \Pi(f, g),$$

where the above three terms correspond to the respective "low-high", "high-low" and "high-high" frequency interactions between f and g. More specifically, $T_f g$ is defined as

$$T_f g := \sum_k f_{<k-k_0} g_k,$$

where k_0 is some universal parameter independent of k. We will be able to take, e.g., $k_0 = 4$ for most purposes.

5.2.3. *Bilinear estimates on Ω.* One important consequence of the bounds for $\mathcal{E}$ and the corresponding inequality on $\mathbb{R}^d$ is the following algebra property for $H^s(\Omega)$, $s \ge 0$,

$$\|fg\|_{H^s(\Omega)} \lesssim_A \|f\|_{H^s(\Omega)} \|g\|_{L^\infty(\Omega)} + \|g\|_{H^s(\Omega)} \|f\|_{L^\infty(\Omega)}. \tag{5.1}$$

In our estimates for the elliptic problems below, the bilinear terms above will frequently appear in the form $\partial_i f \partial_j g$ where f is some function defined on $\mathbb{R}^d$ encoding the regularity of the domain and the desired uniform bound for g is below C^1. For this reason, in order to avoid negative Hölder norms inside a domain, we will need the following paraproduct type estimate, which we will use in the sequel.

Proposition 5.3 (Bilinear paraproduct type estimate on Ω)**.** *Let either i) $s > 0$ and $\alpha_1, \alpha_2, \beta \in [0, 1]$ or ii) $s = 0$, $\alpha_1 = \alpha_2 = 1$ and $\beta \in [0, 1]$. Then we have for any $r \ge 0$,*

$$\begin{aligned}\|\partial_i f \partial_j g\|_{H^s(\Omega)} \lesssim_A \|g\|_{H^{s+2-\alpha_1}(\Omega)} \|f\|_{C^{\alpha_1}(\Omega)} + \|f\|_{H^{s+r+1}(\Omega)} \sup_{k>0} 2^{-k(r+\alpha_2-1)} \|g^1_k\|_{C^{\alpha_2}(\Omega)} \\ + \|f\|_{C^{1,2\epsilon}(\Omega)} \sup_{k>0} 2^{k(s+\beta-\epsilon)} \|g^2_k\|_{H^{1-\beta}(\Omega)},\end{aligned}$$

where $g = g^1_k + g^2_k$ is any sequence of partitions of g in $C^{\alpha_2}(\Omega) + H^{1-\beta}(\Omega)$.

Proof. By Proposition 5.1, it suffices to prove these estimates for f, g defined on $\mathbb{R}^d$. We prove the estimate for $0 < \alpha_1, \alpha_2 < 1$ and $s > 0$ as the other cases are more easily dealt with. We recall that for $0 < \alpha < 1$, the C^α norm on $\mathbb{R}^d$ can be characterized by the equivalent Besov norm,

$$\|u\|_{C^\alpha(\mathbb{R}^d)} \approx \|P_{\leq 0}u\|_{L^\infty(\mathbb{R}^d)} + \sup_{j>0} 2^{\alpha j}\|P_j u\|_{L^\infty(\mathbb{R}^d)}. \tag{5.2}$$

We now decompose $\partial_i f \partial_j g$ into paraproducts,

$$\partial_i f \partial_j g = T_{\partial_i f}\partial_j g + T_{\partial_j g}\partial_i f + \Pi(\partial_i f, \partial_j g). \tag{5.3}$$

We then have the standard estimate

$$\|T_{\partial_i f}\partial_j g\|_{H^s(\mathbb{R}^d)} \lesssim \|f\|_{C^{\alpha_1}(\mathbb{R}^d)}\|\partial_j g\|_{H^{s+1-\alpha_1}(\mathbb{R}^d)},$$

which follows by shifting $1-\alpha_1$ derivatives off of the low frequency factor and onto the high frequency factor in each term. Using the hypothesis $s > 0$, the high-high paraproduct may be estimated by the same term. For the remaining low-high interaction, we write

$$T_{\partial_j g}\partial_i f = \sum_k P_{<k-4}\partial_j g P_k \partial_i f = \sum_k P_{<k-4}\partial_j(g_k^1)P_k\partial_i f + \sum_k P_{<k-4}\partial_j(g_k^2)P_k\partial_i f.$$

Using standard Bernstein type inequalities and square summing, the first term on the right can be easily controlled by

$$\|\partial_i f\|_{H^{s+r}(\mathbb{R}^d)} \sup_{k>0} 2^{-k(r+\alpha_2-1)}\|g_k^1\|_{C^{\alpha_2}(\mathbb{R}^d)},$$

while the latter can be controlled by

$$\|f\|_{C^{1,2\epsilon}(\mathbb{R}^d)} \sup_{k>0} 2^{k(s+\beta-\epsilon)}\|g_k^2\|_{H^{1-\beta}(\mathbb{R}^d)}.$$

□

The following corollary of the above proposition will be used heavily in the higher energy bounds to control product terms on Ω with suitable pointwise control norms.

Corollary 5.4. *Let s and α_1, α_2 be as in Proposition 5.3. Assume that $f \in H^{s+2-\alpha_2}(\Omega) \cap C^{\alpha_1}(\Omega)$ and $g \in H^{s+2-\alpha_1}(\Omega) \cap C^{\alpha_2}(\Omega)$. Then we have*

$$\|\partial_i f \partial_j g\|_{H^s(\Omega)} \lesssim_A \|g\|_{H^{s+2-\alpha_1}(\Omega)}\|f\|_{C^{\alpha_1}(\Omega)} + \|f\|_{H^{s+2-\alpha_2}(\Omega)}\|g\|_{C^{\alpha_2}(\Omega)}.$$

Proof. This follows immediately from Proposition 5.3 by taking $g_j^2 = 0$ and $r = 1 - \alpha_2$. □

5.2.4. *Generalized Moser type estimate.* Next, we prove a Moser type estimate with the same flavor as the above bilinear estimate. The main purpose of this estimate will be to suitably control (extensions of) compositions of functions on Ω with diffeomorphisms of $\mathbb{R}^d$. This will be important for obtaining more refined elliptic estimates where we need to use such diffeomorphisms to flatten the boundary.

Proposition 5.5 (Balanced Moser estimate)**.** *Let $d \geq 1$ be an integer and let $G : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism with $\|DG\|_{C^\epsilon}, \|DG^{-1}\|_{C^\epsilon} \lesssim_A 1$. Let $s \geq 0, r \geq 0$ and $\alpha, \beta \in [0,1]$. Then for every $F \in H^s(\mathbb{R}^d)$ and partition $F = F_j^1 + F_j^2 \in C^\alpha(\mathbb{R}^d) + H^{1-\beta}(\mathbb{R}^d)$, we have*

$$\|F(G)\|_{H^s(\mathbb{R}^d)} \lesssim_A \|F\|_{H^s(\mathbb{R}^d)} + \|G - Id\|_{H^{s+r}} \sup_{j>0} 2^{-j(\alpha+r-1)}\|F_j^1\|_{C^\alpha(\mathbb{R}^d)} + \sup_{j>0} 2^{j(s+\beta-1-\epsilon)}\|F_j^2\|_{H^{1-\beta}(\mathbb{R}^d)}.$$

Remark 5.6. The same estimate holds for $F \in H^s(\Omega)$ by replacing F with its Stein extension.

Proof. The case $0 \leq s \leq 1$ is a consequence of the following standard fact.

Proposition 5.7 (Theorem 3.23 of [35]). *Let $0 \leq s \leq 1$ and let $G : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism with $\|DG\|_{L^\infty} \lesssim_A 1$ and $\|DG^{-1}\|_{L^\infty} \lesssim_A 1$. Then for every $F \in H^s(\mathbb{R}^d)$, we have*

$$\|F(G)\|_{H^s(\mathbb{R}^d)} \approx_A \|F\|_{H^s(\mathbb{R}^d)}.$$

Now, assume $s > 1$. We begin by performing a Littlewood-Paley decomposition,

$$\|F(G)\|_{H^s(\mathbb{R}^d)}^2 \lesssim_{j_0} \|F(G)\|_{L^2(\mathbb{R}^d)}^2 + \sum_{j>j_0} 2^{2js} \|P_j(F(G))\|_{L^2(\mathbb{R}^d)}^2,$$

where $j_0 > 0$ is some fixed constant depending only on A, to be chosen later. We have

$$2^{js}\|P_j(F(G))\|_{L^2(\mathbb{R}^d)} \lesssim 2^{js}\|P_j(F_{<j'}(G))\|_{L^2(\mathbb{R}^d)} + 2^{js}\|P_j(F_{\geq j'}(G))\|_{L^2(\mathbb{R}^d)},$$

where $F_{<j'} := P_{<j'}F$, $F_{\geq j'} := F - F_{<j'}$ and $j' := j - j_1$ with j_1 being some parameter depending only on s which will also be chosen later. For the latter term, by a change of variables and since $s > 0$, we have

$$\sum_{j>j_0} 2^{2js}\|P_j(F_{\geq j'}(G))\|_{L^2(\mathbb{R}^d)}^2 \lesssim_A \sum_{j>j_0}\sum_{k\geq j'} 2^{2(j-k)s}2^{2ks}\|P_kF\|_{L^2(\mathbb{R}^d)}^2 \lesssim_A \|F\|_{H^s(\mathbb{R}^d)}^2.$$

On the other hand, using the fundamental theorem of calculus, we obtain

$$2^{js}\|P_j(F_{<j'}(G))\|_{L^2(\mathbb{R}^d)} \lesssim 2^{js} \sup_{\tau\in[0,1]} \|P_j\left(DF_{<j'}(G_\tau)P_{\geq j'}G\right)\|_{L^2(\mathbb{R}^d)} + 2^{js}\|P_j\left(F_{<j'}(P_{<j'}G)\right)\|_{L^2(\mathbb{R}^d)}, \tag{5.4}$$

where

$$G_\tau = \tau P_{<j'}G + (1-\tau)G.$$

Now, as $\|DG\|_{\dot{C}^\epsilon}, \|DG^{-1}\|_{\dot{C}^\epsilon} \lesssim_A 1$, it follows that $P_{<j'}G$ and G_τ (for $\tau \in [0,1]$) are invertible with $\|P_{<j'}DG\|_{L^\infty}, \|DG_\tau\|_{L^\infty} \lesssim_A 1$ as long as j_0 is large enough (depending only on A and the collar). Now, to control the first term on the right-hand side of (5.4), we split $F_{<j'} = (F_j^1)_{<j'} + (F_j^2)_{<j'}$ and estimate (using the estimate for G_τ^{-1}),

$$\begin{aligned} 2^{js} \sup_{\tau\in[0,1]} \|P_j\left(DF_{<j'}(G_\tau)P_{\geq j'}G\right)\|_{L^2(\mathbb{R}^d)} &\lesssim_A 2^{-j(r+\alpha-1)}\|F_j^1\|_{C^\alpha(\mathbb{R}^d)}2^{j(s+r)}\|P_{\geq j'}G\|_{L^2(\mathbb{R}^d)} \\ &\quad + 2^{j(s-1+\beta-\epsilon)}\|F_j^2\|_{H^{1-\beta}(\mathbb{R}^d)}. \end{aligned} \tag{5.5}$$

Square summing (and possibly relabelling ϵ) gives

$$\begin{aligned} \left(\sum_{j>j_0} 2^{2js} \sup_{\tau\in[0,1]} \|P_j\left(DF_{<j'}(G_\tau)P_{\geq j'}G\right)\|_{L^2(\mathbb{R}^d)}^2\right)^{\frac{1}{2}} &\lesssim_A \sup_{j>0} 2^{-j(r+\alpha-1)}\|F_j^1\|_{C^\alpha(\mathbb{R}^d)}\|G - Id\|_{H^{s+r}} \\ &\quad + \sup_{j>0} 2^{j(s-1+\beta-\epsilon)}\|F_j^2\|_{H^{1-\beta}(\mathbb{R}^d)}. \end{aligned}$$

Next, we control the second term on the right-hand side of (5.4), which is a bit easier. Let k be the largest integer strictly less than s so that $0 < s - k \leq 1$. If $j_1 := j - j'$ is large enough (depending only on k), we have by the chain rule and straightforward paraproduct analysis,

$$2^{js}\|P_jF_{<j'}(P_{<j'}G)\|_{L^2(\mathbb{R}^d)} \lesssim_A 2^{j(s-k)}\|\tilde{P}_j(D^kF_{<j'}(P_{<j'}G))\|_{L^2(\mathbb{R}^d)},$$

where $\tilde{P}_j$ is a slightly fattened Littlewood-Paley projection. We then use the fundamental theorem of calculus to obtain

$$\begin{aligned} 2^{j(s-k)}\|\tilde{P}_j(D^kF_{<j'}(P_{<j'}G))\|_{L^2(\mathbb{R}^d)} &\lesssim_A 2^{j(s-k)} \sup_{\tau\in[0,1]} \|\tilde{P}_j(D^{k+1}F_{<j'}(G_\tau)P_{\geq j'}G)\|_{L^2(\mathbb{R}^d)} \\ &\quad + 2^{j(s-k)}\|\tilde{P}_j(D^kF_{<j'}(G))\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

For the first term, we have simply

$$2^{j(s-k)}\sup_{\tau\in[0,1]}\|\tilde{P}_j(D^{k+1}F_{<j'}(G_\tau)P_{\geq j'}G)\|_{L^2(\mathbb{R}^d)}\lesssim_A 2^{j(s-k-1-\epsilon)}\|D^{k+1}F_{<j'}\|_{L^2(\mathbb{R}^d)}\lesssim 2^{-j\epsilon}\|F\|_{H^s(\mathbb{R}^d)}.$$

For the second term, we have

$$2^{j(s-k)}\|\tilde{P}_j(D^kF_{<j'}(G))\|_{L^2(\mathbb{R}^d)}\lesssim_A 2^{j(s-k)}\|D^kF_{\geq j'}\|_{L^2(\mathbb{R}^d)}+\|\tilde{P}_j((D^kF)(G))\|_{H^{s-k}(\mathbb{R}^d)}.$$

Since $0<s-k\leq 1$, we obtain from Proposition 5.7,

$$\left(\sum_{j>j_0}2^{2j(s-k)}\|\tilde{P}_j(D^kF_{<j'}(P_{<j'}G))\|^2_{L^2(\mathbb{R}^d)}\right)^{\frac{1}{2}}\lesssim_A\|F\|_{H^s(\mathbb{R}^d)},$$

where we used that $s-k\leq 1$ to control $\|(D^kF)(G)\|_{H^{s-k}(\mathbb{R}^d)}$ and that $s-k>0$ to control the l^2 sum of $2^{j(s-k)}\|D^kF_{\geq j'}\|_{L^2(\mathbb{R}^d)}$. Combining everything together completes the proof. □

We also note a much cruder variant of the above proposition where we measure G only in pointwise norms and F in Sobolev based norms. This will only be needed in our construction of regularization operators later on.

Proposition 5.8 (Crude Moser estimate)**.** *Under the assumptions of Proposition 5.5, the following bound holds for every $F\in H^s(\mathbb{R}^d)$,*

$$\|F(G)\|_{H^s(\mathbb{R}^d)}\lesssim_A\|F\|_{H^s(\mathbb{R}^d)}+\|G-Id\|_{C^{s+r+\epsilon}(\mathbb{R}^d)}\|F\|_{H^{1-r}(\mathbb{R}^d)}.$$

Proof. The proof follows almost identical reasoning to Proposition 5.5. The only difference is that we do not partition F in (5.5) and instead estimate

$$\|(DF_{<j'})(G_\tau)\|_{L^2(\mathbb{R}^d)}\lesssim_A\|DF_{<j'}\|_{L^2(\mathbb{R}^d)}\lesssim 2^{jr}\|F\|_{H^{1-r}(\mathbb{R}^d)}.$$

We then invoke Bernstein's inequality to obtain

$$2^{j(r+s)}\|P_{\geq j}G\|_{L^\infty(\mathbb{R}^d)}\lesssim 2^{-j\epsilon}\|G-Id\|_{C^{s+r+\epsilon}},$$

and conclude by summing in j. □

5.3. **Local coordinate parameterizations and Sobolev norms in Λ_*.** With the above estimates in hand, we can begin the process of proving refined versions of the various elliptic, trace and product type estimates on Γ that will be important for establishing our higher energy estimates. Our goal in this subsection is to construct a family of coordinate neighborhoods for Γ_* which will act as a "universal" set of coordinate neighborhoods which we can use to flatten the boundary of nearby hypersurfaces $\Gamma\in\Lambda_*$. We will also use these local coordinates to define Sobolev type norms on Γ which are suitable for proving uniform estimates later in this section. To achieve this, we slightly modify the construction from [41, Appendix A] (but note the difference in our definitions of Λ_*).

5.3.1. *Local coordinates and partition of unity.* As in [41, Appendix A], since Γ_* is compact, for any $\sigma>0$ we can choose $x_i\in\mathbb{R}^d$ and $r,r_i\in(0,\frac{1}{2}]$, $i=1,\dots,m$, such that we have the following two properties:

(i) $B(\Gamma_*,r)\subseteq\cup_{i=1}^m R_i(r_i)$, where $B(S,\epsilon)$ denotes the ϵ neighborhood of S and $R_i(\cdot):=\tilde{R}_i(\cdot)\times I_i(\cdot)\subseteq\mathbb{R}^d$ is a rotated cylinder with perpendicular vertical segment centered at x_i with the given equal radius and length.

(ii) For each i, $z = (\tilde{z}, z_d)$ being the natural Euclidean coordinates on R_i, there exists a function $f_{*i} : \tilde{R}_i(2r_i) \to I_i$ such that

$$\|f_{*i}\|_{C^0} < \sigma r_i, \quad \|Df_{*i}\|_{C^0} < \sigma \text{ and } \Omega_* \cap R_i(2r_i) = \{z_d > f_{*i}(\tilde{z})\}. \tag{5.6}$$

When $\delta > 0$ is small enough, for every $\Gamma \in \Lambda_*$ with corresponding bounded domain Ω, (i) holds with Γ_* replaced by Γ. Moreover, there exist functions $f_i : \tilde{R}_i(2r_i) \to I_i$ satisfying (ii) with Ω_* replaced by Ω such that we can control the Sobolev and Hölder type norms of f_i by the corresponding norms of Γ. Specifically, we have

$$\|f_i\|_{H^s} \lesssim_A 1 + \|\Gamma\|_{H^s}, \qquad \|f_i\|_{C^{k,\alpha}} \lesssim_A 1 + \|\Gamma\|_{C^{k,\alpha}}$$

for $s \geq 0$, integer $k \geq 0$ and $\alpha \in [0, 1)$. Indeed, by performing a computation in local coordinates, the above Sobolev bound follows from the Moser estimate in Proposition 5.5 and the pointwise bound can be verified directly from the chain rule and interpolation. Using these coordinate representations, we intend to construct local coordinate maps on each $\tilde{R}_i(2r_i)$ for Ω which flatten Γ and have uniform estimates in Λ_*. In some of the estimates in this section, by a slight abuse of notation, we write $\|\Gamma\|$ when we really mean $1 + \|\Gamma\|$ in order to declutter the notation. This will not affect any of the analysis for the dynamic problem.

On each $\tilde{R}_i(2r_i)$, let $\phi_i = \gamma_i f_i$, where $\gamma_i(\tilde{z}) = \overline{\gamma}\left(\frac{|\tilde{z}|}{r_i}\right)$ and $\overline{\gamma} : [0, \infty) \to [0, 1]$ is a smooth cutoff supported on $[0, \frac{3}{2}]$ and equal to 1 on $[0, \frac{5}{4}]$. We can extend ϕ_i to a function on $\mathbb{R}^d$ which gains half a degree of regularity in H^s norms and is bounded in suitable pointwise norms. Indeed, let $\tilde{z} \in \mathbb{R}^{d-1}$ and $s \geq \frac{1}{2}$. We define an extension Φ_i of ϕ_i by

$$\Phi_i(z) = \int_{\mathbb{R}^{d-1}} \widehat{\phi_i}(\xi') e^{-(1+|\xi'|^2)z_d^2} e^{2\pi i \xi' \cdot \tilde{z}} d\xi' \text{ for } z = (\tilde{z}, z_d) \in \mathbb{R}^d.$$

We first observe that for each integer $k \geq 0$ and $\alpha \in [0, 1)$, $\|\Phi_i\|_{C^{k,\alpha}(\mathbb{R}^d)} \lesssim_{k,\alpha} \|\phi_i\|_{C^{k,\alpha}(\mathbb{R}^{d-1})}$. One also has the same bounds for $W^{k,\infty}$ for each $k \geq 0$. To see this, we observe that Φ_i can be rewritten as the convolution

$$\Phi_i(z) = c_d e^{-z_d^2} \int_{\mathbb{R}^{d-1}} \phi_i(\tilde{z} + z_d y) e^{-|y|^2} dy,$$

where c_d is a dimensional constant. In this form, the above bounds are easily checked. We also have $\|\Phi_i\|_{H^{s+\frac{1}{2}}(\mathbb{R}^d)} \approx_s \|\phi_i\|_{H^s(\mathbb{R}^{d-1})}$ for every $s \geq 0$, which follows from inspecting the Fourier transform of Φ_i, in a similar fashion as [35, Lemma 3.36].

From the above, we see that if $\sigma > 0$ from (5.6) is small enough, then the map

$$H_i(\tilde{z}, z_d) := (\tilde{z}, z_d + \Phi_i(\tilde{z}, z_d))$$

is a diffeomorphism from $\mathbb{R}^d \to \mathbb{R}^d$ with $\|H_i - Id\|_{C^{k,\alpha}} \lesssim_A \|\Gamma\|_{C^{k,\alpha}}$ and $\|H_i - Id\|_{H^{s+\frac{1}{2}}} \lesssim_A \|\Gamma\|_{H^s}$ for $s \geq 0$, integer $k \geq 0$ and $\alpha \in [0, 1)$. Moreover, for the inverse function $G_i := H_i^{-1}$, the same bounds hold for $G_i - Id$ and its d'th component g_i satisfies the bounds $|\partial_{z_d} g_i| + |(\partial_{z_d} g_i)^{-1}| \lesssim_A 1$. Finally, if $\sigma > 0$ is small enough and Λ_* is a tight enough collar neighborhood we have, in the C^1 topology,

$$\|H_i - Id\|_{C^1} + \|G_i - Id\|_{C^1} \lesssim_A \rho,$$

where $\rho > 0$ is some positive constant which can be made as small as we like (depending on σ and Λ_*). We then have for some uniform $\delta_* > 0$,

$$\left(\tilde{R}_i\left(\frac{5}{4}r_i\right) \times I_i\left(\frac{5}{4}\delta_* r_i\right)\right) \cap \Omega = \left(\tilde{R}_i\left(\frac{5}{4}r_i\right) \times I_i\left(\frac{5}{4}\delta_* r_i\right)\right) \cap \{g_i > 0\}.$$

Partition of unity. Here, we construct a partition of unity for Ω with bounds uniform in Λ_*. We follow essentially the procedure from [41, Appendix A]. Let γ be a smooth cutoff defined on $[0,\infty)$ satisfying $0 \leq \gamma \leq 1$ with γ supported in $[0, \frac{5}{4})$ and equal to 1 on $[0, \frac{9}{8}]$. Moreover, let ζ be a smooth function defined on $[0,\infty)$ taking values in $[\frac{1}{3},\infty)$ with $\zeta = \frac{1}{3}$ on $[0,\frac{1}{3}]$ and $\zeta(x) = x$ for $x \geq \frac{2}{3}$. Define

$$\tilde{\gamma}_{*i}(z) := \gamma(\frac{|\tilde{z}|}{r_i})\gamma(\frac{|z_d|}{\delta_* r_i}), \qquad \eta = \zeta \circ \sum_i (\tilde{\gamma}_{*i} \circ G_i).$$

We then define a partition of unity via

$$\gamma_{*i} := \frac{\tilde{\gamma}_{*i}(G_i)}{\eta}, \qquad \gamma_{*0} := (1 - \sum_i \gamma_{*i})\mathbb{1}_\Omega. \tag{5.7}$$

We see that $\sum_{i\geq 0} \gamma_{*i} = 1$ on Ω and $0 \leq \gamma_{*i} \leq 1$ for each $i \geq 0$. Moreover, by the Moser and Sobolev product estimates, we have

$$\|\gamma_{*i}\|_{H^{s+\frac{1}{2}}} \lesssim_A \|\Gamma\|_{H^s}$$

for $s \geq 0$.

5.3.2. *Sobolev spaces on hypersurfaces in Λ_*.* We can use the above partition of unity to define $C^{k,\alpha}$ and H^s spaces on hypersurfaces $\Gamma \in \Lambda_*$. Indeed, if Γ is C^1 and in H^s, we may define what it means to be in $H^r(\Gamma)$ for $0 \leq r \leq s$ through the inner product,

$$\langle f, g\rangle_{H^r(\Gamma)} := \sum_{i\geq 1} \langle \phi_i f_i, \phi_i g_i\rangle_{H^r(\mathbb{R}^{d-1})},$$

where $\phi_i := \gamma_{*i} \circ H_i(\tilde{z}, 0)$ (note that this is not the same ϕ_i as in the previous subsection), $f_i := f \circ H_i(\tilde{z}, 0)$ and $g_i := g \circ H_i(\tilde{z}, 0)$. If Γ is $C^{k,\alpha}$ we may also define

$$\|f\|_{C^{k,\alpha}(\Gamma)} := \sup_{i\geq 1} \|\phi_i f_i\|_{C^{k,\alpha}(\mathbb{R}^{d-1})}.$$

Finally, for a function v defined on Ω, we write $v_i = \gamma_{*i} v$ and $u_i = v_i(H_i)$.

Using the above and the full generality afforded by Proposition 5.5, we prove a refined product type estimate on the boundary Γ. Precisely, we have the following.

Proposition 5.9 (Product estimates on the boundary)**.** *Let Ω be a bounded domain with boundary $\Gamma \in \Lambda_*$. If f, g are functions on Γ and $g = g_j^1 + g_j^2$ is any sequence of partitions, then for $s \geq 0$ and $r \geq 1$ we have*

$$\begin{aligned}\|fg\|_{H^s(\Gamma)} \lesssim_A \|f\|_{L^\infty(\Gamma)}\|g\|_{H^s(\Gamma)} + (\|f\|_{H^{s+r-1}(\Gamma)} + \|f\|_{L^\infty(\Gamma)}\|\Gamma\|_{H^{s+r}}) \sup_{j>0} 2^{-j(r-1)}\|g_j^1\|_{L^\infty(\Gamma)} \\ + (1 + \|f\|_{C^{2\epsilon}(\Gamma)}) \sup_{j>0} 2^{j(s-\epsilon)}\|g_j^2\|_{L^2(\Gamma)}.\end{aligned}$$

Remark 5.10. If we take $r = 1$ and $g_j^1 = g$, we recover something resembling the standard algebra property,

$$\|fg\|_{H^s(\Gamma)} \lesssim_A \|f\|_{L^\infty(\Gamma)}\|g\|_{L^\infty(\Gamma)}\|\Gamma\|_{H^{s+1}} + \|f\|_{H^s(\Gamma)}\|g\|_{L^\infty(\Gamma)} + \|g\|_{H^s(\Gamma)}\|f\|_{L^\infty(\Gamma)}, \tag{5.8}$$

but with the twist being the additional explicit presence of the H^{s+1} norm of the surface on the right-hand side. We also remark that the proof below will allow for the first term on the right of (5.8) to be replaced by $(\|f\|_{W^{1,\infty}(\Gamma)}\|g\|_{L^\infty(\Gamma)} + \|f\|_{L^\infty(\Gamma)}\|g\|_{W^{1,\infty}(\Gamma)})\|\Gamma\|_{H^s}$, which is perhaps more natural, but we will never actually need this.

Proof. Let $(\gamma_{*i})_i$ be the partition of unity for Ω defined in (5.7). As before, we write $\phi_i(\tilde{z}) := \gamma_{*i}(H_i(\tilde{z},0))$, which is smooth with domain independent bounds since G_i and H_i are inverse. Similarly, we write $f_i = f(H_i(\tilde{z},0))$ and $g_i = g(H_i(\tilde{z},0))$, which are functions defined on the support of ϕ_i. By definition, it suffices to control $\|\phi_i f_i g_i\|_{H^s(\mathbb{R}^{d-1})}$ for each $i \geq 1$. To begin with, let $j' = j-4$ and let P_j and $P_{<j'}$ denote Littlewood-Paley projections on $\mathbb{R}^{d-1}$. Moreover, define $\tilde{\phi}_i$ to be a smooth compactly supported function equal to 1 on the support of γ_{*i} with support properties chosen so that $\tilde{\phi}_i$ is supported in the region where f_i is well-defined. Then a simple paraproduct estimate using the Littlewood-Paley trichotomy gives

$$\|\phi_i f_i g_i\|_{H^s(\mathbb{R}^{d-1})} \lesssim_A \|f\|_{L^\infty(\Gamma)}\|\phi_i g_i\|_{H^s(\mathbb{R}^{d-1})} + \left(\sum_{j>0} 2^{2js}\|P_{<j'}(\phi_i g_i)P_j(f_i\tilde{\phi}_i)\|^2_{L^2(\mathbb{R}^{d-1})}\right)^{\frac{1}{2}}.$$

For the latter term in the above, we estimate

$$\begin{aligned}\left(\sum_{j>0} 2^{2js}\|P_{<j'}(\phi_i g_i)P_j(f_i\tilde{\phi}_i)\|^2_{L^2(\mathbb{R}^{d-1})}\right)^{\frac{1}{2}} &\lesssim_A \|f_i\tilde{\phi}_i\|_{H^{s+r-1}(\mathbb{R}^{d-1})}\sup_{j>0} 2^{-j(r-1)}\|g_j^1\|_{L^\infty(\Gamma)} \\ &\quad + (1+\|f\|_{C^{2\epsilon}(\Gamma)})\sup_{j>0} 2^{j(s-\epsilon)}\|g_j^2\|_{L^2(\Gamma)}.\end{aligned}$$

We are then reduced to showing

$$\|f_i\tilde{\phi}_i\|_{H^{s+r-1}(\mathbb{R}^{d-1})} \lesssim_A \|f\|_{H^{s+r-1}(\Gamma)} + \|f\|_{L^\infty(\Gamma)}\|\Gamma\|_{H^{s+r}}.$$

For this, we note that

$$\|f_i\tilde{\phi}_i\|_{H^{s+r-1}(\mathbb{R}^{d-1})} \leq \sum_{j\geq 1}\|\tilde{\phi}_i\gamma_{*j}(H_i(\tilde{z},0))f_i\|_{H^{s+r-1}(\mathbb{R}^{d-1})}.$$

Let us write $\varphi_{ij} := G_j \circ H_i$. Then we have

$$\|\tilde{\phi}_i\gamma_{*j}(H_i(\tilde{z},0))f_i\|_{H^{s+r-1}(\mathbb{R}^{d-1})} = \|(\phi_j f_j)(\varphi_{ij}(\tilde{z},0))\tilde{\phi}_i\|_{H^{s+r-1}(\mathbb{R}^{d-1})}.$$

We note that φ_{ij} is a diffeomorphism having the same bounds as G_j and H_i. By using the extension Φ from earlier, we may assume that $\phi_j f_j$ is defined on $\mathbb{R}^d$ with $\|\phi_j f_j\|_{H^{s+r-\frac{1}{2}}(\mathbb{R}^d)} \lesssim \|\phi_j f_j\|_{H^{s+r-1}(\mathbb{R}^{d-1})}$ and $\|\phi_j f_j\|_{L^\infty(\mathbb{R}^d)} \lesssim \|\phi_j f_j\|_{L^\infty(\mathbb{R}^{d-1})}$. Therefore, by the trace estimate on $\mathbb{R}^{d-1}$, the fact that φ_{ij} is a diffeomorphism and the balanced Moser estimate, we have

$$\|\tilde{\phi}_i(\phi_j f_j)(\varphi_{ij}(\tilde{z},0))\|_{H^{s+r-1}(\mathbb{R}^{d-1})} \lesssim_A \|(\phi_j f_j)\circ\varphi_{ij}\|_{H^{s+r-\frac{1}{2}}(\mathbb{R}^d)} \lesssim_A \|\phi_j f_j\|_{H^{s+r-1}(\mathbb{R}^{d-1})} + \|\Gamma\|_{H^{s+r}}\|f\|_{L^\infty(\Gamma)}.$$

Since, by definition, we have

$$\|\phi_j f_j\|_{H^{s+r-1}(\mathbb{R}^{d-1})} \leq \|f\|_{H^{s+r-1}(\Gamma)},$$

the proof is complete. □

5.3.3. *Trace estimates.* Now, we prove a refined version of the trace theorem for Γ.

Proposition 5.11 (Balanced trace estimate)**.** *Let Ω be a bounded domain with boundary $\Gamma \in \Lambda_*$. For every $s > \frac{1}{2}$, $r \geq 0$, $\alpha, \beta \in [0,1]$ and every sequence of partitions $v = v_j^1 + v_j^2$, we have*

$$\|v_{|\Gamma}\|_{H^{s-\frac{1}{2}}(\Gamma)} \lesssim_A \|v\|_{H^s(\Omega)} + \|\Gamma\|_{H^{s+r-\frac{1}{2}}}\sup_{j>0} 2^{-j(r+\alpha-1)}\|v_j^1\|_{C^\alpha(\Omega)} + \sup_{j>0} 2^{j(s-1+\beta-\epsilon)}\|v_j^2\|_{H^{1-\beta}(\Omega)}.$$

Proof. For $i \geq 1$, define $\tilde{v}_i = \gamma_{*i}\mathcal{E}v$ where $\mathcal{E}$ is the Stein extension operator for Ω. It suffices to prove the estimate with the left-hand side replaced by $\|\tilde{v}_i(H_i(\tilde{z},0))\|_{H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})}$. Using the trace theorem on $\mathbb{R}^{d-1}$, we have

$$\|\tilde{v}_i(H_i(\tilde{z},0))\|_{H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})} \lesssim \|\tilde{u}_i\|_{H^s(\mathbb{R}^d)},$$

where $\tilde{u}_i := \tilde{v}_i \circ H_i$. We then use Proposition 5.5 and the operator bounds for $\mathcal{E}$ in Proposition 5.1 to conclude. □

5.3.4. *An extension operator depending continuously on the domain.* Another use of the above local coordinates is to construct a family of extension operators which depend continuously in a suitable sense on the domain. This will be important for establishing our continuous dependence result later on. Potentially, something akin to the Stein extension operator could work here, but we opt for the following simpler construction where the dependence on the domain is more transparent.

Proposition 5.12. *Fix a collar neighborhood Λ_* and let $s > \frac{d}{2}+1$. For each bounded domain Ω with H^s boundary $\Gamma \in \Lambda_*$ there exists an extension operator $E_\Omega : H^s(\Omega) \to H^s(\mathbb{R}^d)$ such that for all $v \in H^s(\Omega)$,*

$$\|E_\Omega v\|_{H^s(\mathbb{R}^d)} + \|\Gamma\|_{H^s} \approx_{A,\|v\|_{C^{\frac{1}{2}}(\Omega)}} \|(v,\Gamma)\|_{\mathbf{H}^s}, \qquad \|E_\Omega v\|_{H^s(\mathbb{R}^d)} \lesssim_A \|\Gamma\|_{H^{s-\frac{1}{2}}}\|v\|_{H^s(\Omega)}, \tag{5.9}$$

where the dependence on $\|v\|_{C^{\frac{1}{2}}(\Omega)}$ is polynomial. Moreover, if Ω_n is a sequence of domains with $\Gamma_n \to \Gamma$ in H^s, then for every $v \in H^s(\mathbb{R}^d)$, there holds

$$\|E_{\Omega_n} v_{|\Omega_n} - E_\Omega v_{|\Omega}\|_{H^s(\mathbb{R}^d)} \to 0. \tag{5.10}$$

Remark 5.13. One can loosely think of (5.10) as a strong operator topology convergence for this family of extensions.

Proof. Given a family of domains Ω_n and Ω with boundaries $\Gamma_n, \Gamma \in \Lambda_*$, denote by γ^n_{*i} and γ_{*i} the corresponding partitions of unity, so that

$$v = \sum_i \gamma^n_{*i} v \quad \text{on } \Omega_n \text{ and } v = \sum_i \gamma_{*i} v \quad \text{on } \Omega.$$

Define $u^n_i = (\gamma^n_{*i}v) \circ H^n_i$ on $\mathbb{R}^d_+$. Let k be the largest integer less than or equal to s, and define the half-space extension

$$\begin{cases} \tilde{u}^n_i(\tilde{z}, z_d) = \sum_{j=1}^{k+1} c_j u^n_i(\tilde{z}, -\frac{z_d}{j}) & \text{if } z_d < 0, \\ \tilde{u}^n_i(\tilde{z}, z_d) = u^n_i(\tilde{z}, z_d) & \text{if } z_d \geq 0, \end{cases}$$

where $c_1, \dots, c_{k+1}$ are gotten as in [24, Lemma 6.37] by solving an appropriate Vandermonde system. It is standard to verify that we have $\tilde{u}^n_i \in H^s(\mathbb{R}^d)$.

We define the Ω_n extension of v by

$$\tilde{v}_n = \sum_i \tilde{u}^n_i \circ G^n_i,$$

and similarly let $\tilde{v}$ by the Ω extension of v. To verify the continuous dependence property, we want to verify that if $\Gamma_n \to \Gamma$ in H^s, then $\tilde{v}_n \to \tilde{v}$ in $H^s(\mathbb{R}^d)$. For this, it suffices to prove that $\tilde{u}^n_i \circ G^n_i \to \tilde{u}_i \circ G_i$ in $H^s(\mathbb{R}^d)$ for each i. We note that

$$\|\tilde{u}^n_i \circ G^n_i - \tilde{u}_i \circ G_i\|_{H^s(\mathbb{R}^d)} \leq \|(\tilde{u}^n_i - \tilde{u}_i) \circ G^n_i\|_{H^s(\mathbb{R}^d)} + \|\tilde{u}_i \circ G^n_i - \tilde{u}_i \circ G_i\|_{H^s(\mathbb{R}^d)}. \tag{5.11}$$

The first term on the right-hand side of (5.11) can be shown to go to zero by using standard Moser estimates. The latter term goes to zero by arguing similarly to the proof that translation is continuous in L^p spaces (using a simple density argument to replace $\tilde{u}_i$ by a smooth function).

Finally, the bounds (5.9) follow from the definition of the extension and Proposition 5.5.

□

5.4. **Pointwise elliptic estimates.** Here we establish variants of the $C^{2,\alpha}$ and $C^{1,\alpha}$ estimates for the Dirichlet problem which adequately track the dependence on the domain regularity. In our analysis later, we will mostly use the $C^{1,\alpha}$ estimates with $\alpha = \frac{1}{2}$ or $\alpha = \epsilon$. However, the $C^{2,\alpha}$ estimates will be relevant for proving bounds for our regularization operators, which are defined in Section 6.

As will become apparent later, to obtain the desired pointwise elliptic estimates, it is crucial to use a domain flattening map whose Jacobian has determinant 1. This will be especially necessary for the $C^{1,\alpha}$ estimate, as we must preserve the divergence form of the equation. For this reason, instead of the map H_i, we will use the more familiar domain flattening map

$$F_i(z) = (\tilde{z}, z_d + \phi_i(\tilde{z})), \tag{5.12}$$

whose Jacobian has determinant 1. The tradeoff when using the flattening F_i is that it does not exhibit a $\frac{1}{2}$ gain in regularity for the H^s norm on the interior compared to the boundary, but this will not matter for this section because all domain dependent coefficients will be placed in L^∞ based norms. We let $\Psi_i := F_i^{-1}$, and begin with the $C^{2,\alpha}$ estimates.

Proposition 5.14 ($C^{2,\alpha}$ estimates for the inhomogeneous Dirichlet problem)**.** *Let $0 < \alpha < 1$ and let Ω be a bounded domain with boundary $\Gamma \in \Lambda_*$ having $C^{2,\alpha}$ regularity. Consider the boundary value problem*

$$\begin{cases} \Delta v = g & \text{in } \Omega, \\ v = \psi & \text{on } \Gamma. \end{cases}$$

Then v satisfies the estimate

$$\|v\|_{C^{2,\alpha}(\Omega)} \lesssim_A \|\Gamma\|_{C^{2,\alpha}} \|v\|_{W^{1,\infty}(\Omega)} + \|g\|_{C^\alpha(\Omega)} + \|\psi\|_{C^{2,\alpha}(\Gamma)}.$$

Proof. We write $v_i = \gamma_{*i} v$, $h_i = \Delta v_i$, $f_i = h_i \circ F_i$ and $v_i = u_i \circ \Psi_i$. Omitting some of the subscripts for notational convenience, we see that $u := u_i$ satisfies the equation

$$\begin{cases} \Delta u = \partial_k((\delta^{jk} - a^{jk})\partial_j u) + f, \\ u_{|z_d=0} = (\gamma_{*i}\psi)(H_i(\tilde{z}, 0)), \end{cases} \tag{5.13}$$

where $a^{jk} = (\Psi^j_{x_l} \Psi^k_{x_l})(F_i)$ with repeated indices summed over. Note that to compute the boundary term in (5.13) we used that $F_i(\tilde{z}, 0) = H_i(\tilde{z}, 0)$. By the well-known Schauder estimates for the half-space, we obtain

$$\|u\|_{C^{2,\alpha}} \lesssim_A \|(\delta^{jk} - a^{jk})\partial_j u\|_{C^{1,\alpha}} + \|f\|_{C^\alpha} + \|(\gamma_{*i}\psi)(H_i(\tilde{z}, 0))\|_{C^{2,\alpha}}. \tag{5.14}$$

Using the Besov characterization (5.2) and the paradifferential expansion (5.3), it is straightforward to estimate

$$\|(\delta^{jk} - a^{jk})\partial_j u\|_{C^{1,\alpha}} \lesssim \|\delta^{jk} - a^{jk}\|_{C^\epsilon} \|u\|_{C^{2,\alpha}} + \|\Gamma\|_{C^{2,\alpha}} \|v\|_{W^{1,\infty}(\Omega)}. \tag{5.15}$$

As a^{ij} is close to the identity in C^ϵ, this simplifies the estimate (5.14) to

$$\|u\|_{C^{2,\alpha}} \lesssim_A \|\Gamma\|_{C^{2,\alpha}} \|v\|_{W^{1,\infty}(\Omega)} + \|f\|_{C^\alpha} + \|(\gamma_{*i}\psi)(H_i(\tilde{z}, 0))\|_{C^{2,\alpha}}. \tag{5.16}$$

Clearly, we have $\|f\|_{C^\alpha} \lesssim_A \|h\|_{C^\alpha(\Omega)}$. On the other hand, we have

$$\|u(\Psi_i)\|_{\dot C^{2,\alpha}} \lesssim_A \|(D\Psi_i)^*(D^2u)(\Psi_i)D\Psi_i\|_{\dot C^\alpha} + \|(Du)(\Psi_i)D^2\Psi_i\|_{\dot C^\alpha}. \tag{5.17}$$

We can estimate both terms above by the right-hand side of (5.16). We show how to do this for the first term, as the second term is similar. For this, we may assume that u is defined on all of $\mathbb{R}^d$ by using a suitable extension operator from the half-space to $\mathbb{R}^d$. Then we write as usual $u_{<j}$ to mean $P_{<j}u$ and $u_{\geq j} := u - u_{<j}$. By the Besov characterization of C^α, we need to estimate

$$\sup_{j>0} 2^{j\alpha}\|P_j((D\Psi_i)^*(D^2u)(\Psi_i)D\Psi_i)\|_{L^\infty}.$$

By the standard Littlewood-Paley trichotomy, we first obtain,

$$2^{j\alpha}\|P_j((D\Psi_i)^*(D^2u)(\Psi_i)D\Psi_i)\|_{L^\infty} \lesssim_A \|u\|_{C^{2,\alpha}} + 2^{j\alpha}\|D^2u\|_{L^\infty}\|\tilde P_j B(D\Psi_i, D\Psi_i)\|_{L^\infty},$$

where B is a suitable bilinear form. For the latter term, we split $u = u_{<j} + u_{\geq j}$ and estimate using Bernstein's inequality,

$$\begin{aligned}2^{j\alpha}\|D^2u\|_{L^\infty}\|\tilde P_j B(D\Psi_i, D\Psi_i)\|_{L^\infty} &\lesssim_A \|v\|_{W^{1,\infty}(\Omega)}2^{j(1+\alpha)}\|\tilde P_j B(D\Psi_i, D\Psi_i)\|_{L^\infty} + \|u\|_{C^{2,\alpha}}\\ &\lesssim_A \|\Gamma\|_{C^{2,\alpha}}\|v\|_{W^{1,\infty}(\Omega)} + \|u\|_{C^{2,\alpha}}.\end{aligned}$$

The other term in (5.17) is similarly handled. Combining the above, we obtain

$$\|v_i\|_{C^{2,\alpha}(\Omega)} \lesssim_A \|\Gamma\|_{C^{2,\alpha}}\|v\|_{W^{1,\infty}(\Omega)} + \|h\|_{C^\alpha(\Omega)} + \|(\gamma_{*i}\psi)(H_i(\tilde z, 0))\|_{C^{2,\alpha}}.$$

Expanding

$$h = \Delta(\gamma_{*i}v) = \Delta\gamma_{*i}v + 2\nabla\gamma_{*i}\cdot\nabla v + \gamma_{*i}\Delta v$$

we obtain

$$\|h\|_{C^\alpha(\Omega)} \lesssim_A \|\Gamma\|_{C^{2,\alpha}}\|v\|_{W^{1,\infty}(\Omega)} + \|\nabla\gamma_{*i}\cdot\nabla v\|_{C^\alpha(\Omega)} + \|g\|_{C^\alpha(\Omega)}.$$

The second term on the right-hand side can be estimated crudely by

$$\|\nabla\gamma_{*i}\cdot\nabla v\|_{C^\alpha(\Omega)} \lesssim_A \|v\|_{C^{1,\alpha}(\Omega)} + \|\Gamma\|_{C^{2,\alpha}}\|v\|_{W^{1,\infty}(\Omega)}.$$

Finally, by estimating the term $\|v\|_{C^{1,\alpha}(\Omega)} \lesssim \delta_0\|v\|_{C^{2,\alpha}(\Omega)} + C(\delta_0)\|v\|_{C^0(\Omega)}$ for some δ_0 sufficiently small and absorbing the first term into the left-hand side of the estimate, we conclude the proof. □

By very similar reasoning and the corresponding estimate in the half-space (see Theorem 8.33 in [24]) we also have a $C^{1,\alpha}$ variant if the source term g is replaced by $\nabla\cdot g$. More precisely, we have the following.

Proposition 5.15 ($C^{1,\alpha}$ estimates for the Dirichlet problem)**.** *Let Ω be a bounded $C^{1,\alpha}$ domain with $0 < \alpha < 1$ and with boundary $\Gamma \in \Lambda_*$. Consider the boundary value problem*

$$\begin{cases} \Delta v = \nabla\cdot g_1 + g_2 & \text{in } \Omega,\\ v = \psi & \text{on } \partial\Omega.\end{cases}$$

Then v satisfies the estimate

$$\|v\|_{C^{1,\alpha}(\Omega)} \lesssim_A \|\Gamma\|_{C^{1,\alpha}}(\|v\|_{W^{1,\infty}(\Omega)} + \|g_1\|_{L^\infty(\Omega)}) + \|g_1\|_{C^\alpha(\Omega)} + \|g_2\|_{L^\infty(\Omega)} + \|\psi\|_{C^{1,\alpha}(\Gamma)}.$$

Interpolating and using the straightforward estimate

$$\|v\|_{L^\infty(\Omega)} \lesssim_A \|g_1\|_{L^\infty(\Omega)} + \|g_2\|_{L^\infty(\Omega)} + \|\psi\|_{L^\infty(\Gamma)},$$

we deduce also

$$\|v\|_{C^{1,\epsilon}(\Omega)} \lesssim_A \|g_1\|_{C^\epsilon(\Omega)} + \|g_2\|_{L^\infty(\Omega)} + \|\psi\|_{C^{1,\epsilon}(\Gamma)} \tag{5.18}$$

and

$$\|v\|_{C^{1,\alpha}(\Omega)} \lesssim_A \|\Gamma\|_{C^{1,\alpha}}(\|g_1\|_{C^{\epsilon}(\Omega)} + \|g_2\|_{L^\infty(\Omega)} + \|\psi\|_{C^{1,\epsilon}(\Gamma)}) + \|g_1\|_{C^{\alpha}(\Omega)} + \|g_2\|_{L^\infty(\Omega)} + \|\psi\|_{C^{1,\alpha}(\Gamma)}.$$

Proof. Much of the proof is similar to the $C^{2,\alpha}$ estimate. We only outline the slight changes. First, we note that

$$\begin{aligned}\Delta v_i &= \partial_j(\partial_j\gamma_{*i}v) + \partial_j\gamma_{*i}\partial_j v + \gamma_{*i}\nabla\cdot g_1 + \gamma_{*i}g_2 \\ &= \partial_j(\partial_j\gamma_{*i}v) + \nabla\cdot(\gamma_{*i}g_1) + \partial_j\gamma_{*i}\partial_j v - \nabla\gamma_{*i}\cdot g_1 + \gamma_{*i}g_2 =: \nabla\cdot h_1 + h_2.\end{aligned}$$

Hence, localizing with γ_{*i} preserves the divergence source term to leading order. More precisely, h_2 will be suitable for estimating in L^∞ in the sense that $\|h_2\|_{L^\infty} \lesssim_A \|v\|_{W^{1,\infty}(\Omega)} + \|g_1\|_{L^\infty(\Omega)} + \|g_2\|_{L^\infty(\Omega)}$. The next step is to perform the domain flattening procedure. The most important point here is that since the Jacobian determinant of F_i is 1, the corresponding equation for u (using the notation from the proof of Proposition 5.14) becomes

$$\begin{cases} \partial_k(a^{jk}\partial_j u) = \nabla\cdot\tilde{h}_1 + \tilde{h}_2 & \text{in } \Omega, \\ u_{|z_d=0} = (\gamma_{*i}\psi)(H_i(\tilde{z},0)) & \text{on } \partial\Omega, \end{cases}$$

where

$$\tilde{h}_1 := (h_1\cdot D\Psi_i)(F_i), \qquad \tilde{h}_2 := h_2(F_i).$$

In other words, the divergence structure of the equation is preserved. From this point, the proof follows the same line of reasoning as the $C^{2,\alpha}$ estimates by writing an equation for Δu. The difference is that we use the $C^{1,\alpha}$ norm and the corresponding estimate for the Laplace equation in the half-space when the equation has the above divergence form. □

When g_1 and g_2 are zero in the above proposition, we can interpolate using the maximum principle for $\mathcal{H}$ and the $C^{1,\epsilon}$ bound above to obtain C^α bounds for the harmonic extension with constant depending only on A_Γ.

Corollary 5.16. *Let $0 \leq \alpha < 1$. The following low regularity bound for $\mathcal{H}$ holds uniformly for domains Ω with boundary $\Gamma \in \Lambda_*$,*

$$\|\mathcal{H}g\|_{C^\alpha(\Omega)} \lesssim_A \|g\|_{C^\alpha(\Gamma)}.$$

Proof. By the above and the maximum principle, we have $C^{1,\epsilon}(\Gamma) \to C^{1,\epsilon}(\Omega)$ and $C^0(\Gamma) \to C^0(\Omega)$ bounds for $\mathcal{H}$ that are uniform in Λ_*. By [34, Example 5.15] we also know that $(C^0(\mathbb{R}^n), C^{1,\epsilon}(\mathbb{R}^n))_{\theta,\infty} = C^\alpha(\mathbb{R}^n)$ for an appropriate choice of θ. Therefore, we just have to transfer the interpolation properties on $\mathbb{R}^n$ for $n = d$ and $n = d-1$ to Ω and Γ, respectively, with constants uniform in the collar. For Ω, we argue as in Proposition 5.1, and on Γ we simply unravel the definition of our function spaces via the partition of unity. □

Remark 5.17. Of course, we note that Corollary 5.16 avoids C^1 and Lipschitz regularity, as these do not fall into the interpolation scale.

5.5. **L^2 based balanced elliptic estimates.** In this subsection, we will prove H^s type estimates for various elliptic problems. In the following analysis, we will always be using the coordinate maps H_i and G_i (as opposed to F_i and Ψ_i from the pointwise estimates) to flatten the boundary since we will now need the $\frac{1}{2}$ gain of regularity on Ω in H^s based norms given by this flattening.

5.5.1. *The Dirichlet problem.* We begin our analysis by proving estimates for the inhomogeneous Dirichlet problem

$$\begin{cases} \Delta v = g & \text{in } \Omega, \\ v = \psi & \text{on } \Gamma. \end{cases}$$

We first recall two baseline estimates which will be used heavily in the derivation of the higher regularity bounds below. The first is when $\psi = 0$, in which case v satisfies the H^1 estimate

$$\|v\|_{H^1(\Omega)} \lesssim_A \|g\|_{H^{-1}(\Omega)}. \tag{5.19}$$

On the other hand, for $\frac{1}{2} < s \leq 1$ and $g = 0$, we have

$$\|v\|_{H^s(\Omega)} \lesssim_A \|\psi\|_{H^{s-\frac{1}{2}}(\Gamma)}. \tag{5.20}$$

The bound (5.19) is completely standard. The bound (5.20) was established by Jerison and Kenig in [29], and even holds, in an appropriate sense, at the endpoint $s = \frac{1}{2}$. For our purposes, we will only need the range $\frac{1}{2} < s \leq 1$, but we do need to quantify the dependence of the implicit constant in [29] on the domain. As noted in [48], the implicit domain dependent constant is, as expected, solely dependent on the Lipschitz character of Ω, so is controlled uniformly in the collar. Formally, [48] only quantifies the domain dependence for the inhomogeneous problem $g \neq 0$, $\psi = 0$, but the analogous homogeneous estimate follows immediately from this and the existence of an extension operator $E : H^{s-\frac{1}{2}}(\Gamma) \to H^s(\Omega)$ for $\frac{1}{2} < s \leq 1$ with norm uniform in Λ_*. In this low regularity range of s, such an operator can be constructed by using the partition of unity for Ω and the construction in [41]. We omit the details.

In a small number of places in the higher energy bounds, the following elliptic estimates which hold on C^{1,ϵ_0} (but not quite Lipschitz) domains will be convenient for simplifying the analysis.

Proposition 5.18. *For every $0 < s < \frac{1}{2} + \epsilon_0$, there holds*

$$\|\Delta^{-1} g\|_{H^{s+1}(\Omega)} \lesssim_A \|g\|_{H^{s-1}(\Omega)}, \qquad \|\mathcal{H}\psi\|_{H^{s+1}(\Omega)} \lesssim_A \|\psi\|_{H^{s+\frac{1}{2}}(\Gamma)}.$$

Proposition 5.18 is well-known to specialists; see, e.g., [40]. We remark that bounds of this type hold in the range $s < \frac{1}{2}$ when the domain is Lipschitz; the excess regularity given by a C^{1,ϵ_0} domain is required to extend the range to $s < \frac{1}{2} + \epsilon_0$.

Next, we move to the higher regularity estimates for the Dirichlet problem.

Proposition 5.19 (Higher regularity bounds for the inhomogeneous Dirichlet problem)**.** *Let Ω be a bounded domain with boundary $\Gamma \in \Lambda_*$. Suppose that v solves the Dirichlet problem*

$$\begin{cases} \Delta v = g & \textit{in } \Omega, \\ v = \psi & \textit{on } \partial\Omega, \end{cases}$$

and let $s \geq 2$. Then for $r \geq 0$, $\alpha \in [0,1], \beta \in [0,1]$ and any sequence of partitions $v := v_j^1 + v_j^2$, we have

$$\|v\|_{H^s(\Omega)} \lesssim_A \|g\|_{H^{s-2}(\Omega)} + \|\psi\|_{H^{s-\frac{1}{2}}(\Gamma)} + \|\Gamma\|_{H^{s+r-\frac{1}{2}}} \sup_{j>0} 2^{-j(\alpha-1+r)} \|v_j^1\|_{C^\alpha(\Omega)} + \sup_{j>0} 2^{j(s-1+\beta-\epsilon)} \|v_j^2\|_{H^{1-\beta}(\Omega)}.$$

Proof. Using the partition of unity, it suffices to estimate $v_i := \gamma_{*i} v$ for each $i \geq 0$. Since the case $i = 0$ is essentially an interior regularity estimate, we focus on the case $i \geq 1$. We define

$$h := \Delta v_i = g\gamma_{*i} + v\Delta\gamma_{*i} + 2\nabla v \cdot \nabla\gamma_{*i}.$$

Using the map $H_i = G_i^{-1}$, we can write a variable coefficient equation for $u := v_i \circ H_i$,

$$\begin{cases} -\Delta u = (a^{ij} - \delta^{ij})\partial_i \partial_j u + b_j \partial_j u - f, \\ u|_{\{z_d=0\}} = (\gamma_{*i}\psi)(H_i(\tilde{z}, 0)). \end{cases}$$

Here (dropping the i index from the partition and now using it as a dummy index), we wrote $a^{lm} := (G^l_{x_k} G^m_{x_k}) \circ H$ (where k is summed over), $b_j := (\Delta G^j) \circ H$ and $f = h \circ H$. As a first step, we prove the following estimate for u:

(5.21)

$$\|u\|_{H^s} \lesssim_A \|f\|_{H^{s-2}} + \|\psi\|_{H^{s-\frac{1}{2}}(\Gamma)} + \|\Gamma\|_{H^{s+r-\frac{1}{2}}} \sup_{j>0} 2^{-j(\alpha-1+r)} \|v_j^1\|_{C^\alpha(\Omega)} + \sup_{j>0} 2^{j(s-1+\beta-\epsilon)} \|v_j^2\|_{H^{1-\beta}(\Omega)}.$$

For this, we use the standard elliptic regularity for the half-space to obtain

$$\|u\|_{H^s} \lesssim_A \|u\|_{L^2} + \|f\|_{H^{s-2}} + \|b_i\partial_i u\|_{H^{s-2}} + \|(a^{ij} - \delta^{ij})\partial_i\partial_j u\|_{H^{s-2}} + \|(\gamma_{*i}\psi)(H_i(\tilde{z}, 0))\|_{H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})}. \tag{5.22}$$

By definition, the last term on the right-hand side is controlled by $\|\psi\|_{H^{s-\frac{1}{2}}(\Gamma)}$. Moreover, by a change of variables and the baseline estimates (5.19) and (5.20), we can control, crudely,

$$\|u\|_{L^2} \lesssim_A \|v_i\|_{L^2(\Omega)} \lesssim_A \|h\|_{L^2(\Omega)} + \|\psi\|_{H^{\frac{1}{2}}(\Gamma)} \lesssim_A \|f\|_{L^2} + \|\psi\|_{H^{\frac{1}{2}}(\Gamma)} \lesssim_A \|f\|_{H^{s-2}} + \|\psi\|_{H^{s-\frac{1}{2}}(\Gamma)}. \tag{5.23}$$

For the purpose of estimating the third and fourth terms on the right-hand side, we may assume that $u \in H^s(\mathbb{R}^d)$ with compact support instead of just $u \in H^s(\mathbb{R}^d_+)$ by using any suitable extension for the half-space. We then recall that in a suitably refined collar, we have

$$\|a^{ij} - \delta^{ij}\|_{L^\infty} + \|DG - I\|_{L^\infty} \ll_A 1.$$

Next, we define a partition of u as follows: First write $v_i = \gamma_{*i}v_j^1 + \gamma_{*i}v_j^2$ and then $u = v_i \circ H_i = (\gamma_{*i}v_j^1) \circ H_i + (\gamma_{*i}v_j^2) \circ H_i =: u_j^1 + u_j^2$. To prove (5.21), it suffices now by interpolation and the above estimates to prove the estimate

$$\|b_i\partial_i u\|_{H^{s-2}} + \|(a^{ij} - \delta^{ij})\partial_i\partial_j u\|_{H^{s-2}} \lesssim_A \|u\|_{H^{s-\epsilon}} + \|DG - I\|_{L^\infty}\|u\|_{H^s} + \mathrm{RHS}(5.21). \tag{5.24}$$

We show the details for $b_i\partial_i u$ since it is the more difficult of the two terms to deal with (as it involves two derivatives applied to the domain flattening map) and because the estimate for $(a^{ij} - \delta^{ij})\partial_j\partial_i u$ follows from a similar analysis. Our first aim is to establish the bound

$$\|b_i\partial_i u\|_{H^{s-2}} \lesssim_A \|(\nabla u)(G) \cdot \Delta G\|_{H^{s-2}} + \mathrm{RHS}(5.24), \tag{5.25}$$

which, to leading order, is essentially like doing an H^{s-2} "change of variables". This bound follows immediately from Proposition 5.7 for $2 \le s \le 3$, so we restrict to $s \ge 3$. To simplify notation a bit, we write $w := b_i\partial_i u$. We begin by applying Proposition 5.5 to obtain

$$\|w\|_{H^{s-2}} \lesssim_A \|(\nabla u)(G) \cdot \Delta G\|_{H^{s-2}} + \|\Gamma\|_{H^{s+r-\frac{1}{2}}} \sup_{j>0} 2^{-j(1+r)} \|w_j^1\|_{L^\infty} + \sup_{j>0} 2^{j(s-2-\epsilon)} \|w_j^2\|_{L^2}, \tag{5.26}$$

where $w = w_j^1 + w_j^2$ is a well-chosen partition which needs to be picked so that we can estimate the latter two terms above by RHS(5.24). We take

$$\begin{aligned} w_j^1 &:= (\Delta P_{<j}G \cdot (\nabla P_{<j}u_j^1)(G))(H), \\ w_j^2 &:= (\Delta P_{<j}G \cdot (\nabla P_{<j}u_j^2)(G) + \Delta P_{<j}G \cdot (\nabla P_{\ge j}u)(G) + \Delta P_{\ge j}G \cdot (\nabla u)(G))(H). \end{aligned}$$

It is then easily verified using the above and (5.26) that we have

$$\|w\|_{H^{s-2}} \lesssim_A \|(\nabla u)(G) \cdot \Delta G\|_{H^{s-2}} + \sup_{j>0} 2^{j(s-2-\epsilon)} \|\Delta P_{\ge j}G \cdot (\nabla u)(G)\|_{L^2} + \mathrm{RHS}(5.24).$$

To estimate the latter term on the right, we use that $s-2-\epsilon>0$ to estimate

$$2^{j(s-2-\epsilon)}\|\Delta P_{\geq j}G\cdot(\nabla u)(G)\|_{L^2}\leq \sup_{l\geq 0}2^{l(s-2-\epsilon)}\|\Delta P_l G\cdot(\nabla u)(G)\|_{L^2}.$$

Then splitting $u=P_{<l}u_l^1+\big(P_{<l}u_l^2+P_{\geq l}u\big)$, a change of variables and a simple application of the Bernstein inequalities allows us to control the above term by the right-hand side of (5.24). This establishes (5.25) for $s\geq 3$. Finally, for each $s\geq 2$, it remains to estimate $\|(\nabla u)(G)\Delta G\|_{H^{s-2}}$ by the right-hand side of (5.24). From a simple paradifferential analysis as in Proposition 5.3, we have

$$\begin{aligned}\|(\nabla u)(G)\cdot\Delta G\|_{H^{s-2}}&\lesssim_A \|(\nabla u)(G)\|_{H^{s-1-\epsilon}}+\|T_{(\nabla u)(G)}\Delta G\|_{H^{s-2}}\\&\lesssim_A \|(\nabla u)(G)\|_{H^{s-1-\epsilon}}+\mathrm{RHS}(5.24),\end{aligned}$$

where, above, to estimate the latter term in the first line, we estimated each summand $P_{<j-4}(\nabla u(G))P_j\Delta G$ in the paradifferential expansion of $T_{(\nabla u)(G)}\Delta G$ using the partition $u=P_{<j}u_j^1+\big(P_{<j}u_j^2+P_{\geq j}u\big)$ and Bernstein's inequality. Then, using Proposition 5.5 and this same partition, we have easily

$$\|(\nabla u)(G)\|_{H^{s-1-\epsilon}}\lesssim_A \mathrm{RHS}(5.24).$$

This establishes the bound (5.24) for $b_i\partial_i u$. The bound for $(a^{ij}-\delta^{ij})\partial_i\partial_j u$ follows similar reasoning, but is easier because it involves only one derivative applied to the domain flattening map, and therefore the initial change of variables performed above is not needed. This concludes the estimate (5.21). Our next step is replace u on the left-hand side of (5.21) with v_i and replace f on the right-hand side with g. Recall first that $v_i=u\circ G_i$ and $f=h\circ H_i$. We may assume that v_i and u are defined on $\mathbb{R}^d$ using Stein's extension or a suitable half-space extension in the case of u. Therefore, using the partition $u=u_j^1+u_j^2$ as defined earlier and Proposition 5.5 we obtain

$$\|v_i\|_{H^s(\Omega)}\lesssim_A \|u\|_{H^s}+\|\Gamma\|_{H^{s+r-\frac{1}{2}}}\sup_{j>0}2^{-j(\alpha-1+r)}\|v_j^1\|_{C^\alpha(\Omega)}+\sup_{j>0}2^{j(s+\beta-1-\epsilon)}\|v_j^2\|_{H^{1-\beta}(\Omega)},$$

where we used that $\|G-Id\|_{H^{s+r}}\lesssim_A \|\Gamma\|_{H^{s+r-\frac{1}{2}}}$.

To conclude we now need only show that

$$\|f\|_{H^{s-2}}\lesssim_A \|g\|_{H^{s-2}(\Omega)}+\|\Gamma\|_{H^{s+r-\frac{1}{2}}}\sup_{j>0}2^{-j(\alpha-1+r)}\|v_j^1\|_{C^\alpha(\Omega)}+\sup_{j>0}2^{j(s+\beta-1-\epsilon)}\|v_j^2\|_{H^{1-\beta}(\Omega)}+\sup_i\|v_i\|_{H^{s-\epsilon}(\Omega)}. \tag{5.27}$$

Expanding out $h=\Delta(v\gamma_{*i})$ and using again a paradifferential expansion similar to Proposition 5.3, the identity $g:=\Delta v$ and the splitting $v=v_j^1+v_j^2$ we observe first that

$$\|h\|_{H^{s-2}(\Omega)}\lesssim_A \|g\|_{H^{s-2}(\Omega)}+\|\Gamma\|_{H^{s+r-\frac{1}{2}}}\sup_{j>0}2^{-j(\alpha-1+r)}\|v_j^1\|_{C^\alpha(\Omega)}+\sup_{j>0}2^{j(s+\beta-1-\epsilon)}\|v_j^2\|_{H^{1-\beta}(\Omega)}+\sup_i\|v_i\|_{H^{s-\epsilon}(\Omega)}.$$

Therefore, we need to only show (5.27) with g replaced by h. For this, we first extend h to a function $\tilde{h}:=\mathcal{E}\Delta(\gamma_{*i}v)$ on $\mathbb{R}^d$ using Stein's extension. Then, using the partition $\tilde{h}=h_j^1+h_j^2$ with $h_j^1=\mathcal{E}\Delta P_{<j}(v_j^1\gamma_{*i})$ and $h_j^2=\mathcal{E}\Delta P_{<j}(v_j^2\gamma_{*i})+\mathcal{E}\Delta P_{\geq j}(v\gamma_{*i})$ together with Proposition 5.5, we obtain (5.27) and conclude the proof. □

We also note a much cruder variant of the above estimate which will be useful for constructing regularization operators later on. As with the corresponding Moser bound in Proposition 5.8, the proposition below could be optimized considerably, but such optimizations will not be needed in this article.

Proposition 5.20 (Cruder variant of the Dirichlet estimates). *Let Γ, v, ψ, g and $s \geq 2$ be as in Proposition 5.19, and assume that $\psi = 0$. Then for every $\delta > 0$, we have the estimate*

$$\|v\|_{H^s(\Omega)} \lesssim_{A,\delta} \|g\|_{H^{s-2}(\Omega)} + \|\Gamma\|_{C^{s+\delta}} \|v\|_{H^1(\Omega)}.$$

Proof. We only give a sketch of the proof since it is essentially a much simpler version of Proposition 5.19. One starts by using the cruder flattening (5.12) as in the pointwise elliptic estimates and writing the corresponding equation for u (using the notation in (5.13)). This flattening is a bit more convenient for this estimate because the source terms in (5.13) are simpler. Moreover, we will only need to measure Γ in pointwise norms, and therefore will not need the $\frac{1}{2}$ gain of regularity from the flattening in Proposition 5.19. As in the proof of Proposition 5.19, we then obtain the preliminary bound

$$\|u\|_{H^s} \lesssim_A \|f\|_{H^{s-2}} + \|(\delta^{jk} - a^{jk})\partial_j u\|_{H^{s-1}}.$$

Using simple paraproduct type estimates and a change of variables, it is straightforward to then estimate

$$\|u\|_{H^s} \lesssim_{A,\delta} \|f\|_{H^{s-2}} + \|\Gamma\|_{C^{s+\delta}} \|v\|_{H^1(\Omega)}. \tag{5.28}$$

Then, to conclude, one estimates using Proposition 5.8 with $r = 0$ and $r = 2$,

$$\|v_i\|_{H^s} \lesssim_{A,\delta} \|u\|_{H^s} + \|\Gamma\|_{C^{s+\delta}} \|v\|_{H^1(\Omega)}, \qquad \|f\|_{H^{s-2}} \lesssim_A \|h\|_{H^{s-2}(\Omega)} + \|\Gamma\|_{C^{s+\delta}} \|v\|_{H^1(\Omega)},$$

and then performs a simple paraproduct analysis to finally estimate

$$\|h\|_{H^{s-2}(\Omega)} \lesssim_A \|g\|_{H^{s-2}(\Omega)} + \|\Gamma\|_{C^{s+\delta}} \|v\|_{H^1(\Omega)} + \|v\|_{H^{s-\epsilon}(\Omega)}.$$

Combining the above and interpolating finishes the proof. □

5.5.2. *Harmonic extension bounds.* By taking $g = 0$ in Proposition 5.19, we obtain the following corollary for the harmonic extension operator $\mathcal{H}$.

Proposition 5.21 (Harmonic extension bounds). *Let Ω be a bounded domain with boundary $\Gamma \in \Lambda_*$. Then the following bound holds for the harmonic extension operator $\mathcal{H}$ when $s \geq 2$, $r \geq 0$, $\beta \in [0, \frac{1}{2})$ and $\alpha \in [0, 1)$,*

$$\|\mathcal{H}\psi\|_{H^s(\Omega)} \lesssim_A \|\psi\|_{H^{s-\frac{1}{2}}(\Gamma)} + \|\Gamma\|_{H^{s+r-\frac{1}{2}}} \sup_{j>0} 2^{-j(\alpha-1+r)} \|\psi_j^1\|_{C^\alpha(\Gamma)} + \sup_{j>0} 2^{j(s-1+\beta-\epsilon)} \|\psi_j^2\|_{H^{\frac{1}{2}-\beta}(\Gamma)}.$$

Here, $\psi = \psi_j^1 + \psi_j^2$ is any sequence of partitions.

Proof. First, Proposition 5.19 yields the estimate

$$\|\mathcal{H}\psi\|_{H^s(\Omega)} \lesssim_A \|\psi\|_{H^{s-\frac{1}{2}}(\Gamma)} + \|\Gamma\|_{H^{s+r-\frac{1}{2}}} \sup_{j>0} 2^{-j(\alpha-1+r)} \|\phi_j^1\|_{C^\alpha(\Omega)} + \sup_{j>0} 2^{j(s-1-\epsilon)} \|\phi_j^2\|_{H^1(\Omega)},$$

where $\phi_j^1 = P_{<j}\mathcal{H}\psi_j^1$ and $\phi_j^2 = P_{<j}\mathcal{H}\psi_j^2 + P_{\geq j}\mathcal{H}\psi$. From the C^α bounds for $\mathcal{H}$ in Corollary 5.16 (which hold only for $\alpha \in [0, 1)$), we have $\|\phi_j^1\|_{C^\alpha(\Omega)} \lesssim \|\psi_j^1\|_{C^\alpha(\Gamma)}$. On the other hand, from (5.20), we obtain

$$\sup_{j>0} 2^{j(s-1-\epsilon)} \|\phi_j^2\|_{H^1(\Omega)} \lesssim_A \|\mathcal{H}\psi\|_{H^{s-\epsilon}(\Omega)} + \sup_{j>0} 2^{j(s-1+\beta-\epsilon)} \|\psi_j^2\|_{H^{\frac{1}{2}-\beta}(\Gamma)}.$$

The proof then concludes by interpolation and again (5.20). □

5.5.3. *Curvature estimate.* With the above local coordinates, we can control the surface regularity in terms of the mean curvature. The following estimate is a slight refinement of Lemma 4.7 as well as Propositions A.2 and A.3 in [41].

Proposition 5.22 (Curvature estimate)**.** *Let $s \geq 2$. The following estimates for $\|\Gamma\|_{H^s}$ and the normal n_Γ hold:*

$$\|\Gamma\|_{H^s} + \|n_\Gamma\|_{H^{s-1}(\Gamma)} \lesssim_A 1 + \|\kappa\|_{H^{s-2}(\Gamma)}.$$

Proof. We only sketch the details as the proof is similar to [41]. As in their proof, let $\{f_i \in H^s(\tilde{R}_i(2r_i))\}$ be the local coordinate functions associated to Γ defined earlier. Let $\gamma : [0,\infty) \to [0,1]$ be a smooth cutoff function supported on $[0, \frac{3}{2}]$ with $\gamma = 1$ on $[0, \frac{5}{4}]$. On each $\tilde{R}_i(2r_i)$, we let

$$\gamma_i(\tilde{z}) = \gamma(\frac{|\tilde{z}|}{r_i}), \qquad \kappa_i(\tilde{z}) = \gamma_i(\tilde{z})\kappa(\tilde{z}, f_i(\tilde{z})), \qquad g_i = \gamma_i f_i.$$

Using the mean curvature formula

$$\kappa(\tilde{z}, f(\tilde{z})) = -\partial_j(\frac{\partial_j f}{\sqrt{1+|\nabla f|^2}}) = -\frac{\Delta f}{(1+|\nabla f|^2)^{\frac{1}{2}}} + \frac{\partial_j f \partial_k f \partial_{jk} f}{(1+|\nabla f|^2)^{\frac{3}{2}}},$$

we obtain the following elliptic equation for g_i:

$$\begin{aligned} -\Delta g_i = &-\frac{\partial_{j_1} f_i \partial_{j_2} f_i}{(1+|\nabla f_i|^2)} \partial_{j_1 j_2} g_i + (1+|\nabla f_i|^2)^{\frac{1}{2}} \kappa_i - \Delta\gamma_i f_i - 2D\gamma_i \cdot Df_i \\ &+ \frac{\partial_{j_1} f_i \partial_{j_2} f_i}{1+|\nabla f_i|^2}(\partial_{j_1 j_2}\gamma_i f_i + \partial_{j_1}\gamma_i \partial_{j_2} f_i + \partial_{j_2}\gamma_i \partial_{j_1} f_i). \end{aligned}$$

As $\|Df_i\|_{L^\infty} \ll 1$ the first term on the right-hand side can be viewed perturbatively. A paradifferential type analysis similar to the estimate for u in Proposition 5.19 together with standard Moser and product type estimates then gives

$$\|g_i\|_{H^s} \lesssim_A \delta\|g_i\|_{H^s} + \|f_i\|_{H^{s-\epsilon}} + \|\kappa\|_{H^{s-2}(\Gamma)}$$

for some $\delta > 0$ small enough (depending on Λ_*). We then obtain

$$\|g_i\|_{H^s} \lesssim_A \|f_i\|_{H^{s-\epsilon}} + \|\kappa\|_{H^{s-2}(\Gamma)},$$

and so, we obtain,

$$\sup_i \|f_i\|_{H^s} \lesssim_A 1 + \|\kappa\|_{H^{s-2}(\Gamma)},$$

which completes the proof. □

5.5.4. *Estimates for the Dirichlet-to-Neumann operator.* Here, we use the above estimates to prove refined bounds for the Dirichlet-to-Neumann operator which is defined by $\mathcal{N} := n_\Gamma \cdot (\nabla\mathcal{H})_{|\Gamma}$. We begin with the following baseline ellipticity estimate.

Lemma 5.23. *The Dirichlet-to-Neumann map on Γ satisfies*

$$\|\psi\|_{H^1(\Gamma)} \lesssim_A \|\mathcal{N}\psi\|_{L^2(\Gamma)} + \|\psi\|_{L^2(\Gamma)}.$$

Proof. Let $v = \mathcal{H}\psi$. We begin by proving the standard estimate

$$\int_\Gamma |\nabla v|^2 dS \lesssim_A \|\mathcal{N}\psi\|^2_{L^2(\Gamma)} + \|\psi\|_{L^2(\Gamma)}\|\psi\|_{H^1(\Gamma)}. \tag{5.29}$$

Let X be a smooth vector field on $\mathbb{R}^d$ which is uniformly transversal to all hypersurfaces in Λ_*. That is, $X \cdot n_\Gamma \gtrsim_A 1$ and $|DX| \lesssim_A 1$. Integration by parts then gives

$$\begin{aligned}\int_\Gamma |\nabla v|^2 \, dS &\lesssim_A \int_\Gamma n_\Gamma \cdot X |\nabla v|^2 \, dS \\ &\lesssim_A \|\nabla v\|_{L^2(\Omega)}^2 + 2\int_\Omega X_j \partial_j \nabla v \cdot \nabla v \, dx \\ &\lesssim_A \|\nabla v\|_{L^2(\Omega)}^2 + 2\int_\Gamma (X \cdot \nabla v) \mathcal{N}\psi \, dS.\end{aligned}$$

For the first term, we have from the $H^{\frac{1}{2}} \to H^1$ harmonic extension bound and straightforward interpolation,

$$\|v\|_{H^1(\Omega)}^2 \lesssim_A \|\psi\|_{H^{\frac{1}{2}}(\Gamma)}^2 \lesssim_A \|\psi\|_{L^2(\Gamma)} \|\psi\|_{H^1(\Gamma)}.$$

Combining this with the Cauchy Schwarz inequality for the second term, we obtain (5.29). Using the partition of unity $(\gamma_{*i})_i$, it straightforward to then estimate

$$\|\psi\|_{H^1(\Gamma)} \lesssim_A \|\psi\|_{L^2(\Gamma)} + \|\nabla^\top v\|_{L^2(\Gamma)} \lesssim_A \|\psi\|_{L^2(\Gamma)} + \|\nabla v\|_{L^2(\Gamma)},$$

where $\nabla^\top$ denotes the projection of ∇ onto the tangent space of Γ. Combining this with (5.29) and Cauchy Schwarz concludes the proof. □

We will also need the reverse inequality.

Lemma 5.24. *The Dirichlet-to-Neumann map on Γ satisfies*

$$\|\mathcal{N}\psi\|_{L^2(\Gamma)} \lesssim_A \|\psi\|_{H^1(\Gamma)}.$$

Proof. Using the same notation as in the above lemma and essentially the same argument, we have the estimate

$$\begin{aligned}&\int_\Gamma (X \cdot n_\Gamma)|\nabla^\top \psi|^2 \, dS + \int_\Gamma (X \cdot n_\Gamma)|\mathcal{N}\psi|^2 \, dS = \int_\Gamma (X \cdot n_\Gamma)|\nabla v|^2 \, dS \\ &\geq -C\|\psi\|_{H^1(\Gamma)}^2 + 2\int_\Gamma (X \cdot \nabla v)\mathcal{N}\psi \, dS\end{aligned}$$

for some constant C depending only on A. Writing $X^\top := X - (X \cdot n_\Gamma)n_\Gamma$, we obtain

$$\int_\Gamma (X \cdot n_\Gamma)|\mathcal{N}\psi|^2 \, dS \leq C\|\psi\|_{H^1(\Gamma)}^2 + \int_\Gamma (X \cdot n_\Gamma)|\nabla^\top \psi|^2 \, dS - 2\int_\Gamma X^\top \cdot \nabla v \mathcal{N}\psi \, dS,$$

which by Cauchy Schwarz completes the proof. □

Next, we prove higher regularity versions of these bounds. The first bound below amounts essentially to elliptic regularity estimates for the Neumann boundary value problem.

Proposition 5.25 (Ellipticity for the Dirichlet-to-Neumann operator I)**.** *Let $s \geq \frac{3}{2}$, $\alpha \in [0,1)$ and $\beta \in [0, \frac{1}{2})$. Then we have*

(5.30)
$$\|\psi\|_{H^s(\Gamma)} \lesssim_A \|\psi\|_{L^2(\Gamma)} + \|\mathcal{N}\psi\|_{H^{s-1}(\Gamma)} + \|\Gamma\|_{H^{s+r}} \sup_{j>0} 2^{-j(r+\alpha-1)} \|\psi_j^1\|_{C^\alpha(\Gamma)} + \sup_{j>0} 2^{j(s+\beta-\frac{1}{2}-\epsilon)} \|\psi_j^2\|_{H^{\frac{1}{2}-\beta}(\Gamma)}.$$

Proof. The proof of this is very similar to the Dirichlet problem, so we only sketch the details. Indeed, write $v := \mathcal{H}\psi$. By Proposition 5.11, (5.20) and the $C^\alpha \to C^\alpha$ bound for $\mathcal{H}$, it suffices to control v in $H^{s+\frac{1}{2}}(\Omega)$ by the right-hand side of (5.30). As with the Dirichlet problem, the procedure is to write the Laplace equation for $u = v_i \circ H_i$ and to reduce matters to the standard estimate for the Neumann problem on the half-space (which is available since $s > 1$). The only added technicality is that there are extra source terms coming from the Neumann data (in contrast to the source terms which do not appear for the Dirichlet problem with zero boundary data). By using Proposition 5.11 and an analysis similar to Proposition 5.19, it is straightforward to obtain the preliminary estimate

$$\|\psi\|_{H^s(\Gamma)} \lesssim_A \|v\|_{H^1(\Omega)} + \|\mathcal{N}\psi\|_{H^{s-1}(\Gamma)} + \|\Gamma\|_{H^{s+r}} \sup_{j>0} 2^{-j(r+\alpha-1)}\|v_j^1\|_{C^\alpha(\Omega)} + \sup_{j>0} 2^{j(s+\beta-\frac{1}{2}-\epsilon)}\|v_j^2\|_{H^{1-\beta}(\Omega)},$$

where $v := v_j^1 + v_j^2$ is any partition of v. The first term $\|v\|_{H^1(\Omega)}$ is harmless and can be controlled by $\|\psi\|_{L^2(\Gamma)} + \|\mathcal{N}\psi\|_{L^2(\Gamma)}$ using the $H^{\frac{1}{2}} \to H^1$ bound for $\mathcal{H}$ and Lemma 5.23. We then take $v_j^1 = \mathcal{H}\psi_j^1$ and $v_j^2 = \mathcal{H}\psi_j^2$ and use again the $C^\alpha \to C^\alpha$ bounds for $\mathcal{H}$ and (5.20) to conclude. □

We will also need the following iterated version of the ellipticity bound above.

Proposition 5.26 (Ellipticity for the Dirichlet-to-Neumann operator II)**.** *Let $s \geq \frac{1}{2}$ and let $k \geq 1$ be an integer. Then using the same notation as the previous proposition, we have the bound*

$$\|\psi\|_{H^{s+k}(\Gamma)} \lesssim_A \|\psi\|_{L^2(\Gamma)} + \|\mathcal{N}^k\psi\|_{H^s(\Gamma)} + \|\Gamma\|_{H^{s+k+r}} \sup_{j>0} 2^{-j(\alpha-1+r)}\|\psi_j^1\|_{C^\alpha(\Gamma)} + \sup_{j>0} 2^{j(s+k-\frac{1}{2}+\beta-\epsilon)}\|\psi_j^2\|_{H^{\frac{1}{2}-\beta}(\Gamma)}.$$

Proof. Lemma 5.23 and Proposition 5.25 give us this bound for $k = 1$. For $k \geq 2$, we may assume inductively that the corresponding estimate holds for all $1 \leq m \leq k-1$. We begin by applying Proposition 5.25 to obtain

(5.31)
$$\|\psi\|_{H^{s+k}(\Gamma)} \lesssim_A \|\psi\|_{L^2(\Gamma)} + \|\mathcal{N}\psi\|_{H^{s+k-1}(\Gamma)} + \|\Gamma\|_{H^{s+k+r}} \sup_{j>0} 2^{-j(\alpha-1+r)}\|\psi_j^1\|_{C^\alpha(\Gamma)} + \sup_{j>0} 2^{j(s+k-\frac{1}{2}+\beta-\epsilon)}\|\psi_j^2\|_{H^{\frac{1}{2}-\beta}(\Gamma)}.$$

Using the inductive hypothesis, we have

$$\|\mathcal{N}\psi\|_{H^{s+k-1}(\Gamma)} \lesssim_A \|\mathcal{N}\psi\|_{L^2(\Gamma)} + \|\mathcal{N}^k\psi\|_{H^s(\Gamma)} + \|\Gamma\|_{H^{s+k+r}} \sup_{j>0} 2^{-jr}\|\phi_j^1\|_{L^\infty(\Gamma)} + \sup_{j>0} 2^{j(s+k-1-2\epsilon)}\|\phi_j^2\|_{H^\epsilon(\Gamma)},$$

where $\mathcal{N}\psi := \phi_j^1 + \phi_j^2$ is any partition of $\mathcal{N}\psi$. By Lemma 5.24, the first term on the right can be controlled by $\|\psi\|_{H^1(\Gamma)}$ which can be dispensed with by interpolation (between L^2 and $H^{1+\epsilon}$ to ensure the domain dependent contributions in the estimate are harmless). Therefore, to conclude, we need to choose ϕ_j^1 and ϕ_j^2 so that the latter two terms on the right-hand side of the above are controlled by the right-hand side of (5.31). Using v, v_j^1 and v_j^2 from the previous proposition, we can take $\phi_j^1 = \nabla_n P_{<j} v_j^1$ and $\phi_j^2 = \nabla_n P_{<j} v_j^2 + \nabla_n P_{\geq j} v$. The proof then concludes in a similar way to Proposition 5.25. We omit the details. □

For our energy estimates, we will also need good bounds for the following div-curl system.

Proposition 5.27 (div-curl estimate with Neumann type data)**.** *Let $v \in H^s(\Omega)$ be a vector field defined on Ω and let $s > \frac{3}{2}$, $\alpha, \beta \in [0,1]$. Let $v := v_j^1 + v_j^2$ be any partition of v. Moreover, let $\mathcal{B}v$ denote either the Neumann trace of v, $n_\Gamma \cdot \nabla v$ or the boundary value $\nabla^\top v \cdot n_\Gamma$. Then if v solves the div-curl system,*

$$\begin{cases} \nabla \cdot v = f, \\ \nabla \times v = \omega, \\ \mathcal{B}v = g, \end{cases}$$

then v satisfies the estimate,

$$\|v\|_{H^s(\Omega)} \lesssim_A \|f\|_{H^{s-1}(\Omega)} + \|\omega\|_{H^{s-1}(\Omega)} + \|g\|_{H^{s-\frac{3}{2}}(\Gamma)} + \|v\|_{L^2(\Omega)} + \|\Gamma\|_{H^{s+r-\frac{1}{2}}} \sup_{j>0} 2^{-j(r+\alpha-1)}\|v_j^1\|_{C^\alpha(\Omega)}$$
$$+ \sup_{j>0} 2^{j(s-1+\beta-\epsilon)}\|v_j^2\|_{H^{1-\beta}(\Omega)}.$$

Proof. The proof is very similar to the Dirichlet and Neumann problems in that one flattens the boundary and reduces to the corresponding estimate on the half-space with source terms depending on essentially f, ω, g and the domain regularity. We omit the details of the domain flattening as it is similar to Proposition 5.19. However, for the sake of clarity, it is instructive to explain the div-curl estimate in the case when Ω is the half-space $\{z_d < 0\}$ (particularly in the case of the latter boundary condition involving $\nabla^\top v \cdot n_\Gamma$). We show that it is in essence a statement about elliptic regularity for the Neumann problem. In such a setting, n_Γ takes the form e_d. We compute for each (Euclidean) component v_j of a vector field v on Ω,

$$\Delta v_j = \partial_i \omega_{ij} + \partial_j f.$$

Therefore, in the case of boundary data given by $\mathcal{B}v = n_\Gamma \cdot \nabla v$, the div-curl estimate is simply given by elliptic regularity for the Neumann problem. To understand the case of the other boundary value $\nabla^\top v \cdot n_\Gamma$, we note that the full Neumann data for v is determined by this boundary value and the curl and divergence of v. If $j \neq d$, this is seen from the identity

$$\partial_d v_j = \partial_j v_d + \omega_{dj}.$$

So, by the trace theorem and elliptic regularity for the Neumann problem, we have the desired control of v_j for $j \neq d$. If $j = d$, we have

$$\partial_d v_d = f - \sum_{i=1}^{d-1} \partial_i v_i,$$

which by the trace theorem and the estimate for v_i with $i \neq d$ gives us the estimate for v_d. □

We importantly do not claim that the above div-curl system is well-posed. In fact, the problem is generally over-determined (as, for instance, the curl and divergence fix Δv, which forbids certain choices of Neumann data). Fortunately, we will only need the above estimate in our analysis later when we prove energy estimates and to a lesser extent in our construction of regular solutions. We will not need any existence type statement for the above system, however.

Next, to complement the ellipticity estimates for $\mathcal{N}$, we will also need the reverse estimates which control powers of $\mathcal{N}$ applied to a function in terms of the corresponding Sobolev norms of that function. As a preliminary step, we state the following proposition.

Proposition 5.28 (Normal derivative trace bound)**.** *Let $s > 0$, $r \geq 0$ and $\alpha, \beta \in [0,1]$. The normal trace operator $\nabla_n := n_\Gamma \cdot (\nabla)_{|\Gamma}$ satisfies the bound*

$$\|\nabla_n v\|_{H^s(\Gamma)} \lesssim_A \|v\|_{H^{s+\frac{3}{2}}(\Omega)} + \|\Gamma\|_{H^{s+r+1}} \sup_{j>0} 2^{-j(r-1+\alpha)} \|v_j^1\|_{C^\alpha(\Omega)} + \sup_{j>0} 2^{j(s+\beta+\frac{1}{2}-\epsilon)} \|v_j^2\|_{H^{1-\beta}(\Omega)}.$$

Proof. Using the partition $\nabla v = w_j^1 + w_j^2$ where $w_j^1 := \nabla P_{<j} v_j^1$ and $w_j^2 = \nabla P_{<j} v_j^2 + \nabla P_{\geq j} v$ together with the inequalities $\|n_\Gamma\|_{H^{s+r}(\Gamma)} \lesssim_A \|\Gamma\|_{H^{s+r+1}}$ and $\|n_\Gamma\|_{C^\epsilon(\Gamma)} \lesssim_A 1$, we obtain from Proposition 5.9 and Proposition 5.11 (after possibly relabelling ϵ),

$$\begin{aligned}\|\nabla_n v\|_{H^s(\Gamma)} &\lesssim_A \|(\nabla v)_{|\Gamma}\|_{H^s(\Gamma)} + \|\Gamma\|_{H^{s+r+1}} \sup_{j>0} 2^{-jr} \|w_j^1\|_{L^\infty(\Omega)} + \sup_{j>0} 2^{j(s-2\epsilon)} \|w^2_{j|\Gamma}\|_{L^2(\Gamma)} \\ &\lesssim_A \|v\|_{H^{s+\frac{3}{2}}(\Omega)} + \|\Gamma\|_{H^{s+r+1}} \sup_{j>0} 2^{-jr} \|w_j^1\|_{L^\infty(\Omega)} + \sup_{j>0} 2^{j(s-2\epsilon)} \|w_j^2\|_{H^{\frac{1}{2}+\epsilon}(\Omega)}.\end{aligned}$$

By estimating

$$\|w_j^1\|_{L^\infty(\Omega)} \lesssim_A 2^{j(1-\alpha)} \|v_j^1\|_{C^\alpha(\Omega)}$$

and

$$2^{j(s-2\epsilon)}\|w_j^2\|_{H^{\frac{1}{2}+\epsilon}(\Omega)} \lesssim_A \|v\|_{H^{s+\frac{3}{2}}(\Omega)} + 2^{j(s+\frac{1}{2}+\beta-\epsilon)}\|v_j^2\|_{H^{1-\beta}(\Omega)},$$

we complete the proof. □

We can use Proposition 5.28 and the balanced bounds for $\mathcal{H}$ to prove a refined version of the $H^{s+1}(\Gamma) \to H^s(\Gamma)$ bound for $\mathcal{N}$.

Proposition 5.29 (Dirichlet-to-Neumann operator bound I)**.** *Let $s \geq \frac{1}{2}$, $r \geq 0$, $\alpha \in [0,1)$ and $\beta \in [0,\frac{1}{2})$. Then*

$$\|\mathcal{N}\psi\|_{H^s(\Gamma)} \lesssim_A \|\psi\|_{H^{s+1}(\Gamma)} + \|\Gamma\|_{H^{s+1+r}} \sup_{j>0} 2^{-j(r-1+\alpha)}\|\psi_j^1\|_{C^\alpha(\Gamma)} + \sup_{j>0} 2^{j(s+\frac{1}{2}+\beta-\epsilon)}\|\psi_j^2\|_{H^{\frac{1}{2}-\beta}(\Gamma)}$$

for any sequence of partitions $\psi = \psi_j^1 + \psi_j^2$.

Proof. The proof begins by writing $\mathcal{N} = \nabla_n \mathcal{H}$ and applying Proposition 5.28 to obtain

$$\|\mathcal{N}\psi\|_{H^s(\Gamma)} \lesssim_A \|\mathcal{H}\psi\|_{H^{s+\frac{3}{2}}(\Omega)} + \|\Gamma\|_{H^{s+1+r}} \sup_{j>0} 2^{-j(r-1+\alpha)}\|\mathcal{H}\psi_j^1\|_{C^\alpha(\Omega)} + \sup_{j>0} 2^{j(s+\frac{1}{2}+\beta-\epsilon)}\|\mathcal{H}\psi_j^2\|_{H^{1-\beta}(\Omega)}.$$

Using the $C^\alpha \to C^\alpha$ bounds for $\mathcal{H}$, (5.20) and Proposition 5.21, we conclude the proof. □

Similarly to the ellipticity estimate for $\mathcal{N}$, we will need a higher order version of the above estimate as well.

Proposition 5.30 (Dirichlet-to-Neumann operator bound II)**.** *Let $m \geq 1$ be an integer, let $s \geq \frac{1}{2}$ and let $r \geq 0$, $\alpha \in [0,1)$ and $\beta \in [0,\frac{1}{2})$. Then we have the bound*

$$\|\mathcal{N}^m\psi\|_{H^s(\Gamma)} \lesssim_A \|\psi\|_{H^{s+m}(\Gamma)} + \|\Gamma\|_{H^{s+r+m}} \sup_{j>0} 2^{-j(r+\alpha-1)}\|\psi_j^1\|_{C^\alpha(\Gamma)} + \sup_{j>0} 2^{j(s-\frac{1}{2}+m+\beta-\epsilon)}\|\psi_j^2\|_{H^{\frac{1}{2}-\beta}(\Gamma)}$$

and the closely related bound when $s \geq \frac{3}{2}$,

(5.32)
$$\|\mathcal{H}\mathcal{N}^m\psi\|_{H^{s+\frac{1}{2}}(\Omega)} \lesssim_A \|\psi\|_{H^{s+m}(\Gamma)} + \|\Gamma\|_{H^{s+r+m}} \sup_{j>0} 2^{-j(r+\alpha-1)}\|\psi_j^1\|_{C^\alpha(\Gamma)} + \sup_{j>0} 2^{j(s-\frac{1}{2}+m+\beta-\epsilon)}\|\psi_j^2\|_{H^{\frac{1}{2}-\beta}(\Gamma)}$$

for any partition $\psi = \psi_j^1 + \psi_j^2$.

Proof. We begin with the first bound. The previous proposition handles the case $m = 1$. Suppose $m > 1$ and let us suppose inductively that the bound holds for all integers greater than or equal to 1 and strictly less than m. Then we have from the inductive hypothesis,

$$\|\mathcal{N}^m\psi\|_{H^s(\Gamma)} \lesssim_A \|\mathcal{N}\psi\|_{H^{s+m-1}(\Gamma)} + \|\Gamma\|_{H^{s+m+r}} \sup_{j>0} 2^{-jr}\|\phi_j^1\|_{L^\infty(\Gamma)} + \sup_{j>0} 2^{j(s-1+m-\epsilon)}\|\phi_j^2\|_{H^\epsilon(\Gamma)}, \tag{5.33}$$

where $\mathcal{N}\psi := \phi_j^1 + \phi_j^2$ is the same partition of $\mathcal{N}\psi$ as in the proof of Proposition 5.26. Applying the inductive hypothesis again to the first term on the right and arguing the same way as in Proposition 5.26 to control the latter two terms in favour of ψ, ψ_j^1 and ψ_j^2 concludes the proof of the first estimate. To obtain the latter estimate, we proceed in a similar way as above. For the case $m = 1$, we can use Proposition 5.21 to control $\|\mathcal{H}\mathcal{N}\psi\|_{H^{s+\frac{1}{2}}(\Omega)}$ by the right-hand side of (5.33). Then one concludes the bound for all $m \geq 1$ by induction as above. □

Next, we note a bound for the operator $\nabla^\top$ which follows from similar reasoning to the above.

Proposition 5.31. *Let $s \geq \frac{1}{2}$, $r \geq 0$, $\alpha \in [0,1)$ and $\beta \in [0, \frac{1}{2})$. Then*

$$\|\nabla^\top \psi\|_{H^s(\Gamma)} \lesssim_A \|\psi\|_{H^{s+1}(\Gamma)} + \|\Gamma\|_{H^{s+1+r}} \sup_{j>0} 2^{-j(r-1+\alpha)} \|\psi_j^1\|_{C^\alpha(\Gamma)} + \sup_{j>0} 2^{j(s+\frac{1}{2}+\beta-\epsilon)} \|\psi_j^2\|_{H^{\frac{1}{2}-\beta}(\Gamma)} \tag{5.34}$$

for any sequence of partitions $\psi = \psi_j^1 + \psi_j^2$.

Proof. By writing

$$\nabla^\top \psi = \nabla \mathcal{H}\psi - n_\Gamma \mathcal{N}\psi,$$

the proof follows essentially the same line of reasoning as the proofs of Proposition 5.28 and Proposition 5.29. We omit the details. □

Finally, we note a bound for $\mathcal{N}^m \nabla_n$ which will be needed frequently in the higher energy bounds.

Corollary 5.32. *Let $\alpha, \beta \in [0,1]$, $s \geq \frac{1}{2}$ and $r \geq 0$. We have*

$$\|\mathcal{N}^m \nabla_n v\|_{H^s(\Gamma)} \lesssim_A \|v\|_{H^{s+m+\frac{3}{2}}(\Omega)} + \|\Gamma\|_{H^{s+1+m+r}} \sup_{j>0} 2^{-j(r+\alpha-1)} \|v_j^1\|_{C^\alpha(\Omega)} + \sup_{j>0} 2^{j(s+\beta+\frac{1}{2}+m-\epsilon)} \|v_j^2\|_{H^{1-\beta}(\Omega)}$$

where $v = v_j^1 + v_j^2$ is any sequence of partitions of v.

Proof. We omit most of the details. The proof proceeds by first using Proposition 5.30 with the partition $\nabla_n v = n_\Gamma \cdot w^1_{j|\Gamma} + n_\Gamma \cdot w^2_{j|\Gamma}$ in $L^\infty(\Gamma) + H^\epsilon(\Gamma)$ where w_j^1 and w_j^2 are as in the proof of Proposition 5.28 and then using Proposition 5.28 to estimate $\nabla_n v$ in H^{s+m}. □

5.6. **Moving surface identities.** In this section, we suppose that Ω_t is a one parameter family of domains with boundaries $\Gamma_t \in \Lambda_*$ which flow with a velocity vector field v that is not necessarily divergence free. Our purpose is to collect various identities and commutator estimates involving the material derivative $D_t := \partial_t + v \cdot \nabla$ and functions on Γ_t. We begin by recalling several algebraic identities, many of which were proven in [41].

(i) (Material derivative of the normal).

$$D_t n_{\Gamma_t} = -\left((\nabla v)^*(n_{\Gamma_t})\right)^\top. \tag{5.35}$$

(ii) (Leibniz rule for $\mathcal{N}$).

$$\mathcal{N}(fg) = f\mathcal{N}g + g\mathcal{N}f - 2\nabla_n \Delta^{-1}(\nabla \mathcal{H}f \cdot \nabla \mathcal{H}g). \tag{5.36}$$

(iii) (Commutator with ∇).

$$[D_t, \nabla]g = -(\nabla v)^*(\nabla g). \tag{5.37}$$

(iv) (Commutator with Δ^{-1}).

$$[D_t, \Delta^{-1}]g = \Delta^{-1}\left(2\nabla v \cdot \nabla^2 \Delta^{-1} g + \Delta v \cdot \nabla \Delta^{-1} g\right). \tag{5.38}$$

(v) (Commutator with $\mathcal{H}$).

$$S_0 f := [D_t, \mathcal{H}]f = \Delta^{-1}(2\nabla v \cdot \nabla^2 \mathcal{H}f + \nabla \mathcal{H}f \cdot \Delta v). \tag{5.39}$$

(vi) (Commutator with $\mathcal{N}$).

$$S_1 f := [D_t, \mathcal{N}]f = D_t n_{\Gamma_t} \cdot \nabla \mathcal{H}f - n_{\Gamma_t} \cdot ((\nabla v)^*(\nabla \mathcal{H}f)) + n_{\Gamma_t} \cdot \nabla([D_t, \mathcal{H}]f). \tag{5.40}$$

We also have the general Leibniz type formula,

$$\frac{d}{dt}\int_{\Gamma_t} f dS = \int_{\Gamma_t} D_t f + f(\mathcal{D}\cdot v^\top - \kappa v^\perp)\, dS, \tag{5.41}$$

where $\mathcal{D}$ is the covariant derivative.

5.6.1. *Balanced commutator estimates.* Using the above identities, we now establish refined estimates for commutators involving D_t and the Dirichlet-to-Neumann operator. If we assume that v is divergence free, it is a straightforward calculation to verify that $S_0\psi$ can be rewritten in the form

$$S_0\psi = \Delta^{-1}\nabla\cdot\mathcal{B}(\nabla v, \nabla\mathcal{H}\psi), \tag{5.42}$$

where $\mathcal{B}$ is an $\mathbb{R}^d$-valued bilinear form. Using (5.40), we can write the commutator $[D_t, \mathcal{N}]$ as follows:

$$S_1\psi := [D_t,\mathcal{N}]\psi = \nabla_n S_0\psi - \nabla\mathcal{H}\psi\cdot(\nabla_n v) - \nabla^\top\psi\cdot\nabla v\cdot n_{\Gamma_t}.$$

In the higher energy bounds, we will need an estimate for higher order commutators S_k, given by

$$S_k\psi := [D_t,\mathcal{N}^k]\psi = \sum_{l+m=k-1}\mathcal{N}^l[D_t,\mathcal{N}]\mathcal{N}^m\psi, \tag{5.43}$$

where l,m are non-negative integers and $k\in\mathbb{N}$. From now on, let us write $A = \|v\|_{C^{\frac12+\epsilon}(\Omega)} + \|\Gamma\|_{C^{1,\epsilon}}$. For $s\geq\frac12$, we have the following refined estimates for S_k when v is divergence free, which will be useful for estimating S_kD_ta and S_ka, respectively, in the higher energy bounds.

Proposition 5.33. *Suppose that the flow velocity v is divergence free and let $s\geq\frac12$, $k\geq 1$. Then we have the following bounds for S_k.*

(i) (Variant 1). For any sequence of partitions $\psi = \psi_j^1+\psi_j^2$, there holds

$$\begin{aligned}\|S_k\psi\|_{H^s(\Gamma)} \lesssim_A &\|v\|_{W^{1,\infty}(\Omega)}\|\psi\|_{H^{s+k}(\Gamma)} + \|v\|_{H^{s+\frac32+k}(\Omega)}\|\psi\|_{L^\infty(\Gamma)} + \|\Gamma\|_{H^{s+\frac32+k}}\|\psi\|_{L^\infty(\Gamma)}\\ &+\|v\|_{W^{1,\infty}(\Omega)}\|\Gamma\|_{H^{s+k+\frac32}}\sup_{j>0}2^{-\frac{j}{2}}\|\psi_j^1\|_{L^\infty(\Gamma)} + \|v\|_{W^{1,\infty}(\Omega)}\sup_{j>0}2^{j(s+k-\epsilon)}\|\psi_j^2\|_{H^\epsilon(\Gamma)}.\end{aligned}$$

(ii) (Variant 2).

$$\begin{aligned}\|S_k\psi\|_{H^s(\Gamma)} \lesssim_A &\|v\|_{W^{1,\infty}(\Omega)}\|\psi\|_{H^{s+k}(\Gamma)} + \|\Gamma\|_{H^{s+k+1}}(\|\psi\|_{C^{\frac12}(\Gamma)} + \|v\|_{W^{1,\infty}(\Omega)}\|\psi\|_{L^\infty(\Gamma)})\\ &+\|v\|_{H^{s+k+1}(\Omega)}\|\psi\|_{C^{\frac12}(\Gamma)}.\end{aligned}$$

Proof. We will focus on the first estimate as the second one is similar. From (5.43), we need to prove the estimate in (i) with the left-hand side replaced with $\mathcal{N}^l[D_t,\mathcal{N}]\mathcal{N}^m\psi$ where $l+m=k-1$. We will focus first on the term $\mathcal{N}^l(\nabla_nS_0\mathcal{N}^m\psi)$ which is the most difficult to deal with. Let us write $G := \mathcal{B}(\nabla v, \nabla\mathcal{H}\mathcal{N}^m\psi)$ for notational convenience. We begin by applying Corollary 5.32 and then Proposition 5.19 to obtain (using the identity (5.42)),

$$\begin{aligned}\|\mathcal{N}^l(\nabla_nS_0\mathcal{N}^m\psi)\|_{H^s(\Gamma)} \lesssim_A &\ \|G\|_{H^{s+l+\frac12}(\Omega)} + \|\Gamma\|_{H^{s+\frac32+k}}\sup_{j>0}2^{-j(m+\frac32)}\|\Delta^{-1}\nabla\cdot G_j^1\|_{W^{1,\infty}(\Omega)}\\ &+\sup_{j>0}2^{j(s+l+\frac12-\epsilon)}\|\Delta^{-1}\nabla\cdot G_j^2\|_{H^1(\Omega)},\end{aligned}$$

where $G = G_j^1+G_j^2$ is a partition of G defined by taking $G_j^1 = \mathcal{B}(\nabla P_{<j}v, \nabla P_{<j}\mathcal{H}\mathcal{N}_{<j}^m\psi)$, where $\mathcal{N}_{<j} := \nabla_nP_{<j}\mathcal{H}$. Using the $C^{1,\epsilon}$ estimate for Δ^{-1} and the maximum principle for $\mathcal{H}$, it is straightforward to control

$$2^{-j(m+\frac32)}\|\Delta^{-1}\nabla\cdot G_j^1\|_{W^{1,\infty}(\Omega)} \lesssim_A \|v\|_{C^{\frac12+\epsilon}(\Omega)}\|\psi\|_{L^\infty(\Gamma)} \lesssim_A \|\psi\|_{L^\infty(\Gamma)}.$$

Moreover, using the $H^{-1}\to H_0^1$ estimate for Δ^{-1}, we can control the other term by

$$2^{j(s+l+\frac{1}{2}-\epsilon)}\|\Delta^{-1}\nabla\cdot G_j^2\|_{H^1(\Omega)}\lesssim_A 2^{j(s+l+\frac{1}{2}-\epsilon)}\|v\|_{W^{1,\infty}(\Omega)}\|\nabla P_{<j}\mathcal{H}\mathcal{N}^m_{<j}\psi-\nabla\mathcal{H}\mathcal{N}^m\psi\|_{L^2(\Omega)}$$
$$+\|v\|_{H^{s+\frac{3}{2}+k}(\Omega)}\|\psi\|_{L^\infty(\Gamma)}.$$

Finally, it is straightforward (albeit somewhat technical) to verify that the terms on the right-hand side above can be controlled by the right-hand side of (i) using the $H^\epsilon\to H^{\frac{1}{2}+\epsilon}$ bound (5.20), Proposition 5.30, Proposition 5.9 with $g_j^2=g$ (and the fact that $\|n_\Gamma\|_{C^\epsilon(\Gamma)}\lesssim_A 1$) as well as the $H^{\frac{1}{2}+\epsilon}\to H^\epsilon$ trace estimates. Now, we turn to estimating $\|G\|_{H^{s+l+\frac{1}{2}}(\Omega)}$. By performing a paradifferential expansion as in Proposition 5.3, it is easy to see that

$$\|G\|_{H^{s+1+\frac{1}{2}}(\Omega)}\lesssim_A \|v\|_{W^{1,\infty}(\Omega)}\|\mathcal{H}\mathcal{N}^m\psi\|_{H^{s+l+\frac{3}{2}}(\Omega)}+\|T_{\nabla\mathcal{H}\mathcal{N}^m\psi}\nabla v\|_{H^{s+l+\frac{1}{2}}(\Omega)}.$$

Using Proposition 5.21 and Proposition 5.30, the first term on the right can be controlled by the right-hand side of (i). For the latter term, we need to control the l^2 sum of

$$2^{j(s+l+\frac{1}{2})}\|P_j\nabla vP_{<j-4}\nabla\mathcal{H}\mathcal{N}^m\psi\|_{L^2(\Omega)}.$$

For this, we estimate

$$2^{j(s+l+\frac{1}{2})}\|P_j\nabla vP_{<j-4}\nabla\mathcal{H}\mathcal{N}^m\psi\|_{L^2(\Omega)}\lesssim_A 2^{j(s+k+\frac{1}{2})}\|P_j\nabla v\|_{L^2(\Omega)}\|\psi\|_{L^\infty(\Gamma)}$$
$$+2^{j(s+l+\frac{1}{2})}\|v\|_{W^{1,\infty}(\Omega)}\|P_{<j-4}\nabla\mathcal{H}(\mathcal{N}^m-\mathcal{N}^m_{<j})\psi\|_{L^2(\Omega)}.$$

The first term on the right when summed in l^2 is controlled by the right-hand side of (i). The same is true for the latter term after making use of (5.20) and Proposition 5.30. This concludes the full estimate for $\mathcal{N}^l(\nabla_n S_0\mathcal{N}^m\psi)$. The other terms in $\mathcal{N}^l[D_t,\mathcal{N}]\mathcal{N}^m\psi$ are dealt with similarly. □

6. Regularization operators

Let Ω_* be a smooth, bounded domain with boundary Γ_*. In the following, we let Ω be a bounded domain with boundary $\Gamma\in\Lambda(\Gamma_*,\epsilon,\delta)$ where $\epsilon>0$ and $\delta>0$ are small positive constants. As usual, we will abbreviate the above set of hypersurfaces by Λ_* and consider the volume of the associated domains as part of our implicit constants. We recall from (3.2) that we have the diffeomorphism from Γ_* to Γ given by

$$\Phi_\Gamma(x)=x+\eta_\Gamma(x)\nu(x)$$

which parameterizes Γ as a graph over Γ_*. When constructing solutions to the free boundary Euler equations (and also when proving refined energy estimates), it will be important to have a good regularization operator at each dyadic scale which preserves divergence free functions. More precisely, beyond the obvious regularization properties (to be outlined below in more detail), our operators will need to have the following properties.

(i) (Extension property). There is a $\delta_0>0$ such that the following holds: If Ω_j is a domain containing Ω with boundary $\Gamma_j\in\Lambda_*$ such that $\|\mathrm{dist}(x,\Omega)\|_{L^\infty(\Omega_j)}<\delta_0 2^{-j}$ then there is an associated regularization $\Psi_{\leq j}v$ at the dyadic scale 2^j, defined on Ω_j.

(ii) (Regularization is divergence free). Given Ω_j as above, the regularization $\Psi_{\leq j}v$ satisfies $\nabla\cdot\Psi_{\leq j}v=0$ on Ω_j. Here, v is a divergence free function on Ω.

Remark 6.1. The first point will be convenient later for comparing velocities defined on different domains, which are sufficiently close. The second point is important as our regularization operators will not necessarily commute with derivatives (but will commute with derivatives up to lower order terms).

A more precise description of the above regularization operators is given by the following proposition.

Proposition 6.2. *Fix α_0, let v, Ω and Ω_j be as above and let $A = \|\Gamma\|_{C^{1,\epsilon}}$. Then there exists a regularization operator $\Psi_{\leq j}$ which is bounded from $H^s_{div}(\Omega) \to H^s_{div}(\Omega_j)$ for every $s \geq 0$ with the following properties.*

(i) (Regularization bounds).

$$\|\Psi_{\leq j} v\|_{H^{s+\alpha}(\Omega_j)} \lesssim_A 2^{j\alpha} \|v\|_{H^s(\Omega)}, \qquad 0 \leq \alpha.$$

(ii) (Difference bounds).

$$\|(\Psi_{\leq j+1} - \Psi_{\leq j})v\|_{H^{s-\alpha}(\Omega_{j+1})} \lesssim_A 2^{-j\alpha} \|v\|_{H^s(\Omega)}, \qquad 0 \leq \alpha \leq \min\{s, \alpha_0\}.$$

(iii) (Error bounds).

$$\|(I - \Psi_{\leq j})v\|_{H^{s-\alpha}(\Omega)} \lesssim_A 2^{-j\alpha} \|v\|_{H^s(\Omega)}, \qquad 0 \leq \alpha \leq \min\{s, \alpha_0\}.$$

Proof. We begin with a preliminary step of constructing a regularization operator $\Phi_{\leq j}$ with the above three properties which maps $H^s(\Omega)$ to $H^s(\tilde{\Omega}_j)$ where $\tilde{\Omega}_j$ is a neighborhood of Ω_j, but does not necessarily preserve divergence free functions. To do this, we aim to construct a suitable kernel K^j such that

$$\Phi_{\leq j} v(x) = \int_\Omega K^j(x,y) v(y)\, dy.$$

Here, the kernel $K^j(x,y)$ is of the form

$$K^j(x,y) = \sum_{k=0}^{n} K^j_k(x,y)\chi_k(x),$$

where $(\chi_k)_{k=0}^n$ is a partition of unity of a neighborhood of Ω, obtained by selecting an open cover $\{U_k\}_{k=0}^n$ so that there are vectors $(e_k)_{k=1}^n$ all of the same length with e_k outward oriented and uniformly transversal to $\Gamma \cap U_k$. The remaining set U_0 is then chosen to cover the portion of Ω away from the boundary. Let $e_0 = 0$ and take e_k with $k \in \{1, \dots, n\}$ as above. Such a smooth partition of unity can be constructed with bounds depending only on the properties of Λ_*. To construct K^j we consider a smooth bump function ϕ_k with the following properties:

(i) The support of ϕ_k satisfies $\text{supp}\phi_k \subseteq B(e_k, \delta_1)$, $\delta_1 \ll 1$.
(ii) The average of ϕ_k is 1, i.e., $\int_{\mathbb{R}^d} \phi_k(z)\, dz = 1$.
(iii) ϕ_k has zero moments up to some sufficiently large order N, i.e., $\int_{\mathbb{R}^d} z^\alpha \phi_k(z) dz = 0$, $1 \leq |\alpha| \leq N$.

Then, for each $j > 0$, we consider a regularizing kernel

$$K^j_{0,k}(z) := 2^{jd}\phi_k(2^j z).$$

We then define $K^j_k(x,y) := K^j_{0,k}(x-y)$ for $y \in \Omega$. Note that for fixed $x \in U_k$, $K^j_k(x,y)$ is non-zero only if $2^j(x-y) \in B(e_k, \delta_1)$, i.e., y is within distance $2^{-j}\delta_1$ of $x - 2^{-j}e_k$. This is what will allow us to view our kernel K^j not only for $x \in \Omega$ but also for x in a $\mathcal{O}(2^{-j})$ enlargement of Ω. With this in mind, one can check that the family of kernels K^j satisfy the following:

(i) $K^j : \tilde{\Omega}_j \times \Omega \to \mathbb{R}$, where $\tilde{\Omega}_j := \{x \in \mathbb{R}^d : d(x, \Omega) \leq c2^{-j}\}$ with a small universal constant c.
(ii) $|\partial^\alpha_x \partial^\beta_y K^j(x,y)| \lesssim 2^{j(d+|\alpha|+|\beta|)}$, for multi-indices α, β.
(iii) $\int_\Omega K^j(x,y)\, dy = 1$.
(iv) $\int_\Omega K^j(x,y)(x-y)^\alpha\, dy = 0$, $1 \leq |\alpha| \leq N$.

From the definition of K^j, we see that $\Phi_{\leq j} v$ is defined on a neighborhood of Ω_j if δ_0 from property (i) above is small enough. It is then a straightforward matter to verify that $\Phi_{\leq j}$ satisfies the regularization, difference and error bounds in Proposition 6.2 when s and α are integers (the latter two bounds requiring the moment conditions, with $N = N(\alpha_0)$). The general bound follows by interpolation.

It remains to construct the regularization operator $\Psi_{\leq j}$ which preserves divergence free functions. We first note that without loss of generality we may assume that $\Gamma_j \in \Lambda_*$ with the regularization bound

$$\|\Gamma_j\|_{C^{k,\beta}} \lesssim_{A,k,\beta} 2^{j(\beta+k-1-\epsilon)} \tag{6.1}$$

for each integer $k \geq 1$ and real number $0 \leq \beta < 1$. Indeed, for large enough j, by working in local coordinates and using standard mollification techniques we can use the uniform $C^{1,\epsilon}$ regularity of η_Γ to construct a surface $\tilde{\Gamma}_j \in \Lambda_*$ with the bounds (6.1) such that $\tilde{\Gamma}_j$ is within distance $\lesssim_A 2^{-j(1+\epsilon)}$ of Γ. For some small $c > 0$, we can then define a surface Γ_j via the parameterization $\eta_{\Gamma_j} := \eta_{\tilde{\Gamma}_j} + c2^{-j}$. This defines a domain whose boundary has the required regularization bound and which, if δ_0 is small enough, contains all domains within a $\delta_0 2^{-j}$ neighborhood of Ω. Therefore, it suffices to construct $\Psi_{\leq j}$ in the case when Γ_j satisfies (6.1). We make this assumption for the remainder of the construction.

Next, we correct $\Phi_{\leq j} v$ by a gradient potential. We define for $v \in H^s_{div}(\Omega)$,

$$\Psi_{\leq j} v := \Phi_{\leq j} v - \nabla \Delta^{-1}_{\Omega_j} (\nabla \cdot \Phi_{\leq j} v),$$

where $\Delta^{-1}_{\Omega_j}$ is the solution operator for the Dirichlet problem with zero boundary data associated to the domain Ω_j.

To prove the regularization bounds for $\Psi_{\leq j}$, we note that because v is divergence free, we have

$$\nabla \cdot \Phi_{\leq j} v(x) = \sum_{k=0}^{n} \int \phi_k(y) \nabla \chi_k(x) \cdot (v(x - 2^{-j} y) - v(x)) \, dy. \tag{6.2}$$

In other words, no derivatives fall on v or the kernel when taking the divergence. From the above formula, one can easily verify the following bounds for $\nabla \cdot \Phi_{\leq j} v$ for every $s_1, s_2 \geq 0$:

$$\|\nabla \cdot \Phi_{\leq j} v\|_{H^{s_1}(\Omega_j)} \lesssim_A 2^{-js_2} \|v\|_{H^{s_1+s_2}(\Omega)}.$$

To establish the regularization property of $\Psi_{\leq j}$, we use this and (6.1) together with the balanced Dirichlet estimate Proposition 5.20 to obtain

$$\|\nabla \Delta^{-1}_{\Omega_j} (\nabla \cdot \Phi_{\leq j} v)\|_{H^{s+\alpha}(\Omega_j)} \lesssim_A 2^{j\alpha} \|v\|_{H^s(\Omega)}.$$

Therefore, the regularization bound $\|\Psi_{\leq j} v\|_{H^{s+\alpha}(\Omega_j)} \lesssim_A 2^{j\alpha} \|v\|_{H^s(\Omega)}$ follows immediately. The bounds for $\Psi_{\leq j+1} v - \Psi_{\leq j} v$ and $I - \Psi_{\leq j} v$ are analogous. □

Finally, we note the pointwise analogues of the above estimates.

Proposition 6.3. *Given the assumptions of Proposition 6.2, the regularization operator $\Psi_{\leq j}$ satisfies the following pointwise bounds for $0 \leq \alpha < 2$:*

$$\|\Psi_{\leq j} v\|_{C^\alpha(\Omega_j)} \lesssim_A 2^{j\beta} \|v\|_{C^{\alpha-\beta}(\Omega)},$$

for $0 \leq \beta \leq \alpha$, and

$$\|(I - \Psi_{\leq j}) v\|_{C^\alpha(\Omega)} + \|(\Psi_{\leq j+1} - \Psi_{\leq j}) v\|_{C^\alpha(\Omega_{j+1})} \lesssim_A 2^{-j\beta} \|v\|_{C^{\alpha+\beta}(\Omega)},$$

for $\beta \geq 0$.

Proof. The corresponding bounds for $\Phi_{\leq j}$ are straightforward to directly verify. To estimate the gradient correction, we again may assume without loss of generality the bound (6.1) and then use the pointwise estimates from Proposition 5.14 and Proposition 5.15. □

6.1. **Frequency envelopes.** Let $\Gamma \in \Lambda_*$ and let $s > \frac{d}{2}+1$. Suppose that $v \in H^s(\Omega)$ and suppose that $\Gamma \in H^s$ is parameterized in collar coordinates by $x \mapsto x+\eta_\Gamma(x)\nu(x)$. At this point, we define $A := \|\Gamma\|_{C^{1,\epsilon}}+\|v\|_{C^{\frac{1}{2}}(\Omega)}$. Using the extension operator from Proposition 5.12, we have the following Littlewood-Paley decomposition for a function v defined on Ω:

$$v = \sum_{j\geq 0} P_j v,$$

where by abuse of notation $P_j v$ is interpreted to mean $P_j E_\Omega v$ where E_Ω is as in Proposition 5.12 and P_0 is to be interpreted as $P_{\leq 0}$. We also have a corresponding Littlewood-Paley type decomposition for functions on Γ_*. Indeed, denote by $\langle D\rangle_* := (I-\Delta_{\Gamma_*})^{\frac{1}{2}}$. For functions on Γ_*, we then write for $j>0$, $P_j := \varphi(2^{-j}\langle D\rangle_*) - \varphi(2^{-j+1}\langle D\rangle_*)$ and $P_0 := \varphi(\langle D\rangle_*)$ where $\varphi : \mathbb{R}\to\mathbb{R}$ with $\varphi = 1$ on the unit ball and with support in $B_2(0)$. We then have from Proposition 5.12 the almost orthogonality

$$\|(v,\Gamma)\|_{\mathbf{H}^s}^2 \approx_A \sum_{j\geq 0} 2^{2js}\left(\|P_j v\|_{L^2(\mathbb{R}^d)}^2 + \|P_j\eta_\Gamma\|_{L^2(\Gamma_*)}^2\right).$$

The above equivalence will allow us to define $\mathbf{H}^s$ frequency envelopes for states $(v,\Gamma)\in\mathbf{H}^s$ with the l^2 decay required to establish our continuous dependence result as well as the continuity of solutions with values in $\mathbf{H}^s$ later on.

Remark 6.4. To define the Littlewood-Paley decomposition above, we use the extension E_Ω from Proposition 5.12 (as opposed to, e.g., the Stein extension) because of its transparent continuous dependence on the domain. This will be important for establishing continuous dependence of solutions to the free boundary Euler equations with respect to the data when we have to compare frequency envelopes for different initial data.

Definition 6.5 (Frequency envelopes)**.** Let $s > \frac{d}{2}+1$, $\Gamma \in \Lambda_*$ and $(v,\Gamma)\in\mathbf{H}^s$. An $\mathbf{H}^s$ *frequency envelope* for the pair (v,Γ) is a positive sequence c_j such that for each $j\geq 0$,

$$\|P_j v\|_{H^s(\mathbb{R}^d)} + \|P_j\eta_\Gamma\|_{H^s(\Gamma_*)} \lesssim_A c_j\|(v,\Gamma)\|_{\mathbf{H}^s}, \qquad \|c_j\|_{l^2}\lesssim_A 1.$$

We say that the sequence $(c_j)_j$ is admissible if $c_0 \approx_A 1$ and it is slowly varying,

$$c_j \leq 2^{\delta|j-k|}c_k, \qquad j,k\geq 0, \qquad 0<\delta\ll 1.$$

We can always define an admissible frequency envelope by the formula

$$c_j = 2^{-\delta j} + (1+\|(v,\Gamma)\|_{\mathbf{H}^s})^{-1}\max_k 2^{-\delta|j-k|}\left(\|P_k v\|_{H^s(\mathbb{R}^d)} + \|P_k\eta_\Gamma\|_{H^s(\Gamma_*)}\right). \tag{6.3}$$

Unless otherwise stated, we will take this as our formula for c_j. The following proposition will be useful in our construction of rough solutions later on as well as for proving continuity of the data-to-solution map.

Proposition 6.6. *Let* $\Gamma\in\Lambda_*$ *and let* $s>\frac{d}{2}+1$. *Suppose that* $(v,\Gamma)\in\mathbf{H}^s$ *and let* $(c_j)_j$ *be its associated admissible frequency envelope. Then there exists a family of regularized domains* Ω_j *with boundaries* $\Gamma_j\in\Lambda_*$ *and* $\Gamma_j\in H^s$ *along with associated divergence free regularizations* $v_j := \Psi_{\leq j}v$ *defined on a* 2^{-j} *enlargement of* $\Omega_j\cup\Omega$ *such that the following holds.*

(i) *(Good pointwise approximation).*

$$(v_j, \Gamma_j) \to (v, \Gamma) \quad \textit{in } C^1 \times C^{1,\frac{1}{2}} \quad \textit{as } j \to \infty.$$

(ii) *(Uniform bound).*

$$\|(v_j, \Gamma_j)\|_{\mathbf{H}^s} \lesssim_A \|(v, \Gamma)\|_{\mathbf{H}^s}.$$

(iii) *(Higher regularity).*

$$\|(v_j, \Gamma_j)\|_{\mathbf{H}^{s+\alpha}} \lesssim_A 2^{j\alpha} c_j \|(v, \Gamma)\|_{\mathbf{H}^s}, \quad \alpha > 1.$$

(iv) *(Low frequency difference bounds). On a 2^{-j} enlarged neighborhood of $\Omega_j \cup \Omega_{j+1}$, there holds*

$$\|(v_j, \eta_{\Gamma_j}) - (v_{j+1}, \eta_{\Gamma_{j+1}})\|_{L^2 \times L^2} \lesssim_A 2^{-js} c_j \|(v, \Gamma)\|_{\mathbf{H}^s}.$$

Proof. We define Γ_j by the graph parameterization $\eta_{\Gamma_j} = P_{\leq j}\eta_\Gamma$ (using the projections defined above). By Sobolev embedding, we have $|\eta_{\Gamma_j} - \eta_\Gamma| \lesssim 2^{-\frac{3}{2}j}$, and so the existence of the required divergence free regularization $v_j := \Psi_{\leq j} v$ comes from Proposition 6.2.

Next, we turn to verifying the above four properties. We focus on the bounds for v_j as the bounds for Γ_j are similar (and simpler). Properties (i) and (ii) are clear from Sobolev embedding and Proposition 6.2. Next, we turn to property (iii). We begin by establishing this property for $\Phi_{\leq j} v$ and then we will upgrade to the full divergence free regularization $v_j = \Psi_{\leq j} v$. We write w^l as shorthand for $P_l w$ and begin by splitting

$$\|\Phi_{\leq j} v\|_{H^{s+\alpha}} \leq \sum_{l \leq j} \|\Phi_{\leq j} v^l\|_{H^{s+\alpha}} + \sum_{l > j} \|\Phi_{\leq j} v^l\|_{H^{s+\alpha}}.$$

For $l \leq j$, we estimate

$$\|\Phi_{\leq j} v^l\|_{H^{s+\alpha}} \lesssim_A \|v^l\|_{H^{s+\alpha}} \lesssim_A 2^{l\alpha} c_l \|(v, \Gamma)\|_{\mathbf{H}^s} \lesssim_A 2^{j\alpha} c_j 2^{(\alpha-\delta)(l-j)} \|(v, \Gamma)\|_{\mathbf{H}^s}.$$

For $l > j$, we estimate

$$\|\Phi_{\leq j} v^l\|_{H^{s+\alpha}} \lesssim_A 2^{j(\alpha+s)} \|v^l\|_{L^2} \lesssim_A 2^{j\alpha} c_j 2^{(j-l)(s-\delta)} \|(v, \Gamma)\|_{\mathbf{H}^s}.$$

Summing up each contribution gives

$$\|\Phi_{\leq j} v\|_{H^{s+\alpha}} \lesssim_A 2^{j\alpha} c_j \|(v, \Gamma)\|_{\mathbf{H}^s}.$$

To obtain the corresponding bound for $\Psi_{\leq j}$, we simply note that by Proposition 5.20,

$$\|\nabla \Delta^{-1} \nabla \cdot \Phi_{\leq j} v\|_{H^{s+\alpha}} \lesssim_A \|\Phi_{\leq j} v\|_{H^{s+\alpha}} + 2^{j(s+\alpha-\epsilon)} \|\nabla \cdot \Phi_{\leq j} v\|_{L^2}.$$

By (6.2), we have $2^{j(s+\alpha)} \|\nabla \cdot \Phi_{\leq j} v\|_{L^2} \lesssim_A 2^{j\alpha} \|v\|_{H^s}$. Therefore, if we choose δ in the definition of c_j so that $2^{-j\epsilon} \leq c_j$, we have

$$\|\Psi_{\leq j} v\|_{H^{s+\alpha}} \lesssim_A 2^{j\alpha} c_j \|(v, \Gamma)\|_{\mathbf{H}^s}.$$

This establishes property (iii) for $\Psi_{\leq j} v$. The proof of property (iv) is similar except now one can use the difference and error bounds in Proposition 6.2. We omit the details. $\square$

7. Higher energy bounds

Let $k > \frac{d}{2}+1$ be an integer. Our aim in this section is to establish control of the $\mathbf{H}^k$ norm of (v,Γ) in terms of the initial data where the growth of these norms is dictated by the pointwise control parameters A and B below. To accomplish this, we will first construct a coercive energy functional $(v,\Gamma)\mapsto E^k(v,\Gamma)$ associated to each integer $k>\frac{d}{2}+1$ and then we will prove energy estimates for $E^k(v,\Gamma)$ to obtain estimates for $\|(v,\Gamma)\|_{\mathbf{H}^k}$ when (v,Γ) is a solution to the free boundary Euler equations. More precisely, we prove the following theorem.

Theorem 7.1. *Let $s\in\mathbb{R}$ with $s>\frac{d}{2}+1$ and let $k>\frac{d}{2}+1$ be an integer. Fix a collar neighborhood $\Lambda(\Gamma_*,\epsilon,\delta)$ with $\delta>0$ sufficiently small. Then for Γ restricted to Λ_* there exists an energy functional $(v,\Gamma)\mapsto E^k(v,\Gamma)$ such that*

(i) *(Energy coercivity).*

$$E^k(v,\Gamma)\approx_A 1+\|(v,\Gamma)\|^2_{\mathbf{H}^k}. \tag{7.1}$$

(ii) *(Energy propagation). If, in addition to the above, $(v,\Gamma)=(v(t),\Gamma_t)$ is a solution to the free boundary Euler equations, then $E^k(t):=E^k(v(t),\Gamma_t)$ satisfies*

$$\frac{d}{dt}E^k\lesssim_A B\log(1+\|(v,\Gamma)\|_{\mathbf{H}^s})E^k.$$

Here, $A:=1+|\Omega|+\|v\|_{C^{\frac{1}{2}+\epsilon}(\Omega)}+\|\Gamma\|_{C^{1,\epsilon}}$ and $B:=1+\|v\|_{W^{1,\infty}(\Omega)}+\|\Gamma\|_{C^{1,\frac{1}{2}}}$.

By Grönwall's inequality, this gives the single and double exponential bounds

$$\|(v(t),\Gamma_t)\|^2_{\mathbf{H}^k}\lesssim_A \exp\left(\int_0^t C_AB(s)\log(1+\|(v,\Gamma)\|_{\mathbf{H}^s})ds\right)(1+\|(v_0,\Gamma_0)\|^2_{\mathbf{H}^k}).$$

$$\|(v(t),\Gamma_t)\|^2_{\mathbf{H}^k}\lesssim_A \exp\left(\log(C_A(1+\|(v_0,\Gamma_0)\|^2_{\mathbf{H}^k}))\exp\int_0^t C_AB(s)\,ds\right)$$

for all integers $k>\frac{d}{2}+1$.

Remark 7.2. It is important to note that the first part of Theorem 7.1 does not make any reference to the dynamical problem.

7.1. **Constructing the energy functional.** Before establishing the above theorem, we motivate our choice of energy. At this point, the discussion will be heuristic only. There are two quantities to control; namely, the H^k norms of v and Γ. However, these are coupled via the nonlinear evolution, so they must be measured in tandem. We achieve this by working instead with well-chosen *good variables*, which are selected as follows:

i) The vorticity ω. If v is a divergence free vector field on Ω, then in Euclidean coordinates, we have the following relation for Δv_i:

$$\Delta v_i=-\partial_j\omega_{ij},$$

where ω denotes the curl of v. Therefore, v is controlled by ω and a suitable boundary value. However, it turns out to be simpler to view v as the solution to a div-curl system, again with a boundary condition whose choice will be addressed shortly.

ii) The Taylor coefficient a. This variable is used to describe the regularity of the boundary. Precisely, as we will see later, we have the approximate relation

$$\mathcal{N}a\approx a\kappa$$

where κ represents the mean curvature of Γ. Thus, as long as the Taylor sign condition remains satisfied, the H^k norm of Γ should be comparable at leading order to the H^{k-1} norm of a.

iii) The material derivative of the Taylor coefficient, $D_t a$. At leading order this provides information about v via the approximate paradifferential relation

$$D_t a \approx \mathcal{N} T_n v,$$

for a suitable representation of the paraproduct above. This will provide the needed boundary condition for the div-curl system for v.

Thus, at the principal level we have the correspondence

$$v \leftrightarrow (\omega, D_t a), \qquad \Gamma \leftrightarrow a,$$

which will be the basis for our coercivity property. For the first part, it is better to think of v as solving a div-curl system. One might try to think of a rotational/irrotational decomposition $v = v_{rot} + v_{ir}$, where the two components solve div-curl systems as follows:

$$\begin{cases} \operatorname{curl} v_{rot} = \omega, \\ \nabla \cdot v_{rot} = 0, \\ v_{rot} \cdot n_\Gamma = 0 \quad \text{on } \Gamma, \end{cases} \qquad \begin{cases} \operatorname{curl} v_{ir} = 0, \\ \nabla \cdot v_{ir} = 0, \\ v_{ir} \cdot n_\Gamma = v \cdot n_\Gamma \quad \text{on } \Gamma. \end{cases}$$

Unfortunately, such a decomposition is not well-suited for our present problem, essentially due to the fact that in our setting n_Γ has less regularity than v on the free boundary; namely, H^{k-1} versus $H^{k-\frac{1}{2}}$. Hence, we cannot use such a decomposition directly, though a paradifferential form of it will appear later in our existence proof. Instead, we will bypass this difficulty by associating the $D_t a$ variable with $\nabla^\top v \cdot n_\Gamma$, the normal component of the tangential derivatives on the boundary, which will then play the role of the boundary condition in the div-curl system for v. This, in turn, yields the v part of the coercivity bound.

Now we turn our attention to the dynamical side, which ultimately determines the choice of the good variables. There we separate the good variables differently, into the vorticity $\omega \in H^{k-1}(\Omega)$ on one hand, which will provide the interior component of the energy, and the pair $(a, D_t a)$ in $H^{k-1}(\Gamma) \times H^{k-\frac{3}{2}}(\Gamma)$, which carries the boundary component of the energy. For the vorticity, this is immediately clear from the equation

$$D_t \omega_{ij} = -\omega_{ik} \partial_j v_k + \omega_{jk} \partial_i v_k, \tag{7.2}$$

which results from taking curl of (1.1). Based on the transport structure of the vorticity, it is natural to include the quantity $\|\omega\|^2_{H^{k-1}(\Omega)}$ as a component of the energy. On the other hand, it turns out that $\|(a, D_t a)\|^2_{H^{k-1}(\Gamma) \times H^{k-\frac{3}{2}}(\Gamma)}$ can be controlled by the linearized energy $E_{lin}(w_k, s_k)$, where s_k and w_k solve the linearized equation to leading order with

$$\begin{cases} w_k = \nabla \mathcal{H} \mathcal{N}^{k-2} D_t a, \\ s_k = \mathcal{N}^{k-1} a. \end{cases}$$

The derivation for this is a bit more involved than for the vorticity and will be handled later.

With the above discussion in mind, we define our energy as follows:

$$E^k(v, \Gamma) := 1 + \|v\|^2_{L^2(\Omega)} + \|\omega\|^2_{H^{k-1}(\Omega)} + E_{lin}(w_k, s_k). \tag{7.3}$$

In the sequel, we will sometimes refer to $\|\omega\|^2_{H^{k-1}(\Omega)}$ as the rotational part of the energy, denoted by $E^k_r(v,\Gamma)$, and $E_{lin}(w_k,s_k)$ as the irrotational part of the energy, denoted by $E^k_i(v,\Gamma)$.

Remark 7.3. This definition of the energy has to be interpreted in a suitable way when v and Γ do not solve the free boundary Euler equations. Indeed, it is important that, a priori, the definition of the energy functional does not depend on the dynamics of the problem. Therefore, for a bounded connected domain Ω with $(v,\Gamma)\in\mathbf{H}^k$, we define p through the boundary condition $p_{|\Gamma}=0$ and the Laplace equation

$$\Delta p = -\mathrm{tr}(\nabla v)^2.$$

The Taylor sign term is then defined via

$$a := -n_\Gamma\cdot\nabla p_{|\Gamma}.$$

Moreover, we define D_tp through the Dirichlet boundary condition $D_tp_{|\Gamma}=0$ and ΔD_tp given by

$$\Delta D_tp = 4\mathrm{tr}(\nabla^2 p\cdot\nabla v) + 2\mathrm{tr}((\nabla v)^3) + \Delta v\cdot\nabla p =: F. \tag{7.4}$$

In other words, $D_tp=\Delta^{-1}F$. This is the definition of D_tp which is compatible with the dynamical problem. We then define $D_t\nabla p$ by

$$D_t\nabla p := -\nabla v\cdot\nabla p + \nabla D_tp$$

and then D_ta by

$$D_ta := -n_\Gamma\cdot D_t\nabla p_{|\Gamma}.$$

With these definitions, the energy functional (7.3) is well-defined, irrespective of whether the state (v,Γ) evolves dynamically.

Remark 7.4. We note that the energy functional (7.3) is essentially the same as that from [17]. The main difference, so far, is in the derivation of this energy. Indeed, our approach was to identify Alinhac style good unknowns, whereas [17] first derives a wave-type equation for a and then applies powers of the Dirichlet-to-Neumann operator to this equation, as if it were a vector field. However, as can be immediately inferred from the low regularity of our control norms, the way we treat the energy is very different from [17].

7.2. **Coercivity of the energy functional.** We begin by establishing the coercivity part of Theorem 7.1. That is, we want to show that

$$E^k(v,\Gamma)\approx_A 1+\|(v,\Gamma)\|^2_{\mathbf{H}^k}.$$

We begin by collecting some preliminary estimates for the various quantities that will appear in our analysis.

7.3. L^∞ **estimates for coercivity.** Here we will establish some L^∞ based estimates for p and D_tp in terms of the control parameter A. The A control parameter involves only the physical variables v and Γ. The variables p and D_tp are related to these variables through solving a suitable Laplace equation. We will therefore need to make use of the Schauder type estimates in Proposition 5.15 to control these terms (in suitable pointwise norms) by A. For this, we have the following lemma.

Lemma 7.5. *Given the assumptions of Theorem 7.1, the following pointwise estimates for p and D_tp hold.*

(i) ($C^{1,\epsilon}$ estimate for p).

$$\|p\|_{C^{1,\epsilon}(\Omega)}\lesssim_A 1.$$

(ii) (Partition bound for D_tp). There exists a sequence of partitions $D_tp =: F^1_j+F^2_j$ such that

$$\|F^1_j\|_{W^{1,\infty}(\Omega)}\lesssim_A 2^{j(\frac12-\epsilon)},\qquad \|F^2_j\|_{H^1(\Omega)}\lesssim_A 2^{-j(k-1-\epsilon)}(\|v\|_{H^{k-\epsilon}(\Omega)}+\|p\|_{H^{k+\frac12-\epsilon}(\Omega)}).$$

One can loosely think of the partition of D_tp in the second part of Lemma 7.5 as a splitting of D_tp into low and high frequency parts at a dyadic scale 2^j. The high frequency part will typically be best estimated in L^2 based norms, and the low frequency part in L^∞ based norms. In particular, one can think of the estimate for F_j^1 as an estimate for the "low frequency part" of D_tp in $C^{\frac{1}{2}+\epsilon}$. This will serve as a substitute for what would be a $C^{\frac{1}{2}+\epsilon}$ estimate for the inhomogenenous Dirichlet problem, which is not available to us (except for harmonic functions). The usefulness of this will be made more transparent later.

Proof. We begin with some notation. For any integer $l > 0$, we write $\Phi_l := \Phi_{\leq l+1} - \Phi_{\leq l}$ and $\Psi_l := \Psi_{\leq l+1} - \Psi_{\leq l}$. We also write Φ_0 and Ψ_0 to mean $\Phi_{\leq 0}$ and $\Psi_{\leq 0}$, respectively. For a vector or scalar valued function f defined on Ω, we write f^l and $f^{\leq l}$ as shorthand for $\Phi_l f$ and $\Phi_{\leq l} f$, respectively. If in addition, f is a divergence free vector field, we instead use f^l and $f^{\leq l}$ to mean $\Psi_l f$ and $\Psi_{\leq l} f$, respectively. This will ensure that the divergence free structure of f is preserved. We abuse notation and write $f^l g^{\leq l}$ to mean

$$f^l g^{\leq l} := \sum_{l\geq 0}\sum_{0\leq m\leq l} f^l g^m - \frac{1}{2}\sum_{l\geq 0} f^l g^l.$$

This definition ensures (with the convention that $\Phi_0 = \Phi_{\leq 0}$ and $\Psi_0 = \Psi_{\leq 0}$) that we have the decomposition

$$fg = f^l g^{\leq l} + f^{\leq l} g^l, \tag{7.5}$$

which can be thought of as a kind of crude bilinear paraproduct decomposition where $f^l g^{\leq l}$ selects the portion of fg where f is at higher or comparable frequency compared to g. Likewise, we can define trilinear expressions of the form $f^l g^{\leq l} h^{\leq l}$ in such a way that we have $fgh = f^l g^{\leq l} h^{\leq l} + f^{\leq l} g^l h^{\leq l} + f^{\leq l} g^{\leq l} h^l$, and similarly for quadrilinear expressions. Now, we begin with the first part of the lemma. Expanding using (7.5) we see that

$$p = -\Delta^{-1}\mathrm{tr}(\nabla v)^2 = -2\Delta^{-1}\partial_j(v_i^l \partial_i v_j^{\leq l}). \tag{7.6}$$

Importantly, because v^l is divergence free, we were able to write $\mathrm{tr}(\nabla v)^2$ as the divergence of a bilinear expression in v and ∇v, where the high frequency factor is undifferentiated. This will allow us to make use of the lower regularity $C^{1,\alpha}$ estimates in Proposition 5.15 and simultaneously allow us to rebalance derivatives in the bilinear expression for v. This theme of writing multilinear expressions in divergence form with the highest frequency factor undifferentiated will appear several times in the sequel in more complicated forms. In this case, we have from Proposition 5.15,

$$\|p\|_{C^{1,\epsilon}(\Omega)} \lesssim_A \|v_i^l \partial_i v_j^{\leq l}\|_{C^\epsilon(\Omega)} \lesssim_A \|v\|^2_{C^{\frac{1}{2}+\epsilon}(\Omega)} \lesssim_A 1.$$

Next, we turn to the estimate for D_tp, which is the more difficult part. From (7.4), we can write in Euclidean coordinates,

$$D_tp = 4\Delta^{-1}(\partial_i\partial_j p \partial_i v_j) + 2\Delta^{-1}(\partial_j v_k \partial_k v_i \partial_i v_j) + \Delta^{-1}(\partial_i\partial_i v_j \partial_j p). \tag{7.7}$$

In order to make full use of Proposition 5.15, we will again need to write D_tp in the form $\Delta^{-1}\nabla\cdot f$ for some vector field f in a way which allows us to also rebalance derivatives, as we did in the estimate for p. We start by estimating the first term in (7.7). We first write $\partial_i\partial_j p\partial_i v_j = \nabla\cdot(\partial_i p \partial_i v)$ and use the partition

$$\Delta^{-1}\nabla\cdot(\partial_i p\partial_i v) = T_j^1 + T_j^2,$$

where $T_j^1 = \Delta^{-1}\nabla\cdot(\partial_i p \partial_i \Phi_{<j} v)$. From Proposition 5.15 and the $C^{1,\epsilon}$ estimate for p above, we have

$$\|T_j^1\|_{W^{1,\infty}(\Omega)} \lesssim_A \|\nabla p\|_{C^\epsilon(\Omega)}\|\nabla\Phi_{<j} v\|_{L^\infty(\Omega)} + \|\nabla p\|_{L^\infty(\Omega)}\|\nabla\Phi_{<j} v\|_{C^\epsilon(\Omega)} \lesssim_A 2^{j(\frac{1}{2}-\epsilon)}.$$

We also see from (5.19),

$$\|T_j^2\|_{H^1(\Omega)} \lesssim_A 2^{-j(k-1-\epsilon)}\|\nabla p\|_{L^\infty(\Omega)}\|v\|_{H^{k-\epsilon}(\Omega)} \lesssim_A 2^{-j(k-1-\epsilon)}\|v\|_{H^{k-\epsilon}(\Omega)}.$$

Next, we turn to the second term in (7.7). We start by performing a trilinear frequency decomposition. Using the symmetry of the indices, we have

$$\partial_j v_k \partial_k v_i \partial_i v_j = 3\partial_j v_k^l \partial_k v_i^{\leq l} \partial_i v_j^{\leq l}. \tag{7.8}$$

To best balance derivatives, we would like to write this in the form $\nabla \cdot \mathcal{T}(v^l, \nabla v^{\leq l}, \nabla v^{\leq l})$ where $\mathcal{T}$ is an appropriate trilinear expression. To do this, we can use the symmetry of the expression and the fact that v is divergence free to write

$$\begin{aligned}
\partial_j v_k^l \partial_k v_i^{\leq l} \partial_i v_j^{\leq l} &= \partial_j (v_k^l \partial_k v_i^{\leq l} \partial_i v_j^{\leq l}) - v_k^l \partial_k \partial_j v_i^{\leq l} \partial_i v_j^{\leq l} \\
&= \partial_j (v_k^l \partial_k v_i^{\leq l} \partial_i v_j^{\leq l}) - \frac{1}{2} v_k^l \partial_k (\partial_j v_i^{\leq l} \partial_i v_j^{\leq l}) \\
&= \partial_j (v_k^l \partial_k v_i^{\leq l} \partial_i v_j^{\leq l}) - \frac{1}{2} \partial_k (v_k^l \partial_j v_i^{\leq l} \partial_i v_j^{\leq l}).
\end{aligned} \tag{7.9}$$

We partition the last line above into $Q_j^1 + Q_j^2$ where

$$Q_j^1 := \partial_m (v_k^l \partial_k \Phi_{<j} v_i^{\leq l} \partial_i v_m^{\leq l}) - \frac{1}{2} \partial_k (v_k^l \partial_m \Phi_{<j} v_i^{\leq l} \partial_i v_m^{\leq l}).$$

We then obtain in a straightforward way using Proposition 5.15 and summing in l,

$$\|\Delta^{-1} Q_j^1\|_{W^{1,\infty}(\Omega)} \lesssim_A 2^{j(\frac{1}{2}-\epsilon)}\|v\|^3_{C^{\frac{1}{2}+\epsilon}(\Omega)} \lesssim_A 2^{j(\frac{1}{2}-\epsilon)},$$

and from the $H^{-1} \to H^1$ estimate for the Dirichlet problem and Proposition 6.2,

$$\|\Delta^{-1} Q_j^2\|_{H^1(\Omega)} \lesssim_A 2^{-j(k-1-\epsilon)}\|v\|_{H^{k-\epsilon}(\Omega)}.$$

Finally, the last term in (7.7) can be handled by writing

$$\partial_i \partial_i v_j \partial_j p = \partial_i (\partial_i v_j \partial_j p) - \partial_i v_j \partial_i \partial_j p$$

and partitioning each term similarly to the first term in (7.7). Collecting all of the above partitions together completes the proof of the lemma. □

The following simple consequence of the above lemma will be useful for estimating $D_t a$ in pointwise norms.

Corollary 7.6. *Given the assumptions of Lemma 7.5, there exists a sequence of partitions $D_t \nabla p = G_j^1 + G_j^2$ such that*

$$\|G_j^1\|_{L^\infty(\Omega)} \lesssim_A 2^{j(\frac{1}{2}-\epsilon)}, \qquad \|G_j^2\|_{H^{\frac{1}{2}+\epsilon}(\Omega)} \lesssim_A 2^{-j(k-\frac{3}{2}-2\epsilon)}(\|v\|_{H^{k-\epsilon}(\Omega)} + \|p\|_{H^{k+\frac{1}{2}-\epsilon}(\Omega)} + \|D_t \nabla p\|_{H^{k-1-\epsilon}(\Omega)}).$$

Proof. This follows from Lemma 7.5 by taking

$$G_j^1 = \Phi_{<j}(-\nabla \Phi_{<j} v \cdot \nabla p + \nabla F_j^1), \qquad G_j^2 = \Phi_{<j}(-\nabla \Phi_{\geq j} v \cdot \nabla p) + \Phi_{<j} \nabla F_j^2 + \Phi_{\geq j} D_t \nabla p.$$

□

7.4. L^2 **based estimates for** a **and** $D_t a$. Our next step will be to control $(a, D_t a)$ in $H^{k-1}(\Gamma)\times H^{k-\frac{3}{2}}(\Gamma)$ by the energy plus some lower order terms. Let us define for the rest of this section the lower order quantity

$$\Lambda_{k-\epsilon} := \|\Gamma\|_{H^{k-\epsilon}} + \|v\|_{H^{k-\epsilon}(\Omega)} + \|p\|_{H^{k+\frac{1}{2}-\epsilon}(\Omega)} + \|D_t\nabla p\|_{H^{k-1-\epsilon}(\Omega)},$$

where $\epsilon > 0$ is any small, but fixed, positive constant.

Lemma 7.7. *We have*

$$\|a\|_{H^{k-1}(\Gamma)} + \|D_t a\|_{H^{k-\frac{3}{2}}(\Gamma)} \lesssim_A (E^k)^{\frac{1}{2}} + \Lambda_{k-\epsilon}.$$

Proof. To control a in $H^{k-1}(\Gamma)$, we use the ellipticity estimate for the Dirichlet-to-Neumann operator from Proposition 5.26 to obtain

$$\|a\|_{H^{k-1}(\Gamma)} \lesssim_A \|a\|_{L^2(\Gamma)} + \|\mathcal{N}^{k-1}a\|_{L^2(\Gamma)} + \|\Gamma\|_{H^{k-\epsilon}}\|a\|_{C^\epsilon(\Gamma)} \lesssim_A (E^k)^{\frac{1}{2}} + \Lambda_{k-\epsilon}.$$

To estimate $D_t a$ in $H^{k-\frac{3}{2}}(\Gamma)$, we consider the partition $D_t\nabla p := G_j^1 + G_j^2$ from Corollary 7.6 and estimate using Proposition 5.26,

$$\|D_t a\|_{H^{k-\frac{3}{2}}(\Gamma)} \lesssim_A \|\mathcal{N}^{k-2}D_t a\|_{H^{\frac{1}{2}}(\Gamma)} + \|\Gamma\|_{H^{k-\epsilon}}\sup_{j>0} 2^{-j(\frac{1}{2}-\epsilon)}\|n_\Gamma\cdot G_j^1\|_{L^\infty(\Gamma)} + \sup_{j>0} 2^{j(k-2\epsilon-\frac{3}{2})}\|n_\Gamma\cdot G_j^2\|_{H^\epsilon(\Gamma)} + \Lambda_{k-\epsilon}.$$

From the trace theorem,

$$\|\mathcal{N}^{k-2}D_t a\|_{H^{\frac{1}{2}}(\Gamma)} \lesssim_A \|\mathcal{H}\mathcal{N}^{k-2}D_t a\|_{H^1(\Omega)}.$$

Since $k \geq 3$ and

$$\int_\Gamma \mathcal{N}^{k-2}D_t a\, dS = \int_\Gamma n_\Gamma\cdot\nabla\mathcal{H}\mathcal{N}^{k-3}D_t a\, dS = 0,$$

we conclude by a Poincare type inequality that

$$\|\mathcal{H}\mathcal{N}^{k-2}D_t a\|_{H^1(\Omega)} \lesssim_A \|\nabla\mathcal{H}\mathcal{N}^{k-2}D_t a\|_{L^2(\Omega)} \lesssim_A (E^k)^{\frac{1}{2}}.$$

From Corollary 7.6, we have

$$\sup_{j>0} 2^{-j(\frac{1}{2}-\epsilon)}\|n_\Gamma\cdot G_j^1\|_{L^\infty(\Gamma)} \lesssim_A 1.$$

On the other hand, from the trace theorem and Corollary 7.6,

$$2^{j(k-\frac{3}{2}-2\epsilon)}\|n_\Gamma\cdot G_j^2\|_{H^\epsilon(\Gamma)} \lesssim_A \Lambda_{k-\epsilon},$$

which completes the proof. □

With our preliminary estimates in hand, let us proceed with the proof of the first (and harder) half of the coercivity estimate; namely,

$$\|(v,\Gamma)\|_{\mathbf{H}^k} \lesssim_A (E^k)^{\frac{1}{2}}.$$

Let us begin by proving the estimate

$$\|p\|_{H^{k+\frac{1}{2}}(\Omega)} + \|\Gamma\|_{H^k} \lesssim_A (E^k)^{\frac{1}{2}} + \Lambda_{k-\epsilon}. \tag{7.10}$$

We start by recalling from Proposition 5.22 that we have

$$\|\Gamma\|_{H^k} + \|n_\Gamma\|_{H^{k-1}(\Gamma)} \lesssim_A 1 + \|\kappa\|_{H^{k-2}(\Gamma)},$$

where κ is the mean curvature of Γ. Therefore, to establish (7.10), it suffices to establish the same estimate except with $\|p\|_{H^{k+\frac{1}{2}}(\Omega)} + \|\kappa\|_{H^{k-2}(\Gamma)}$ on the left-hand side. To do this, we begin by relating the curvature to the pressure via the formula

$$\kappa = a^{-1}\Delta p - a^{-1}D^2p(n_\Gamma, n_\Gamma). \tag{7.11}$$

Here, we used the fact that $\Delta_\Gamma p = 0$ on Γ. We now estimate each term on the right-hand side of (7.11). For the first term, we use the Laplace equation for p and the bilinear frequency decomposition for $\Delta p = -\mathrm{tr}(\nabla v)^2$ as in Lemma 7.5 together with Proposition 5.9 to obtain

$$\|a^{-1}\Delta p\|_{H^{k-2}(\Gamma)} \lesssim_A \|\mathrm{tr}(\nabla v)^2\|_{H^{k-2}(\Gamma)} + (\|a^{-1}\|_{H^{k-1-\epsilon}(\Gamma)} + \|\Gamma\|_{H^{k-\epsilon}}) \sup_{j>0} 2^{-j(1-\epsilon)} \|\Phi_{<j}\partial_k(v_i^l \partial_i v_k^{\leq l})\|_{L^\infty(\Omega)}$$
$$+ \sup_{j>0} 2^{j(k-2-\epsilon)} \|\Phi_{\geq j}\mathrm{tr}(\nabla v)^2\|_{L^2(\Gamma)}.$$

Using the trace theorem, the product estimates Proposition 5.9 and Corollary 5.4, the latter two terms can be controlled by $C_A\Lambda_{k-\epsilon}$ where C_A is a constant depending polynomially on A only. On the other hand, $\|\mathrm{tr}(\nabla v)^2\|_{H^{k-2}(\Gamma)}$ can be controlled using the balanced trace estimate Proposition 5.11 as well as Corollary 5.4 as follows:

$$\begin{aligned}\|\mathrm{tr}(\nabla v)^2\|_{H^{k-2}(\Gamma)} &\lesssim_A \|\mathrm{tr}(\nabla v)^2\|_{H^{k-\frac{3}{2}}(\Omega)} + \|\Gamma\|_{H^{k-\epsilon}} \sup_{j>0} 2^{-j(1-\epsilon)} \|\Phi_{<j}\partial_k(v_i^l \partial_i v_k^{\leq l})\|_{L^\infty(\Omega)} \\ &\quad + \sup_{j>0} 2^{j(k-\frac{3}{2}-\epsilon)} \|\Phi_{\geq j}\mathrm{tr}(\nabla v)^2\|_{L^2(\Omega)} \\ &\lesssim_A \Lambda_{k-\epsilon}.\end{aligned}$$

To estimate $a^{-1}D^2p(n_\Gamma, n_\Gamma)$ in $H^{k-2}(\Gamma)$, we proceed similarly by starting with Proposition 5.9 and Lemma 7.5 to obtain

$$\|a^{-1}D^2p(n_\Gamma, n_\Gamma)\|_{H^{k-2}(\Gamma)} \lesssim_A \|D^2p(n_\Gamma, n_\Gamma)\|_{H^{k-2}(\Gamma)} + \sup_{j>0} 2^{j(k-2-\epsilon)} \|\Phi_{\geq j}D^2p\|_{L^2(\Gamma)}$$
$$+ (\|a^{-1}\|_{H^{k-1-\epsilon}(\Gamma)} + \|\Gamma\|_{H^{k-\epsilon}}) \sup_{j>0} 2^{-j(1-\epsilon)} \|\Phi_{<j}D^2p\|_{L^\infty(\Omega)}.$$

Similarly to the previous estimate, the latter two terms are controlled by $C_A\Lambda_{k-\epsilon}$. For the term involving $D^2p(n_\Gamma, n_\Gamma)$, we use Proposition 5.9 again, combined with the estimates $\|n_\Gamma\|_{H^{k-1-\epsilon}(\Gamma)} \lesssim_A \|\Gamma\|_{H^{k-\epsilon}}$ and $\|n_\Gamma\|_{C^\epsilon(\Gamma)} \lesssim_A 1$ to obtain (similarly to the above estimate but with a^{-1} replaced by n_Γ)

$$\|D^2p(n_\Gamma, n_\Gamma)\|_{H^{k-2}(\Gamma)} \lesssim_A \|D^2p\|_{H^{k-2}(\Gamma)} + \Lambda_{k-\epsilon}.$$

Proposition 5.11 and the same partition of D^2p above then yields

$$\|D^2p\|_{H^{k-2}(\Gamma)} \lesssim_A \|\nabla p\|_{H^{k-\frac{1}{2}}(\Omega)} + \Lambda_{k-\epsilon}.$$

To complete the proof of (7.10), we now only need to control ∇p in $H^{k-\frac{1}{2}}$. For this, we use the div-curl estimate Proposition 5.27 for ∇p as well as Corollary 5.4, Proposition 5.9 and Proposition 5.31 to obtain

$$\begin{aligned}\|\nabla p\|_{H^{k-\frac{1}{2}}(\Omega)} &\lesssim_A \|\nabla p\|_{L^2(\Omega)} + \|\nabla^\top a\|_{H^{k-2}(\Gamma)} + \|\mathrm{tr}(\nabla v)^2\|_{H^{k-\frac{3}{2}}(\Omega)} + \|\Gamma\|_{H^{k-\epsilon}}\|\nabla p\|_{C^\epsilon(\Omega)} \\ &\lesssim_A (E^k)^{\frac{1}{2}} + \|a\|_{H^{k-1}(\Gamma)} + \Lambda_{k-\epsilon} \\ &\lesssim_A (E^k)^{\frac{1}{2}} + \Lambda_{k-\epsilon},\end{aligned} \tag{7.12}$$

where we used Lemma 7.7 to go from the second to third line. From this, we finally obtain the estimate (7.10). To close the coercivity estimate, it remains to control v in $H^k(\Omega)$ and $D_t\nabla p$ in $H^{k-1}(\Omega)$ by the energy. We first reduce to the estimate

$$\|v\|_{H^k(\Omega)} \lesssim_A (E^k)^{\frac{1}{2}} + \|D_t\nabla p\|_{H^{k-1}(\Omega)} + \Lambda_{k-\epsilon}.$$

For this, we start by relating the boundary term $\nabla^\top v \cdot n_\Gamma$ to $D_t\nabla p$. Indeed, we have

$$D_t\nabla p = \nabla D_t p - \nabla v \cdot \nabla p.$$

Since $\nabla p = -an_\Gamma$ and $D_t p = 0$ on Γ, we obtain

$$\nabla^\top v \cdot n_\Gamma = a^{-1}(D_t \nabla p)^\top,$$

and so, since v is divergence free, we have from the div-curl estimate in Proposition 5.27,

$$\begin{aligned} \|v\|_{H^k(\Omega)} &\lesssim_A \|v\|_{L^2(\Omega)} + \|\omega\|_{H^{k-1}(\Omega)} + \|a^{-1}(D_t\nabla p)^\top\|_{H^{k-\frac{3}{2}}(\Gamma)} + \|\Gamma\|_{H^{k-\epsilon}}\|v\|_{C^{\frac{1}{2}+\epsilon}(\Omega)} \\ &\lesssim_A \|a^{-1}(D_t\nabla p)^\top\|_{H^{k-\frac{3}{2}}(\Gamma)} + (E^k)^{\frac{1}{2}} + \Lambda_{k-\epsilon}. \end{aligned} \tag{7.13}$$

To estimate the first term on the right-hand side of (7.13), we use the decomposition $D_t\nabla p = G_j^1 + G_j^2$ from Corollary 7.6. By the balanced product and trace estimates Proposition 5.9 and Proposition 5.11 and a similar analysis to the estimate for $\|\kappa\|_{H^{k-2}(\Gamma)}$, we obtain

$$\begin{aligned} \|a^{-1}(D_t\nabla p)^\top\|_{H^{k-\frac{3}{2}}(\Gamma)} &\lesssim_A \|D_t\nabla p\|_{H^{k-1}(\Omega)} + (\|a^{-1}\|_{H^{k-1-\epsilon}(\Gamma)} + \|\Gamma\|_{H^{k-\epsilon}})\sup_{j>0} 2^{-j(\frac{1}{2}-\epsilon)}\|G_j^1\|_{L^\infty(\Omega)} \\ &+ \sup_{j>0} 2^{j(k-\frac{3}{2}-2\epsilon)}\|G_j^2\|_{H^{\frac{1}{2}+\epsilon}(\Omega)} \lesssim_A \|D_t\nabla p\|_{H^{k-1}(\Omega)} + \Lambda_{k-\epsilon}. \end{aligned}$$

Finally, we need to show that

$$\|D_t\nabla p\|_{H^{k-1}(\Omega)} \lesssim_A (E^k)^{\frac{1}{2}} + \Lambda_{k-\epsilon}.$$

For this, we will use the div-curl decomposition for $D_t\nabla p$. The divergence and curl are given by

$$\begin{cases} \nabla\cdot D_t\nabla p = 3\mathrm{tr}(\nabla^2 p\cdot\nabla v) + 2\mathrm{tr}(\nabla v)^3 \ \text{ in } \Omega, \\ \nabla\times D_t\nabla p = \nabla^2 p\cdot\nabla v - (\nabla v)^*\cdot\nabla^2 p \ \text{ in } \Omega. \end{cases}$$

Hence, using the div-curl estimate and the partition $D_t\nabla p = G_j^1 + G_j^2$ from Corollary 7.6 in conjunction with Corollary 5.4, we obtain

$$\begin{aligned} \|D_t\nabla p\|_{H^{k-1}(\Omega)} &\lesssim_A \|p\|_{H^{k+\frac{1}{2}}(\Omega)}\|v\|_{C^{\frac{1}{2}}(\Omega)} + \|p\|_{C^{1,\epsilon}(\Omega)}\|v\|_{H^{k-\epsilon}(\Omega)} + \|\mathrm{tr}(\nabla v)^3\|_{H^{k-2}(\Omega)} + \|\nabla^\top(D_t\nabla p)\cdot n_\Gamma\|_{H^{k-\frac{5}{2}}(\Gamma)} \\ &+ \|\Gamma\|_{H^{k-\epsilon}}\sup_{j>0} 2^{-j(\frac{1}{2}-\epsilon)}\|G_j^1\|_{L^\infty(\Omega)} + \sup_{j>0} 2^{j(k-\frac{3}{2}-2\epsilon)}\|G_j^2\|_{H^{\frac{1}{2}+\epsilon}(\Omega)} + \Lambda_{k-\epsilon}. \end{aligned}$$

Estimating G_j^1 and G_j^2 as before and then using (7.12) gives

$$\|D_t\nabla p\|_{H^{k-1}(\Omega)} \lesssim_A (E^k)^{\frac{1}{2}} + \|\nabla^\top(D_t\nabla p)\cdot n_\Gamma\|_{H^{k-\frac{5}{2}}(\Gamma)} + \|v\|_{H^{k-\epsilon}(\Omega)} + \|\Gamma\|_{H^{k-\epsilon}} + \|\mathrm{tr}(\nabla v)^3\|_{H^{k-2}(\Omega)} + \Lambda_{k-\epsilon}.$$

Using a trilinear frequency decomposition as in Lemma 7.5, we obtain easily

$$\|\mathrm{tr}(\nabla v)^3\|_{H^{k-2}(\Omega)} \lesssim_A \|v\|^2_{C^{\frac{1}{2}+\epsilon}(\Omega)}\|v\|_{H^{k-\epsilon}(\Omega)} \lesssim_A \Lambda_{k-\epsilon}.$$

It remains to estimate the boundary term. We compute

$$\nabla^\top(D_t\nabla p)\cdot n_\Gamma = -\nabla^\top D_t a - D_t\nabla p\cdot\nabla^\top n_\Gamma. \tag{7.14}$$

By Proposition 5.9, Proposition 5.31 and using the decomposition $D_t\nabla p = G_j^1 + G_j^2$, the terms in (7.14) are controlled in a similar fashion to the above terms by

$$\|\nabla^\top(D_t\nabla p)\cdot n_\Gamma\|_{H^{k-\frac{5}{2}}(\Gamma)} \lesssim_A \|D_t a\|_{H^{k-\frac{3}{2}}(\Gamma)} + \Lambda_{k-\epsilon} \lesssim_A (E^k)^{\frac{1}{2}} + \Lambda_{k-\epsilon},$$

where we used Lemma 7.7 in the last inequality. Combining everything together, we have

$$\|D_t\nabla p\|_{H^{k-1}(\Omega)} + \|\Gamma\|_{H^k} + \|v\|_{H^k(\Omega)} + \|p\|_{H^{k+\frac{1}{2}}(\Omega)} \lesssim_A (E^k)^{\frac{1}{2}} + \Lambda_{k-\epsilon}.$$

Using the definition of $\Lambda_{k-\epsilon}$ and interpolating gives

$$\|D_t\nabla p\|_{H^{k-1}(\Omega)} + \|\Gamma\|_{H^k} + \|v\|_{H^k(\Omega)} + \|p\|_{H^{k+\frac{1}{2}}(\Omega)} \lesssim_A (E^k)^{\frac{1}{2}} + \|v\|_{L^2(\Omega)} + \|p\|_{H^1(\Omega)} + \|D_t\nabla p\|_{L^2(\Omega)}.$$

We can use the H^1 estimate for the Laplace equation for p to estimate

$$\|p\|_{H^1(\Omega)} \lesssim_A \|v\|_{H^1(\Omega)}.$$

Moreover, by writing $D_t\nabla p = \nabla D_t p - \nabla v \cdot \nabla p$, writing $D_t p$ in the form $\Delta^{-1}\nabla \cdot f$ as in the proof of Lemma 7.5 and using the $H^{-1} \to H^1$ estimate for Δ^{-1}, we have

$$\|D_t \nabla p\|_{L^2(\Omega)} \lesssim_A \|v\|_{H^1(\Omega)}.$$

Therefore, by interpolation we have

$$\|D_t\nabla p\|_{H^{k-1}(\Omega)} + \|\Gamma\|_{H^k} + \|v\|_{H^k(\Omega)} + \|p\|_{H^{k+\frac{1}{2}}(\Omega)} \lesssim_A (E^k)^{\frac{1}{2}}. \tag{7.15}$$

This finally establishes the desired estimate

$$\|(v,\Gamma)\|_{\mathbf{H}^k} \lesssim_A (E^k)^{\frac{1}{2}}.$$

Next, we show the easier part of the coercivity bound; namely,

$$(E^k)^{\frac{1}{2}} \lesssim_A 1 + \|(v,\Gamma)\|_{\mathbf{H}^k}.$$

Clearly, the only nontrivial part is to control the irrotational energy. More precisely, we have to show that

$$\|\nabla \mathcal{H}\mathcal{N}^{k-2} D_t a\|_{L^2(\Omega)} + \|a^{\frac{1}{2}}\mathcal{N}^{k-1} a\|_{L^2(\Gamma)} \lesssim_A 1 + \|(v,\Gamma)\|_{\mathbf{H}^k}. \tag{7.16}$$

To establish this, we will need the following L^2 based estimates for p and $D_t p$.

Lemma 7.8. *The following estimate holds:*

$$\|p\|_{H^{k+\frac{1}{2}}(\Omega)} + \|D_t p\|_{H^k(\Omega)} \lesssim_A \|(v,\Gamma)\|_{\mathbf{H}^k}.$$

Proof. First, from the balanced Dirichlet estimate in Proposition 5.19, as well as Corollary 5.4 and Lemma 7.5, we have

$$\|p\|_{H^{k+\frac{1}{2}}(\Omega)} \lesssim_A \|\mathrm{tr}(\nabla v)^2\|_{H^{k-\frac{3}{2}}(\Omega)} + \|\Gamma\|_{H^k}\|p\|_{W^{1,\infty}(\Omega)} \lesssim_A \|(v,\Gamma)\|_{\mathbf{H}^k}.$$

To estimate $D_t p$, recall that we can write $D_t p$ in the form $\Delta^{-1}\nabla \cdot f$. Indeed, similarly to Lemma 7.5, we can start by writing

$$D_t p = \Delta^{-1}\partial_i(\partial_i v_j \partial_j p) + 3\Delta^{-1}\partial_i(\partial_j p \partial_j v_i) + 2\Delta^{-1}\mathrm{tr}(\nabla v)^3 =: F_1 + F_2 + F_3. \tag{7.17}$$

We now will use Proposition 5.19 to estimate each term. We begin with F_1. We use the partition $F_1 = H_j^1 + H_j^2$ where $H_j^1 := \Delta^{-1}\partial_i(\partial_i \Phi_{\leq j} v_k \partial_k p)$ and Proposition 5.19 to obtain,

$$\|F_1\|_{H^k(\Omega)} \lesssim_A \|\nabla p \cdot \nabla v\|_{H^{k-1}(\Omega)} + \|\Gamma\|_{H^k} \sup_{j>0} 2^{-\frac{j}{2}} \|H_j^1\|_{W^{1,\infty}(\Omega)} + \sup_{j>0} 2^{j(k-1)} \|H_j^2\|_{H^1(\Omega)}.$$

Using Corollary 5.4 and the $H^{k+\frac{1}{2}}$ estimate for p above, we obtain

$$\|\nabla p \cdot \nabla v\|_{H^{k-1}(\Omega)} \lesssim_A \|v\|_{C^{\frac{1}{2}+\epsilon}(\Omega)} \|p\|_{H^{k+\frac{1}{2}}(\Omega)} + \|p\|_{C^{1,\epsilon}(\Omega)}\|v\|_{H^k(\Omega)} \lesssim_A \|(v,\Gamma)\|_{\mathbf{H}^k}.$$

We also have from Proposition 5.15 and the properties of $\Phi_{\leq j}$,

$$\sup_{j>0} 2^{-\frac{j}{2}} \|H_j^1\|_{W^{1,\infty}(\Omega)} \lesssim_A \|p\|_{C^{1,\epsilon}(\Omega)} \|v\|_{C^{\frac{1}{2}+\epsilon}(\Omega)} \lesssim_A 1,$$

and from the $H^{-1} \to H^1$ estimate for Δ^{-1} and Lemma 7.5, we have

$$\sup_{j>0} 2^{j(k-1)} \|H_j^2\|_{H^1(\Omega)} \lesssim_A \sup_{j>0} 2^{j(k-1)} \|\nabla p\|_{L^\infty(\Omega)} \|\nabla \Phi_{>j} v\|_{L^2(\Omega)} \lesssim_A \|v\|_{H^k(\Omega)}.$$

Hence,

$$\|F_1\|_{H^k(\Omega)} \lesssim_A \|(v,\Gamma)\|_{\mathbf{H}^k}. \tag{7.18}$$

By a very similar analysis, we obtain the same bound (7.18) for F_2. To estimate F_3, one uses the decomposition of $\operatorname{tr}(\nabla v)^3$ from (7.8) and (7.9) and then partitions one of the factors $\nabla v^{\leq l} = \nabla \Phi_{<j} v^{\leq l} + \nabla \Phi_{\geq j} v^{\leq l}$. After that, an estimate similar to F_1 yields the bound (7.18) for the term F_3. Therefore,

$$\|D_t p\|_{H^k(\Omega)} \lesssim_A \|(v,\Gamma)\|_{\mathbf{H}^k},$$

as desired. □

Now, returning to the proof of (7.16), for the term $\|a^{\frac{1}{2}}\mathcal{N}^{k-1}a\|_{L^2(\Gamma)}$, we have from Lemma 5.24 and Proposition 5.30,

$$\|a^{\frac{1}{2}}\mathcal{N}^{k-1}a\|_{L^2(\Gamma)} \lesssim_A \|a\|_{H^{k-1}(\Gamma)} + \|a\|_{L^\infty(\Gamma)}\|\Gamma\|_{H^k} \lesssim_A \|a\|_{H^{k-1}(\Gamma)} + \|\Gamma\|_{H^k}.$$

Then from Proposition 5.9, Proposition 5.11 and Lemma 7.8, we have

$$\|a\|_{H^{k-1}(\Gamma)} \lesssim_A \|p\|_{H^{k+\frac{1}{2}}(\Omega)} + \|\Gamma\|_{H^k} \lesssim_A \|(v,\Gamma)\|_{\mathbf{H}^k}.$$

To control the other part of the energy, we first note that by (5.20) we have

$$\|\nabla \mathcal{H}\mathcal{N}^{k-2}D_t a\|_{L^2(\Omega)} \lesssim_A \|\mathcal{N}^{k-2}D_t a\|_{H^{\frac{1}{2}}(\Gamma)}.$$

Then we apply Proposition 5.30, Proposition 5.11 and Proposition 5.9, in that order, to obtain

$$\begin{aligned}\|\mathcal{N}^{k-2}D_t a\|_{H^{\frac{1}{2}}(\Gamma)} &\lesssim_A \|D_t\nabla p\|_{H^{k-1}(\Omega)} + \|\Gamma\|_{H^k}\sup_{j>0} 2^{-\frac{j}{2}}\|G_j^1\|_{L^\infty(\Omega)} + \sup_{j>0} 2^{j(k-\frac{3}{2}-2\epsilon)}\|G_j^2\|_{H^{\frac{1}{2}+\epsilon}(\Omega)} \\ &\lesssim_A \|D_t\nabla p\|_{H^{k-1}(\Omega)} + \|\Gamma\|_{H^k},\end{aligned}$$

where $D_t\nabla p = G_j^1 + G_j^2$ is the partition from Corollary 7.6. We then write $D_t\nabla p = -\nabla v\cdot\nabla p + \nabla D_t p$ and use Corollary 5.4 and Lemma 7.8 to obtain

$$\|D_t\nabla p\|_{H^{k-1}(\Omega)} \lesssim_A \|(v,\Gamma)\|_{\mathbf{H}^k}.$$

This completes the proof of (7.16) and thus the proof of part (i) of Theorem 7.1. Next, we turn to part (ii), which is the energy propagation bound.

7.5. L^∞ **estimates for propagation.** Now, we turn to the energy propagation bounds. As in the coercivity estimate, we will need certain L^∞ based estimates for p and $D_t p$, but in norms that have essentially $\frac{1}{2}$ more degrees of regularity compared to Lemma 7.5.

Lemma 7.9. *Given the assumptions of Theorem 7.1, the following pointwise estimates for p and $D_t p$ hold.*

(i) ($C^{1,\frac{1}{2}}$ estimate for p).

$$\|p\|_{C^{1,\frac{1}{2}}(\Omega)} \lesssim_A B.$$

(ii) ($W^{1,\infty}$ estimate for $D_t p$). Let $s\in\mathbb{R}$ with $s > \frac{d}{2}+1$. Then

$$\|D_t p\|_{W^{1,\infty}(\Omega)} \lesssim_A \log(1+\|(v,\Gamma)\|_{\mathbf{H}^s})B.$$

Proof. We begin with the $C^{1,\frac{1}{2}}$ estimate. We have from Proposition 5.15, using the decomposition from (7.6) and a similar analysis to the $C^{1,\epsilon}$ estimate for p,

$$\|p\|_{C^{1,\frac{1}{2}}(\Omega)} \lesssim_A \|\Gamma\|_{C^{1,\frac{1}{2}}}(\|p\|_{C^{1,\epsilon}(\Omega)} + \|v_i^l\partial_i v_j^{\leq l}\|_{C^\epsilon(\Omega)}) + \|v_i^l\partial_i v_j^{\leq l}\|_{C^{\frac{1}{2}}(\Omega)} \lesssim_A \|\Gamma\|_{C^{1,\frac{1}{2}}} + \|v\|_{W^{1,\infty}(\Omega)} \lesssim_A B.$$

Now, we turn to the more difficult $W^{1,\infty}$ estimate for $D_t p$. Again, we first recall from (7.7) that we have

$$D_t p = 4\Delta^{-1}(\partial_i\partial_j p \partial_i v_j) + 2\Delta^{-1}(\partial_j v_k \partial_k v_i \partial_i v_j) + \Delta^{-1}(\partial_i\partial_i v_j \partial_j p). \tag{7.19}$$

Using a very similar analysis to Lemma 7.5 (except without the partition of $D_t p$), we can estimate the second term in (7.19) in $W^{1,\infty}$ by

$$\|\Delta^{-1}(\partial_j v_k \partial_k v_i \partial_i v_j)\|_{W^{1,\infty}(\Omega)} \lesssim_A B.$$

For the first term in (7.19) we have the decomposition

$$\Delta^{-1}(\partial_i\partial_j p \partial_i v_j) = \Delta^{-1}(\partial_i\partial_j p^l \partial_i v_j^{\leq l}) + \Delta^{-1}(\partial_i\partial_j p^{\leq l}\partial_i v_j^l). \tag{7.20}$$

The first term in (7.20) can be estimated similarly using Proposition 5.15 by

$$\|\Delta^{-1}(\partial_i\partial_j p^l \partial_i v_j^{\leq l})\|_{W^{1,\infty}(\Omega)} = \|\Delta^{-1}\partial_j(\partial_i p^l \partial_i v_j^{\leq l})\|_{W^{1,\infty}(\Omega)} \lesssim_A \|p\|_{C^{1,\epsilon}(\Omega)}\|v\|_{W^{1,\infty}(\Omega)} \lesssim_A B. \tag{7.21}$$

For the latter term in (7.20), we write

$$\Delta^{-1}(\partial_i\partial_j p^{\leq l}\partial_i v_j^l) = \Delta^{-1}\partial_i(\partial_i\partial_j p^{\leq l} v_j^l) - \Delta^{-1}\partial_j(\partial_i\partial_i p^{\leq l} v_j^l) \tag{7.22}$$

and use the fact that the pressure term is at low frequency compared to v and a similar analysis to the above to estimate

$$\|\Delta^{-1}(\partial_i\partial_j p^{\leq l}\partial_i v_j^l)\|_{W^{1,\infty}(\Omega)} \lesssim_A B. \tag{7.23}$$

We now focus on the last term in (7.19) which will be responsible for the logarithmic loss in the estimate. We begin by writing

$$\partial_i\partial_i v_j \partial_j p = \partial_i\partial_i v_j^l \partial_j p^{\leq l} + \partial_i\partial_i v_j^{\leq l}\partial_j p^l. \tag{7.24}$$

For the second term on the right-hand side of (7.24), we write

$$\partial_i\partial_i v_j^{\leq l}\partial_j p^l = \partial_j(\partial_i\partial_i v_j^{\leq l} p^l).$$

Again, similarly to the above, we have

$$\|\Delta^{-1}\partial_j(\partial_i\partial_i v_j^{\leq l} p^l)\|_{W^{1,\infty}(\Omega)} \lesssim_A B. \tag{7.25}$$

Now, for the first term on the right of (7.24) we have,

$$\partial_i\partial_i v_j^l \partial_j p^{\leq l} = \Delta(v_j^l \partial_j p^{\leq l}) + \partial_j(v_j^l \partial_i\partial_i p^{\leq l}) - 2\partial_i(v_j^l \partial_j\partial_i p^{\leq l}). \tag{7.26}$$

The latter two terms in (7.26) are estimated similarly to (7.25). We focus our attention on the first term, which corresponds to estimating $\Delta^{-1}\Delta(v_j^l\partial_j p^{\leq l})$ in $W^{1,\infty}$. We begin by writing

$$\Delta^{-1}\Delta(v_j^l \partial_j p^{\leq l}) = v_j^l \partial_j p^{\leq l} - \mathcal{H}(v_j^l \partial_j p^{\leq l}). \tag{7.27}$$

For the first term in (7.27) we note that

$$\nabla(v_j^l \partial_j p^{\leq l}) = v_j^l \partial_j \nabla p^{\leq l} + \nabla v_j^l \partial_j p^{\leq l}.$$

From the $C^{1,\epsilon}$ bound for p from Lemma 7.5, we clearly have $\|v_j^l \partial_j \nabla p^{\leq l}\|_{L^\infty(\Omega)} \lesssim_A B$. On the other hand, we have the same estimate for $\nabla v_j^l \partial_j p^{\leq l}$ because

$$\nabla v_j^l \partial_j p^{\leq l} = \nabla v_j \partial_j p - \nabla v_j^{\leq l}\partial_j p^l.$$

This yields the estimate $\|v_j^l\partial_j p^{\leq l}\|_{W^{1,\infty}(\Omega)} \lesssim_A B$. It remains to estimate $\mathcal{H}(v_j^l\partial_j p^{\leq l})$, which is where we incur the logarithmic loss. By the maximum principle, it suffices to estimate $\|\nabla\mathcal{H}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)}$. We begin by showing that for each $m \geq 0$

$$\|\Phi_m\nabla\mathcal{H}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)} \lesssim_A B, \tag{7.28}$$

with implicit constant independent of m. Indeed, we have

$$\|\Phi_m\nabla\mathcal{H}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)} \lesssim \|\Phi_m\nabla\mathcal{H}\Phi_{\leq m}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)} + \|\Phi_m\nabla\mathcal{H}\Phi_{>m}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)}.$$

For the first term, we have from the regularization properties of Φ_m and the $C^{1,\epsilon}$ estimate from Proposition 5.15,

$$\begin{aligned}\|\Phi_m\nabla\mathcal{H}\Phi_{\leq m}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)} &\lesssim 2^{-\epsilon m}\|\mathcal{H}\Phi_{\leq m}(v_j^l\partial_j p^{\leq l})\|_{C^{1,\epsilon}(\Omega)} \lesssim_A 2^{-\epsilon m}\|\Phi_{\leq m}(v_j^l\partial_j p^{\leq l})\|_{C^{1,\epsilon}(\Omega)}\\ &\lesssim_A \|v_j^l\partial_j p^{\leq l}\|_{W^{1,\infty}(\Omega)}.\end{aligned}$$

Therefore, similarly to the estimate for $\nabla(v_j^l\partial_j p^{\leq l})$, we have

$$\|\Phi_m\nabla\mathcal{H}\Phi_{\leq m}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)} \lesssim_A B.$$

For the other term, we have from the regularization properties of $\Phi_{\leq m}$ and $\Phi_{\geq m}$ and the maximum principle,

$$\begin{aligned}\|\Phi_m\nabla\mathcal{H}\Phi_{>m}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)} &\lesssim_A 2^m\|\mathcal{H}\Phi_{>m}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)} \leq 2^m\|\Phi_{>m}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)}\\ &\lesssim_A \|v_j^l\partial_j p^{\leq l}\|_{W^{1,\infty}(\Omega)}.\end{aligned}$$

Combining everything gives (7.28). Now, to prove the full estimate, we fix an integer $m_0 > 0$ to be chosen later and estimate using (7.28),

$$\|\nabla\mathcal{H}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)} \lesssim_A m_0 B + \|\Phi_{\geq m_0}\nabla\mathcal{H}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)}. \tag{7.29}$$

For the latter term, since $s > \frac{d}{2} + 1$, we obtain by Sobolev embedding, the regularization properties of $\Phi_{\geq m_0}$ and the elliptic estimate for $\mathcal{H}$, the estimate

$$\|\Phi_{\geq m_0}\nabla\mathcal{H}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)} \lesssim_A 2^{-m_0\delta_0}\|\mathcal{H}(v_j^l\partial_j p^{\leq l})\|_{H^{s-\epsilon}(\Omega)} \lesssim_A 2^{-m_0\delta_0}\|(v,\Gamma)\|_{\mathbf{H}^s}^r,$$

where $r \geq 1$ is some integer and $\delta_0 > 0$ is a constant depending on k. Taking $m_0 \approx r\delta_0^{-1}\log(1 + \|(v,\Gamma)\|_{\mathbf{H}^s})$ and combining everything above with (7.29) then yields

$$\|\nabla\mathcal{H}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)} \lesssim_A B\log(1 + \|(v,\Gamma)\|_{\mathbf{H}^s}).$$

This completes the proof of the lemma. □

Remark 7.10. It is perhaps worth remarking that by using Proposition 5.15 and the maximum principle to estimate $\|\nabla\mathcal{H}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)}$ in the above proof in C^ϵ, we can also easily obtain the bound

$$\|D_t p\|_{W^{1,\infty}(\Omega)} \lesssim_A \|v\|_{C^{1,\epsilon}(\Omega)}.$$

Of course, we do not want this in our energy estimates as it would force us to forfeit the scale invariant control parameter B.

7.6. **Proof of energy propagation.** Now, we turn to the second part of Theorem 7.1. Using (7.2) and the coercivity bound (7.1) it is straightforward to verify the following energy estimate for the rotational component of the energy:

$$\frac{d}{dt}E_r^k(v(t),\Gamma_t) \lesssim_A BE^k(v(t),\Gamma_t).$$

The main bulk of the work will be in establishing a propagation bound for the irrotational part of the energy. Namely, we want to show that

$$\frac{d}{dt}E_i^k(v(t),\Gamma_t) \lesssim_A B\log(1+\|(v,\Gamma)\|_{\mathbf{H}^s})E^k(v(t),\Gamma_t).$$

To do this, we start by deriving a wave-type equation for a. The general procedure for deriving this equation is similar to [17]. However, we need to more precisely identify the source terms in order to obtain estimates with the required pointwise control parameters A and B.

We begin our derivation with the simple commutator identity

$$D_t\nabla p = -\nabla v\cdot\nabla p + \nabla D_t p.$$

Applying D_t and performing some elementary algebraic manipulations gives

$$\begin{aligned} D_t^2\nabla p &= -\nabla D_t v\cdot\nabla p + D_t\nabla D_t p + \nabla v\cdot(\nabla v\cdot\nabla p) - \nabla v\cdot D_t\nabla p \\ &= \frac{1}{2}\nabla|\nabla p|^2 + \nabla D_t^2 p + 2\nabla v\cdot(\nabla v\cdot\nabla p) - 2\nabla v\cdot\nabla D_t p, \end{aligned}$$

where in the last line, we used the Euler equations to write $-\nabla D_t v\cdot\nabla p = \frac{1}{2}\nabla|\nabla p|^2$. As $\Delta p = -\mathrm{tr}(\nabla v)^2$ is lower order, it is natural to further split $\nabla|\nabla p|^2$ as

$$\frac{1}{2}\nabla|\nabla p|^2 = \frac{1}{2}\nabla\mathcal{H}|\nabla p|^2 + \frac{1}{2}\nabla\Delta^{-1}\Delta|\nabla p|^2.$$

From this, we obtain the equation

$$D_t^2\nabla p - \frac{1}{2}\nabla\mathcal{H}|\nabla p|^2 = \frac{1}{2}\nabla\Delta^{-1}\Delta|\nabla p|^2 + \nabla D_t^2 p + 2\nabla v\cdot(\nabla v\cdot\nabla p) - 2\nabla v\cdot\nabla D_t p =: g. \tag{7.30}$$

It will be seen later that g can be thought of as a perturbative source term. In an effort to convert (7.30) into an equation for $D_t a$, we take the normal component of the trace on Γ_t to obtain

$$D_t^2\nabla p\cdot n_{\Gamma_t} - \frac{1}{2}\mathcal{N}(a^2) = g\cdot n_{\Gamma_t}, \tag{7.31}$$

where we used the dynamic boundary condition $p|_{\Gamma_t} = 0$ to write $|\nabla p_{|\Gamma_t}|^2 = a^2$. Since D_t is tangent to Γ_t, we have

$$D_t^2 a = -D_t^2\nabla p\cdot n_{\Gamma_t} - D_t\nabla p\cdot D_t n_{\Gamma_t} = -D_t^2\nabla p\cdot n_{\Gamma_t} + a|D_t n_{\Gamma_t}|^2. \tag{7.32}$$

Note that for the latter equality in (7.32), we wrote $D_t\nabla p = -D_t(an_{\Gamma_t})$ and used that $D_t n_{\Gamma_t}$ is tangent to Γ_t. Combining (7.31) and (7.32), we obtain the equation

$$D_t^2 a + \frac{1}{2}\mathcal{N}(a^2) = -g\cdot n_{\Gamma_t} + a|D_t n_{\Gamma_t}|^2,$$

which can be further reduced using the Leibniz type formula for $\mathcal{N}$ from (5.36) to the equation

$$D_t^2 a + a\mathcal{N}a = f, \tag{7.33}$$

where

$$f := -g\cdot n_{\Gamma_t} + a|D_t n_{\Gamma_t}|^2 + n_{\Gamma_t}\cdot\nabla\Delta^{-1}(|\nabla\mathcal{H}a|^2).$$

To propagate $(a, D_t a)$ in $H^{k-1}(\Gamma_t) \times H^{k-\frac{3}{2}}(\Gamma_t)$, one natural idea, in view of the ellipticity of $\mathcal{N}$, would be to use the spectral theorem to apply $\mathcal{N}^{k-\frac{3}{2}}$ to the above equation, and then read off the associated energy for the leading order wave-like equation. This is essentially the approach used in [17]. However, there is a much better choice for our purposes, which comes from instead applying $\nabla\mathcal{H}\mathcal{N}^{k-2}$ to the above equation. The benefit to this is twofold. The most important advantage is that we only have to work with integer powers of $\mathcal{N}$, which will allow us to make use of the balanced elliptic estimates from the previous sections. Secondly, this choice allows us to reinterpret the desired estimate for $(a, D_t a)$ in $H^{k-1}(\Gamma_t) \times H^{k-\frac{3}{2}}(\Gamma_t)$ as an L^2 type estimate for the linearized equation (2.8) with perturbative source terms. Indeed, by defining the variables

$$
\begin{aligned}
&w := \nabla\mathcal{H}\mathcal{N}^{k-2} D_t a,\\
&s := \mathcal{N}^{k-1} a,\\
&q := \mathcal{H}(a\mathcal{N}^{k-1} a),
\end{aligned}
$$

we may interpret (w, s, q) to leading order as a solution to the linearized system (2.8). To verify this, note that we clearly have $\nabla \cdot w = 0$. Moreover, we observe that $q|_{\Gamma_t} = as$ and that $w|_{\Gamma_t} \cdot n_{\Gamma_t} = \mathcal{N}^{k-1} D_t a$. Hence,

$$
D_t s - w|_{\Gamma_t} \cdot n_{\Gamma_t} = [D_t, \mathcal{N}^{k-1}]a =: \mathcal{R}.
$$

We also note that in Ω_t, by using the equation (7.33) for a and the Leibniz formula for $\mathcal{N}$,

$$
D_t w + \nabla q = \mathcal{Q},
$$

where

$$
\mathcal{Q} := -\nabla v \cdot w + \nabla[D_t, \mathcal{H}](\mathcal{N}^{k-2} D_t a) + \nabla\mathcal{H}[D_t, \mathcal{N}^{k-2}]D_t a + \nabla\mathcal{H}\mathcal{N}^{k-2} f - \nabla\mathcal{H}[\mathcal{N}^{k-2}, a]\mathcal{N} a. \tag{7.34}
$$

To summarize the above in a compact form, we can write

$$
\begin{cases}
D_t w + \nabla q = \mathcal{Q} \text{ in } \Omega_t,\\
\nabla \cdot w = 0 \text{ in } \Omega_t,\\
D_t s - w \cdot n_{\Gamma_t} = \mathcal{R} \text{ on } \Gamma_t,\\
q = as \text{ on } \Gamma_t.
\end{cases}
$$

The linearized energy estimate from Proposition 2.2 combined with Cauchy-Schwarz and Lemma 7.9 immediately gives the preliminary bound

$$
\frac{d}{dt} E_i^k \lesssim_A B \log(1 + \|(v, \Gamma)\|_{\mathbf{H}^s}) E^k + (\|\mathcal{R}\|_{L^2(\Gamma_t)} + \|\mathcal{Q}\|_{L^2(\Omega_t)})(E^k)^{\frac{1}{2}}.
$$

It remains to control the source terms $\mathcal{Q}$ and $\mathcal{R}$. This will be where the bulk of the work is situated. Our goal is to show that

$$
\|\mathcal{Q}\|_{L^2(\Omega_t)} + \|\mathcal{R}\|_{L^2(\Gamma_t)} \lesssim_A B \log(1 + \|(v, \Gamma)\|_{\mathbf{H}^s})(E^k)^{\frac{1}{2}}.
$$

We begin with the estimate for $\mathcal{Q}$. We proceed term by term. Clearly, we have

$$
\|\nabla v \cdot w\|_{L^2(\Omega_t)} \lesssim B(E^k)^{\frac{1}{2}}.
$$

To handle the second term in the definition of $\mathcal{Q}$, we begin by recalling the simple commutator identity from (5.42),

$$
[D_t, \mathcal{H}]\psi = \Delta^{-1}\nabla \cdot \mathcal{B}(\nabla v, \nabla\mathcal{H}\psi),
$$

where $\mathcal{B}$ is an $\mathbb{R}^d$-valued bilinear form. We then estimate using the $H^{-1} \to H^1$ bound for Δ^{-1} to obtain

$$\|\nabla[D_t, \mathcal{H}](\mathcal{N}^{k-2} D_t a)\|_{L^2(\Omega_t)} \lesssim_A B \|\nabla \mathcal{H} \mathcal{N}^{k-2} D_t a\|_{L^2(\Omega_t)} \lesssim_A B (E^k)^{\frac{1}{2}}.$$

For the third term in (7.34), we use the $H^{\frac{1}{2}}(\Gamma_t) \to H^1(\Omega_t)$ bound for $\mathcal{H}$ to obtain

$$\|\nabla \mathcal{H}([D_t, \mathcal{N}^{k-2}] D_t a)\|_{L^2(\Omega_t)} \lesssim_A \|[D_t, \mathcal{N}^{k-2}] D_t a\|_{H^{\frac{1}{2}}(\Gamma_t)}.$$

Then, from the commutator estimate Proposition 5.33 we obtain

$$\begin{aligned}
\|[D_t, \mathcal{N}^{k-2}] D_t a\|_{H^{\frac{1}{2}}(\Gamma_t)} \lesssim_A \ & \|v\|_{H^k(\Omega_t)} \|D_t a\|_{L^\infty(\Gamma_t)} + \|v\|_{W^{1,\infty}(\Omega_t)} \|D_t a\|_{H^{k-\frac{3}{2}}(\Gamma_t)} + \|D_t a\|_{L^\infty(\Gamma_t)} \|\Gamma\|_{H^k} \\
& + \|v\|_{W^{1,\infty}(\Omega_t)} \|\Gamma\|_{H^k(\Omega_t)} \sup_{j>0} 2^{-\frac{j}{2}} \|G_j^1 \cdot n_{\Gamma_t}\|_{L^\infty(\Gamma_t)} \\
& + \|v\|_{W^{1,\infty}(\Omega_t)} \sup_{j>0} 2^{j(k-\frac{3}{2}-2\epsilon)} \|G_j^2 \cdot n_{\Gamma_t}\|_{H^\epsilon(\Gamma_t)},
\end{aligned}$$

where G_j^1 and G_j^2 are as in Corollary 7.6. Using Lemma 7.9, the energy coercivity, Lemma 7.7 and (7.15), we have

$$\|[D_t, \mathcal{N}^{k-2}] D_t a\|_{H^{\frac{1}{2}}(\Gamma_t)} \lesssim_A B \log(1 + \|(v, \Gamma)\|_{\mathbf{H}^s}) (E^k)^{\frac{1}{2}}.$$

Next, we turn to the estimate for $\nabla \mathcal{H} \mathcal{N}^{k-2} f$, which involves the most work. We recall that

$$f := -g \cdot n_{\Gamma_t} + a |D_t n_{\Gamma_t}|^2 + \nabla_n \Delta^{-1}(|\nabla \mathcal{H} a|^2),$$

where g is defined as in (7.30). Using the identities $D_t n_{\Gamma_t} = -((Dv)^* n_{\Gamma_t})^\top = -(Dv)^* n_{\Gamma_t} + n_{\Gamma_t}(n_{\Gamma_t} \cdot (Dv)^* n_{\Gamma_t})$ and $|\nabla \mathcal{H} a|^2 = \frac{1}{2} \Delta |\mathcal{H} a|^2$, we may reorganize f into the expression

$$f = \frac{1}{2} \nabla_n \Delta^{-1} \Delta (\mathcal{H} a)^2 - \frac{1}{2} \nabla_n \Delta^{-1} \Delta |\nabla p|^2 - \nabla_n D_t^2 p + M_1 + M_2, \tag{7.35}$$

where M_1 is a multilinear expression in n_{Γ_t}, ∇p, ∇v with exactly two factors of ∇v (e.g., from (5.35), the term $a|D_t n_{\Gamma_t}|^2$), and M_2 is a multilinear expression in ∇p, ∇v, $\nabla D_t p$ and n_{Γ_t} with a single factor of each of $\nabla D_t p$ and ∇v (e.g., the term $n_{\Gamma_t} \cdot \nabla D_t p \cdot \nabla v$). We will abuse notation slightly and refer to terms of the first type as $M_1(\nabla v, \nabla v)$ and terms of the second type as $M_2(\nabla D_t p, \nabla v)$. Next, we estimate each term in $\nabla \mathcal{H} \mathcal{N}^{k-2} f$, with the expression (7.35) for f substituted in.

From Corollary 5.32, we have

$$\begin{aligned}
\|\nabla \mathcal{H} \mathcal{N}^{k-2} \nabla_n \Delta^{-1} \Delta (\mathcal{H} a)^2\|_{L^2(\Omega_t)} &\lesssim_A \|\mathcal{N}^{k-2} \nabla_n \Delta^{-1} \Delta (\mathcal{H} a)^2\|_{H^{\frac{1}{2}}(\Gamma_t)} \\
&\lesssim_A \|\Gamma_t\|_{H^k} \|\Delta^{-1} \Delta (\mathcal{H} a)^2\|_{C^{\frac{1}{2}}(\Omega_t)} + \|\Delta^{-1} \Delta (\mathcal{H} a)^2\|_{H^k(\Omega_t)}.
\end{aligned}$$

By writing $\Delta^{-1} \Delta (\mathcal{H} a)^2 = (\mathcal{H} a)^2 - \mathcal{H}(\mathcal{H} a)^2$ and using the $C^{\frac{1}{2}}$ estimate for $\mathcal{H}$ from Corollary 5.16 twice together with the maximum principle, we have

$$\|\Delta^{-1} \Delta (\mathcal{H} a)^2\|_{C^{\frac{1}{2}}(\Omega_t)} \lesssim_A \|\mathcal{H} a\|_{L^\infty(\Omega_t)} \|\mathcal{H} a\|_{C^{\frac{1}{2}}(\Omega_t)} \lesssim_A \|a\|_{C^{\frac{1}{2}}(\Gamma_t)} \lesssim_A B.$$

From Proposition 5.19, we obtain also

$$\|\Delta^{-1} \Delta (\mathcal{H} a)^2\|_{H^k(\Omega_t)} \lesssim_A B \|\Gamma_t\|_{H^k} + \|\Delta (\mathcal{H} a)^2\|_{H^{k-2}(\Omega_t)}.$$

Then using that $\Delta (\mathcal{H} a)^2 = 2 |\nabla \mathcal{H} a|^2$, we obtain from Corollary 5.4,

$$\|\Delta (\mathcal{H} a)^2\|_{H^{k-2}(\Omega_t)} \lesssim \|\mathcal{H} a\|_{C^{\frac{1}{2}}(\Omega_t)} \|\mathcal{H} a\|_{H^{k-\frac{1}{2}}(\Omega_t)} \lesssim_A B \|\mathcal{H} a\|_{H^{k-\frac{1}{2}}(\Omega_t)}.$$

Then from Proposition 5.21, Lemma 7.7 and the energy coercivity bound (7.15), we obtain

$$\|\mathcal{H}a\|_{H^{k-\frac{1}{2}}(\Omega_t)} \lesssim_A \|a\|_{H^{k-1}(\Gamma_t)} + \|\Gamma\|_{H^k}\|a\|_{L^\infty(\Omega_t)} \lesssim_A (E^k)^{\frac{1}{2}}.$$

Therefore,

$$\|\Delta(\mathcal{H}a)^2\|_{H^{k-2}(\Omega_t)} \lesssim_A B(E^k)^{\frac{1}{2}}.$$

Next, we turn to the term $\nabla_n\Delta^{-1}\Delta|\nabla p|^2$ in (7.35). The procedure here is similar. Like with the previous estimate, we obtain

$$\|\nabla\mathcal{H}\mathcal{N}^{k-2}\nabla_n\Delta^{-1}\Delta|\nabla p|^2\|_{L^2(\Omega_t)} \lesssim_A \|\Gamma_t\|_{H^k}\|\Delta^{-1}\Delta(|\nabla p|^2)\|_{C^{\frac{1}{2}}(\Omega_t)} + \|\Delta(|\nabla p|^2)\|_{H^{k-2}(\Omega_t)} \tag{7.36}$$

and also

$$\|\Delta^{-1}\Delta(|\nabla p|^2)\|_{C^{\frac{1}{2}}(\Omega_t)} \lesssim_A B.$$

Moreover, by expanding $\Delta|\nabla p|^2$ (and some simple manipulations), we have

$$\|\Delta(|\nabla p|^2)\|_{H^{k-2}(\Omega_t)} \lesssim \||\nabla^2 p|^2\|_{H^{k-2}(\Omega_t)} + \||\Delta p|^2\|_{H^{k-2}(\Omega_t)} + \|\nabla p\Delta p\|_{H^{k-1}(\Omega_t)}.$$

Using Corollary 5.4 and Lemma 7.9, we have for the first two terms

$$\||\nabla^2 p|^2\|_{H^{k-2}(\Omega_t)} + \||\Delta p|^2\|_{H^{k-2}(\Omega_t)} \lesssim_A \|\nabla p\|_{C^{\frac{1}{2}}(\Omega_t)}\|p\|_{H^{k+\frac{1}{2}}(\Omega_t)} \lesssim_A B\|p\|_{H^{k+\frac{1}{2}}(\Omega_t)}.$$

To handle the other term, we use the Laplace equation for p to write

$$\|\nabla p\Delta p\|_{H^{k-1}(\Omega_t)} = \|\nabla p\partial_i v_j\partial_j v_i\|_{H^{k-1}(\Omega_t)}. \tag{7.37}$$

Then from (5.1), Corollary 5.4, Lemma 7.5 and Lemma 7.8, we have

$$\begin{aligned}\|\nabla p\partial_i v_j\partial_j v_i\|_{H^{k-1}(\Omega_t)} &\lesssim_A \|v\|_{W^{1,\infty}(\Omega_t)}\|\nabla p\partial_i v_j\|_{H^{k-1}(\Omega_t)} + \|\nabla p\partial_i v_j\|_{L^\infty(\Omega_t)}\|v\|_{H^k(\Omega_t)}\\ &\lesssim_A \|v\|_{W^{1,\infty}(\Omega_t)}(\|v\|_{C^{\frac{1}{2}}(\Omega_t)}\|p\|_{H^{k+\frac{1}{2}}(\Omega_t)} + \|p\|_{W^{1,\infty}(\Omega_t)}\|v\|_{H^k(\Omega_t)})\\ &\lesssim_A B\|(v,\Gamma)\|_{\mathbf{H}^k}.\end{aligned} \tag{7.38}$$

Combining the above with the energy coercivity (7.15), we obtain

$$\|\Delta(|\nabla p|^2)\|_{H^{k-2}(\Omega_t)} \lesssim_A B(E^k)^{\frac{1}{2}}.$$

Next, we turn to the estimate for M_1. We first write $M_1 = M_1'\mathcal{B}$ where M_1' is an $\mathbb{R}$-valued multilinear expression in n_{Γ_t} and ∇p and $\mathcal{B}$ is an $\mathbb{R}$-valued bilinear expression in ∇v. We use the bilinear frequency decomposition $\mathcal{B}(\nabla v,\nabla v) = \mathcal{B}(\nabla v^l,\nabla v^{\leq l}) + \mathcal{B}(\nabla v^{\leq l},\nabla v^l)$ and consider the partition $\mathcal{B} = \mathcal{B}_j^1 + \mathcal{B}_j^2$ where $\mathcal{B}_j^1 := \mathcal{B}(\nabla\Phi_{<j}v^l,\nabla v^{\leq l}) + \mathcal{B}(\nabla v^{\leq l},\nabla\Phi_{<j}v^l)$. Then using this partition, the trace inequality, energy coercivity and Proposition 5.30, we have

(7.39)

$$\begin{aligned}\|\nabla\mathcal{H}\mathcal{N}^{k-2}M_1\|_{L^2(\Omega_t)} &\lesssim_A \|M_1\|_{H^{k-\frac{3}{2}}(\Gamma_t)} + \|\Gamma_t\|_{H^k}\sup_{j>0}2^{-\frac{j}{2}}\|\mathcal{B}_j^1\|_{L^\infty(\Omega_t)} + \sup_{j>0}2^{j(k-\frac{3}{2}-2\epsilon)}\|\mathcal{B}_j^2\|_{H^{\frac{1}{2}+\epsilon}(\Omega_t)}\\ &\lesssim_A \|M_1\|_{H^{k-\frac{3}{2}}(\Gamma_t)} + \|v\|_{W^{1,\infty}(\Omega_t)}\|v\|_{C^{\frac{1}{2}+\epsilon}(\Omega_t)}\|\Gamma_t\|_{H^k} + \|v\|_{W^{1,\infty}(\Omega_t)}\|v\|_{H^k(\Omega_t)}\\ &\lesssim_A \|M_1\|_{H^{k-\frac{3}{2}}(\Gamma_t)} + B(E^k)^{\frac{1}{2}}.\end{aligned}$$

Using the same partition as above and Proposition 5.9, Proposition 5.11 and Lemma 7.5, we have

$$\begin{aligned}\|M_1\|_{H^{k-\frac{3}{2}}(\Gamma_t)} \lesssim_A\ & \|\nabla v\|_{L^\infty(\Omega_t)}\|v\|_{H^k(\Omega_t)} + (\|\Gamma_t\|_{H^k} + \|M_1'(\nabla p, n_{\Gamma_t})\|_{H^{k-1}(\Gamma_t)})\sup_{j>0}2^{-\frac{j}{2}}\|\mathcal{B}_j^1\|_{L^\infty(\Omega_t)}\\ &+ \sup_{j>0}2^{j(k-\frac{3}{2}-2\epsilon)}\|\mathcal{B}_j^2\|_{\mathcal{H}^{\frac{1}{2}+\epsilon}(\Omega_t)}.\end{aligned}$$

Estimating as in (7.39), this simplifies to

$$\|M_1\|_{H^{k-\frac{3}{2}}(\Gamma_t)} \lesssim_A B(E^k)^{\frac{1}{2}} + B\|M_1'(\nabla p, n_{\Gamma_t})\|_{H^{k-1}(\Gamma_t)}.$$

By Proposition 5.9, Proposition 5.11, Lemma 7.8 and the energy coercivity, we have also

$$\|M_1'(\nabla p, n_{\Gamma_t})\|_{H^{k-1}(\Gamma_t)} \lesssim_A (E^k)^{\frac{1}{2}},$$

from which we deduce

$$\|\nabla \mathcal{HN}^{k-2} M_1\|_{L^2(\Omega_t)} \lesssim_A B(E^k)^{\frac{1}{2}}.$$

Next, we estimate M_2. This estimate is similar to M_1. One starts by writing $M_2 = M_2'\mathcal{B}$ where M_2' is multilinear in ∇p and n_{Γ_t} while $\mathcal{B}$ is bilinear in ∇v and $\nabla D_t p$. Using the partition $\mathcal{B} = \mathcal{B}_j^1 + \mathcal{B}_j^2$ with $\mathcal{B}_j^1 := \mathcal{B}(\nabla \Phi_{<j} v^l, \nabla (D_t p)^{\leq l}) + \mathcal{B}(\nabla v^{\leq l}, \nabla \Phi_{<j}(D_t p)^l)$ and a similar analysis to M_1, we have

$$\begin{aligned}\|\nabla \mathcal{HN}^{k-2} M_2\|_{L^2(\Omega_t)} \lesssim_A \|v\|_{W^{1,\infty}(\Omega_t)}\|D_t p\|_{H^k(\Omega_t)} + \|D_t p\|_{W^{1,\infty}(\Omega_t)}(\|\Gamma_t\|_{H^k} + \|M_2'(\nabla p, n_{\Gamma_t})\|_{H^{k-1}(\Gamma_t)}) \\ + \|D_t p\|_{W^{1,\infty}(\Omega_t)}\|v\|_{H^k(\Omega_t)}.\end{aligned}$$

Then using the $W^{1,\infty}$ bound for $D_t p$ from Lemma 7.9 and the H^k bound for $D_t p$ from Lemma 7.8, we have

$$\|\nabla \mathcal{HN}^{k-2} M_2\|_{L^2(\Omega_t)} \lesssim_A B \log(1 + \|(v, \Gamma)\|_{\mathbf{H}^s})(E^k)^{\frac{1}{2}}.$$

Now we turn to the estimate for the term involving $D_t^2 p$. As usual, we first aim to write it in the form $\Delta^{-1}\nabla \cdot f$ but in such a way that f involves favorable frequency interactions. This presents some mild technical challenges as $D_t^2 p$ will have terms which are up to quadrilinear in ∇v. To deal with this, we have the following lemma.

Lemma 7.11. *There exist bilinear, trilinear and quadrilinear expressions $\mathcal{B}$, $\mathcal{T}$ and $\mathcal{M}$ taking values in $\mathbb{R}^d$ such that*

$$\Delta D_t^2 p = -2\Delta|\nabla p|^2 + \nabla \cdot \mathcal{B}(\nabla D_t p, \nabla v) + \nabla \cdot \mathcal{T}(\nabla p, \nabla v, \nabla v) + \nabla \cdot \mathcal{M}(v^m, \nabla v^{\leq m}, \nabla v^{\leq m}, \nabla v^{\leq m}).$$

Proof. First, using that v is divergence free, it is straightforward to verify

$$\Delta D_t^2 p = \partial_i(\partial_j D_t p \partial_j v_i) + \partial_i(\partial_i v_j \partial_j D_t p) + D_t \Delta D_t p = \nabla \cdot \mathcal{B} + D_t \Delta D_t p.$$

Next, we expand $D_t \Delta D_t p$. We start with the Laplace equation for $D_t p$ from (7.17),

$$\Delta D_t p = 3\partial_j(\partial_i p \partial_i v_j) + \partial_i(\partial_i v_j \partial_j p) + 2\partial_j v_k \partial_k v_i \partial_i v_j.$$

Using that v is divergence free, we have the commutator identity $[\partial_i, D_t]f = \partial_j(\partial_i v_j f)$. Combining this with the Euler equations, we obtain

$$\begin{aligned}D_t(3\partial_j(\partial_i p \partial_i v_j) + \partial_i(\partial_i v_j \partial_j p)) &= \nabla \cdot \mathcal{B} + \nabla \cdot \mathcal{T} - 4\partial_j(\partial_i p \partial_i \partial_j p) \\ &= \nabla \cdot \mathcal{B} + \nabla \cdot \mathcal{T} - 2\Delta|\nabla p|^2.\end{aligned}$$

It remains to expand $2D_t(\partial_j v_k \partial_k v_i \partial_i v_j)$. From the Euler equation and symmetry, we have

$$2D_t(\partial_j v_k \partial_k v_i \partial_i v_j) = 6D_t(\partial_j v_k)\partial_k v_i \partial_i v_j = -6\partial_j \partial_k p \partial_k v_i \partial_i v_j - 6\partial_j v_l \partial_l v_k \partial_k v_i \partial_i v_j.$$

We rearrange the first term as

$$\begin{aligned}-6\partial_j \partial_k p \partial_k v_i \partial_i v_j &= -6\partial_j(\partial_k p \partial_k v_i \partial_i v_j) + 6\partial_k p \partial_j \partial_k v_i \partial_i v_j = -6\partial_j(\partial_k p \partial_k v_i \partial_i v_j) + 3\partial_k p \partial_k(\partial_j v_i \partial_i v_j) \\ &= -6\partial_j(\partial_k p \partial_k v_i \partial_i v_j) + 3\partial_k(\partial_k p \partial_j v_i \partial_i v_j) - 3\partial_k \partial_k p \partial_j v_i \partial_i v_j \\ &= \nabla \cdot \mathcal{T} + 3|\Delta p|^2,\end{aligned} \tag{7.40}$$

where in the last line we used the Laplace equation for p. On the other hand, for the second term, by symmetry of the indices, we have the quadrilinear frequency decomposition,

$$\begin{aligned}-6\partial_j v_l\partial_l v_k\partial_k v_i\partial_i v_j &= -24\partial_j v_l^m\partial_l v_k^{\leq m}\partial_k v_i^{\leq m}\partial_i v_j^{\leq m}\\ &= \nabla\cdot\mathcal{M} + 24v_l^m\partial_l\partial_j v_k^{\leq m}\partial_k v_i^{\leq m}\partial_i v_j^{\leq m} + 24v_l^m\partial_l v_k^{\leq m}\partial_j\partial_k v_i^{\leq m}\partial_i v_j^{\leq m}.\end{aligned}$$

By symmetry and the fact that v is divergence free, the second term on the right-hand side can be rearranged as

$$24v_l^m\partial_l\partial_j v_k^{\leq m}\partial_k v_i^{\leq m}\partial_i v_j^{\leq m} = 8v_l^m\partial_l(\partial_j v_k^{\leq m}\partial_k v_i^{\leq m}\partial_i v_j^{\leq m}) = \nabla\cdot\mathcal{M}.$$

For the third term on the right-hand side, we have

$$\begin{aligned}24v_l^m\partial_l v_k^{\leq m}\partial_j\partial_k v_i^{\leq m}\partial_i v_j^{\leq m} &= 12v_l^m\partial_l v_k^{\leq m}\partial_k(\partial_j v_i^{\leq m}\partial_i v_j^{\leq m}) = \nabla\cdot\mathcal{M} - 12\partial_k v_l^m\partial_l v_k^{\leq m}\partial_j v_i^{\leq m}\partial_i v_j^{\leq m}\\ &= \nabla\cdot\mathcal{M} - 3\partial_k v_l\partial_l v_k\partial_j v_i\partial_i v_j \qquad (7.41)\\ &= \nabla\cdot\mathcal{M} - 3|\Delta p|^2,\end{aligned}$$

where we used the Laplace equation for p in the last line. Combining (7.40) and (7.41) to cancel the $3|\Delta p|^2$ terms then completes the proof of the lemma. □

Now, we return to the estimate for $\nabla\mathcal{H}\mathcal{N}^{k-2}\nabla_n D_t^2 p$. We use Lemma 7.11 and estimate each term separately. The term $-2\nabla\mathcal{H}\mathcal{N}^{k-2}\nabla_n\Delta^{-1}\Delta|\nabla p|^2$ can be estimated identically to (7.36). Let us then turn to the estimate for $\nabla\mathcal{H}\mathcal{N}^{k-2}\nabla_n\Delta^{-1}(\nabla\cdot\mathcal{B})$. We use a partition $\mathcal{B} = \mathcal{B}_j^1 + \mathcal{B}_j^2$ where $\mathcal{B}_j^1$ is defined as follows: First, we perform the frequency decomposition,

$$\mathcal{B} = \mathcal{B}(\nabla(D_t p)^l, \nabla v^{\leq l}) + \mathcal{B}(\nabla(D_t p)^{\leq l}, \nabla v^l)$$

and then define

$$\mathcal{B}_1^j := \mathcal{B}(\nabla\Phi_{\leq j}(D_t p)^l, \nabla v^{\leq l}) + \mathcal{B}(\nabla(D_t p)^{\leq l}, \nabla\Phi_{\leq j} v^l).$$

Then Corollary 5.32 and Proposition 5.19 gives

$$\begin{aligned}\|\nabla\mathcal{H}\mathcal{N}^{k-2}\nabla_n\Delta^{-1}(\nabla\cdot\mathcal{B})\|_{L^2(\Omega_t)} &\lesssim_A \|\mathcal{B}\|_{H^{k-1}(\Omega_t)} + \|\Gamma_t\|_{H^k}\sup_{j>0} 2^{-\frac{j}{2}}\|\Delta^{-1}(\nabla\cdot\mathcal{B}_1^j)\|_{W^{1,\infty}(\Omega_t)}\\ &\quad + \sup_{j>0} 2^{j(k-1-\epsilon)}\|\Delta^{-1}(\nabla\cdot\mathcal{B}_2^j)\|_{H^1(\Omega_t)}.\end{aligned}$$

From Sobolev product estimates and the H^k and L^∞ estimates for $D_t p$,

$$\|\mathcal{B}\|_{H^{k-1}(\Omega_t)} \lesssim_A \|v\|_{W^{1,\infty}(\Omega_t)}\|D_t p\|_{H^k(\Omega_t)} + \|D_t p\|_{W^{1,\infty}(\Omega_t)}\|v\|_{H^k(\Omega_t)} \lesssim_A B\log(1+\|(v,\Gamma)\|_{\mathbf{H}^s})(E^k)^{\frac{1}{2}}.$$

Using Proposition 5.15, we also estimate

$$2^{-\frac{j}{2}}\|\Delta^{-1}(\nabla\cdot\mathcal{B}_1^j)\|_{C^{1,\epsilon}(\Omega_t)} \lesssim_A \|D_t p\|_{W^{1,\infty}(\Omega_t)}\|v\|_{C^{\frac{1}{2}+\epsilon}(\Omega_t)} \lesssim_A B\log(1+\|(v,\Gamma)\|_{\mathbf{H}^s}).$$

Finally, using the error bounds for $\Phi_{>j}$ and the L^∞ and H^k estimates for $D_t p$ from Lemma 7.8 we see that

$$2^{j(k-1-\epsilon)}\|\Delta^{-1}(\nabla\cdot\mathcal{B}_2^j)\|_{H^1(\Omega_t)} \lesssim_A \|v\|_{W^{1,\infty}(\Omega_t)}\|D_t p\|_{H^k(\Omega_t)} + \|D_t p\|_{W^{1,\infty}(\Omega_t)}\|v\|_{H^k(\Omega_t)} \lesssim_A B\log(1+\|(v,\Gamma)\|_{\mathbf{H}^s})(E^k)^{\frac{1}{2}}.$$

Hence,

$$\|\nabla\mathcal{H}\mathcal{N}^{k-2}\nabla_n\Delta^{-1}(\nabla\cdot\mathcal{B})\|_{L^2(\Omega_t)} \lesssim_A B\log(1+\|(v,\Gamma)\|_{\mathbf{H}^s})(E^k)^{\frac{1}{2}}.$$

The estimates for $\nabla\mathcal{H}\mathcal{N}^{k-2}\nabla_n\Delta^{-1}(\nabla\cdot\mathcal{T})$ and $\nabla\mathcal{H}\mathcal{N}^{k-2}\nabla_n\Delta^{-1}(\nabla\cdot\mathcal{M})$ are very similar. The main difference is that we use the partition $\mathcal{T} = \mathcal{T}_1^j + \mathcal{T}_2^j$ with

$$\mathcal{T}_1^j = 2\mathcal{T}(\nabla p, \nabla\Phi_{\leq j} v^l, \nabla v^{\leq l})$$

and the partition $\mathcal{M} = \mathcal{M}_1^j + \mathcal{M}_2^j$ with

$$\mathcal{M}_1^j := \mathcal{M}(v^m, \nabla\Phi_{\leq j} v^{\leq m}, \nabla v^{\leq m}, \nabla v^{\leq m}).$$

Ultimately, we obtain

$$\|\nabla\mathcal{H}\mathcal{N}^{k-2}\nabla_n D_t^2 p\|_{L^2(\Omega_t)} \lesssim_A B\log(1+\|(v,\Gamma)\|_{\mathbf{H}^s})(E^k)^{\frac{1}{2}}$$

which when combined with the previous analysis gives

$$\|\nabla\mathcal{H}\mathcal{N}^{k-2} f\|_{L^2(\Omega_t)} \lesssim_A B\log(1+\|(v,\Gamma)\|_{\mathbf{H}^s})(E^k)^{\frac{1}{2}}$$

as desired. The last term in the estimate for $\mathcal{Q}$ that we need to control is $\nabla\mathcal{H}[\mathcal{N}^{k-2}, a]\mathcal{N}a$. For this, we have the following technical lemma.

Lemma 7.12. *We have the following estimate:*

$$\|\nabla\mathcal{H}[\mathcal{N}^{k-2}, a]\mathcal{N}a\|_{L^2(\Omega_t)} \lesssim_A B\log(1+\|(v,\Gamma)\|_{\mathbf{H}^s})(E^k)^{\frac{1}{2}}.$$

Proof. Thanks to the $H^{\frac{1}{2}}(\Gamma_t) \to H^1(\Omega_t)$ bound for $\mathcal{H}$, it suffices to estimate $\|[\mathcal{N}^{k-2}, a]\mathcal{N}a\|_{H^{\frac{1}{2}}(\Gamma_t)}$. We begin by using the Leibniz formula (5.36) to expand the commutator,

$$[\mathcal{N}^{k-2}, a]\mathcal{N}a = \sum_{n+m=k-3} \mathcal{N}^n(\mathcal{N}a\mathcal{N}^{m+1}a) - 2\mathcal{N}^n\nabla_n\Delta^{-1}(\nabla\mathcal{H}a\cdot\nabla\mathcal{H}\mathcal{N}^{m+1}a). \tag{7.42}$$

We focus on the latter term in (7.42) first as it is a bit more delicate to deal with. To simplify notation slightly, we write

$$a_j := \mathcal{H}\mathcal{N}^j a, \qquad F := \nabla a_0\cdot\nabla a_{m+1}, \qquad \mathcal{N}_{<j} := n_{\Gamma_t}\cdot\nabla\Phi_{<j}\mathcal{H}, \qquad \mathcal{N}_{\geq j} := n_{\Gamma_t}\cdot\nabla\Phi_{\geq j}\mathcal{H}.$$

Using Corollary 5.32 and then Proposition 5.19, we have

$$\begin{aligned}\|\mathcal{N}^n(\nabla_n\Delta^{-1}F)\|_{H^{\frac{1}{2}}(\Gamma_t)} \lesssim_A \|F\|_{H^n(\Omega_t)} + \|\Gamma\|_{H^k}\sup_{j>0} 2^{-j(m+\frac{3}{2})}\|\Delta^{-1}F_j^1\|_{W^{1,\infty}(\Omega_t)}\\ + \sup_{j>0} 2^{j(n+1-\epsilon)}\|\Delta^{-1}F_j^2\|_{H^1(\Omega_t)},\end{aligned}$$

where $F = F_j^1 + F_j^2$ is a suitable partition of F to be chosen. To find a suitable partition, we start with a bilinear frequency decomposition similar to before. We define $a_j^l := \Phi_l a_j$ and $a_j^{\leq l} = \Phi_{\leq l} a_j$.

Remark 7.13. We note that the regularization operator $\Phi_{\leq l}$ does not preserve the harmonic property of a_j. However, using the definition of $\Phi_{\leq l}$ (see Section 6), the operator defined by $C_{\leq l} := [\Delta, \Phi_{\leq l}]$ is readily seen to satisfy the bounds,

$$\|C_{\leq l}\|_{C^\alpha\to L^\infty} \lesssim_A 2^{l(1-\alpha)} \qquad \|C_{\leq l}\|_{H^\alpha\to L^2} \lesssim_A 2^{l(1-\alpha)}, \qquad 0\leq\alpha\leq 1 \tag{7.43}$$

for $\alpha, l \geq 0$. That is, $C_{\leq l}$ behaves like a differential operator of order 1 localized at dyadic scale $\lesssim 2^l$.

Now, using the same convention as before in this section (where repeated indices are summed over) we have

$$F = \nabla a_0^l\cdot\nabla a_{m+1}^{\leq l} + \nabla a_0^{\leq l}\cdot\nabla a_{m+1}^l =: F' + F''.$$

We can write F' and F'' to leading order as the divergence of some vector field. Using that a_0 and a_{m+1} are harmonic, we have

$$\begin{aligned}F' &= \nabla\cdot(a_0^l\nabla a_{m+1}^{\leq l}) - a_0^l C_{\leq l}a_{m+1} =: G' + H',\\ F'' &= \nabla\cdot(a_{m+1}^l\nabla a_0^{\leq l}) - a_{m+1}^l C_{\leq l}a_0 =: G'' + H''.\end{aligned} \tag{7.44}$$

We will focus on F' first. To choose a partition of F', we need to choose a suitable partition of G' and H'. We show the details for G' and remark later on the minor changes needed to deal with H'. We write $G' = (G')^1_j + (G')^2_j$ with

$$(G')^1_j = \nabla \cdot (a^l_0 \nabla \Phi_{\leq l} a_{m+1,\leq j}), \qquad a_{m+1,\leq j} := \Phi_{\leq j}(\mathcal{H}\mathcal{N}^{m+1}_{<j} a).$$

From Proposition 5.15, iterating the maximum principal and using the C^α bounds for $\mathcal{H}$ and the properties of $\Phi_{<j}$, we have

$$2^{-j(m+\frac{3}{2})}\|\Delta^{-1}(G')^1_j\|_{W^{1,\infty}(\Omega_t)} \lesssim_A \|a\|_{C^\epsilon(\Gamma_t)}\|a\|_{C^{\frac{1}{2}}(\Gamma_t)} \lesssim_A B,$$

where we used Lemma 7.5 and Lemma 7.9 in the last inequality. For $(G')^2_j$, we can write

$$(G')^2_j = \nabla \cdot (a^l_0 \nabla b^{\leq l}_{m+1,j}) + \sum_{0\leq i\leq m} \nabla \cdot (a^l_0 \nabla b^{\leq l}_{i,j})$$

where

$$b^{\leq l}_{m+1,j} := \Phi_{\leq l}\Phi_{\geq j} a_{m+1}, \qquad b^{\leq l}_{i,j} := \Phi_{\leq l}\Phi_{<j}\mathcal{H}\mathcal{N}^i_{<j}\mathcal{N}_{\geq j}\mathcal{N}^{m-i}a.$$

Using Corollary 5.16, the properties of the kernel Φ and the $H^{-1} \to H^1$ bound for Δ^{-1}, we obtain for each $0 \leq i \leq m$,

$$\begin{aligned} 2^{j(n+1-\epsilon)}\|\Delta^{-1}\nabla \cdot (a^l_0 \nabla b^{\leq l}_{i,j})\|_{H^1(\Omega_t)} &\lesssim_A 2^{j(n+1-\epsilon)}\|a^l_0\|_{L^\infty(\Omega_t)}\|b^{\leq l}_{i,j}\|_{H^1(\Omega_t)} \\ &\lesssim_A 2^{j(n+1-\epsilon)}\|a\|_{C^{\frac{1}{2}}(\Gamma_t)}\|\mathcal{H}\mathcal{N}^i_{<j}\mathcal{N}_{\geq j}\mathcal{N}^{m-i}a\|_{H^{\frac{1}{2}+\epsilon}(\Omega_t)}. \end{aligned} \tag{7.45}$$

Repeatedly using the $H^\epsilon \to H^{\frac{1}{2}+\epsilon}$ estimate (5.20), the properties of Φ, the bound $\|n_{\Gamma_t}\|_{C^\epsilon(\Gamma_t)} \lesssim_A 1$ and the trace inequality, we can estimate

$$\begin{aligned} 2^{j(n+1-\epsilon)}\|\mathcal{H}\mathcal{N}^i_{<j}\mathcal{N}_{\geq j}\mathcal{N}^{m-i}a\|_{H^{\frac{1}{2}+\epsilon}(\Omega_t)} &\lesssim_A 2^{j(n+1+i-\epsilon)}\|\nabla\Phi_{\geq j}\mathcal{H}\mathcal{N}^{m-i}a\|_{H^{\frac{1}{2}+\epsilon}(\Omega_t)} \\ &\lesssim_A \|\mathcal{H}\mathcal{N}^{m-i}a\|_{H^{n+i+\frac{5}{2}}(\Omega_t)}. \end{aligned}$$

Using Proposition 5.30, Lemma 7.5, Lemma 7.7 and (7.15), we have

$$\|\mathcal{H}\mathcal{N}^{m-i}a\|_{H^{n+i+\frac{5}{2}}(\Gamma_t)} \lesssim_A \|a\|_{H^{k-1}(\Gamma_t)} + \|\Gamma\|_{H^k}\|a\|_{C^\epsilon(\Gamma_t)} \lesssim_A (E^k)^{\frac{1}{2}}.$$

If $n \geq 1$, then doing a similar analysis for the term $\nabla \cdot (a^l_0 \nabla b^{\leq l}_{m+1,j})$ and combining this with (7.45) and the bound $\|a\|_{C^{\frac{1}{2}}(\Gamma_t)} \lesssim_A B$, we obtain

$$2^{j(n+1-\epsilon)}\|\Delta^{-1}(G')^2_j\|_{H^1(\Omega_t)} \lesssim_A B(E^k)^{\frac{1}{2}}.$$

If $n = 0$, the term $\nabla \cdot (a^l_0 \nabla b^{\leq l}_{m+1,j})$ is instead treated slightly differently. For this, we estimate similarly to before,

$$2^{j(1-\epsilon)}\|\Delta^{-1}\nabla \cdot (a^l_0 \nabla b^{\leq l}_{m+1,j})\|_{H^1(\Omega_t)} \lesssim_A \|a\|_{C^{\frac{1}{2}}(\Gamma_t)}\|\mathcal{H}\mathcal{N}^{m+1}a\|_{H^{\frac{3}{2}}(\Omega_t)}.$$

Then we use Proposition 5.18 to estimate the last term as

$$\|\mathcal{H}\mathcal{N}^{m+1}a\|_{H^{\frac{3}{2}}(\Omega_t)} \lesssim_A \|\mathcal{N}^{m+1}a\|_{H^1(\Gamma_t)},$$

and then estimate this term by $(E^k)^{\frac{1}{2}}$ similarly to the above. Next, one readily verifies analogous bounds for H', G'' and H'' by using the similar decompositions,

$$(H')^1_j = -a^l_0 C_{\leq l}(a_{m+1,\leq j}), \qquad (G'')^1_j = \nabla \cdot (\Phi_l(a_{m+1,\leq j})\nabla a^{\leq l}_0), \qquad (H'')^1_j = -C_{\leq l}a_0\Phi_l(a_{m+1,\leq j}). \tag{7.46}$$

From these bounds, ultimately, we obtain

$$\|\mathcal{N}^n(\nabla_n \Delta^{-1}F)\|_{H^{\frac{1}{2}}(\Gamma_t)} \lesssim_A \|F\|_{H^n(\Omega_t)} + B(E^k)^{\frac{1}{2}}.$$

It remains to estimate F in H^n. We begin by looking at each summand in the bilinear frequency decomposition for F,

$$F_l := \nabla\Phi_l a_0 \cdot \nabla\Phi_{\leq l} a_{m+1} + \nabla\Phi_{\leq l} a_0 \cdot \nabla\Phi_l a_{m+1}.$$

For the latter term, we have

$$\|\nabla\Phi_{<l} a_0 \cdot \nabla\Phi_l a_{m+1}\|_{H^n(\Omega_t)} \lesssim_A \|a\|_{C^{\frac{1}{2}}(\Gamma_t)} \|a_{m+1}\|_{H^{n+\frac{3}{2}}(\Omega_t)},$$

which when $n \geq 1$, we know from the above can be controlled by $B(E^k)^{\frac{1}{2}}$. For $n = 0$, we have the same bound by simply using Proposition 5.18. For the other term, we can further decompose

$$a_{m+1} = a^1_{m+1,l} + a^2_{m+1,l} \tag{7.47}$$

where $a^1_{m+1,l} = \mathcal{H}\mathcal{N}^{m+1}_{<l} a$. We then have from the properties of $\Phi_{\leq l}$ and the control of $\|\mathcal{H}a\|_{H^{n+m+\frac{5}{2}}(\Omega_t)}$ by the energy (as above),

$$\|\nabla\Phi_l a_0 \cdot \nabla\Phi_{\leq l} a_{m+1}\|_{H^n(\Omega_t)} \lesssim_A \|a\|_{C^{\frac{1}{2}}(\Gamma_t)} (E^k)^{\frac{1}{2}}.$$

As ∇a_{m+1} is not at top order, we can easily verify using the decomposition above that we also have the following cruder bound for each l

$$\|F_l\|_{H^n(\Omega_t)} \lesssim_A 2^{-\delta l} \|(v,\Gamma)\|^r_{\mathbf{H}^s} (E^k)^{\frac{1}{2}}, \tag{7.48}$$

for some integer $r > 1$ and small constant $\delta > 0$. Arguing as in Lemma 7.9, we can combine the above two bounds to estimate

$$\|F\|_{H^n(\Omega_t)} \lesssim_A B \log(1 + \|(v,\Gamma)\|_{\mathbf{H}^s})(E^k)^{\frac{1}{2}}.$$

This handles the latter term in (7.42). Now, we turn to the first term. We have to estimate $\|\mathcal{N}^n(\mathcal{N}a\mathcal{N}^{m+1}a)\|_{H^{\frac{1}{2}}(\Gamma_t)}$ where $n, m \geq 0$ and $n + m = k - 3$. Here, we only sketch the details as the procedure for this estimate is relatively similar to the previous term. We start by writing

$$\mathcal{N}a\mathcal{N}^{m+1}a = (\mathcal{H}n_{\Gamma_t} \cdot \nabla a_0)(\mathcal{H}n_{\Gamma_t} \cdot \nabla a_m) =: K_{|\Gamma_t}.$$

Then we apply Proposition 5.30 and Proposition 5.11 to estimate

$$\|\mathcal{N}^n K_{|\Gamma_t}\|_{H^{\frac{1}{2}}(\Gamma_t)} \lesssim_A \|K\|_{H^{n+1}(\Omega_t)} + \|\Gamma\|_{H^k} \sup_{j>0} 2^{-j(m+\frac{3}{2})} \|K^1_j\|_{L^\infty(\Omega_t)} + \sup_{j>0} 2^{j(n+\frac{1}{2}-2\epsilon)} \|K^2_j\|_{H^{\frac{1}{2}+\epsilon}(\Omega_t)}$$

where $K = K^1_j + K^2_j$ and

$$K^1_j := \Phi_{<j}((\mathcal{H}n_{\Gamma_t} \cdot \nabla\Phi_{<j} a_0)(\mathcal{H}n_{\Gamma_t} \cdot \nabla\Phi_{<j} \mathcal{H}\mathcal{N}^m_{<j} a)). \tag{7.49}$$

Similarly to the above, we can estimate

$$2^{-j(m+\frac{3}{2})} \|K^1_j\|_{L^\infty(\Omega_t)} \lesssim_A B.$$

We also have an estimate of the form

$$\begin{aligned} 2^{j(n+\frac{1}{2}-2\epsilon)} \|K^2_j\|_{H^{\frac{1}{2}+\epsilon}(\Omega_t)} &\lesssim_A \|K\|_{H^{n+1}(\Omega_t)} + B(E^k)^{\frac{1}{2}} + 2^{j(n+1-\epsilon)} \|\mathcal{B}(\nabla\Phi_{\geq j} a_0, \nabla a_m)\|_{L^2(\Omega_t)} \\ &\lesssim_A \|K\|_{H^{n+1}(\Omega_t)} + B(E^k)^{\frac{1}{2}} + \sup_{l>0} 2^{l(n+1-\epsilon)} \|\mathcal{B}(\nabla\Phi_l a_0, \nabla a_m)\|_{L^2(\Omega_t)} \end{aligned}$$

for some bilinear expression $\mathcal{B}$. Using a decomposition of a_m similar to (7.47), we have

$$2^{l(n+1-\epsilon)} \|\mathcal{B}(\nabla\Phi_l a_0, \nabla a_m)\|_{L^2(\Omega_t)} \lesssim_A B \log(1 + \|(v,\Gamma)\|_{\mathbf{H}^s})(E^k)^{\frac{1}{2}}.$$

Therefore, we have

$$\|\mathcal{N}^n K_{|\Gamma_t}\|_{H^{\frac{1}{2}}(\Gamma_t)} \lesssim_A B \log(1 + \|(v,\Gamma)\|_{\mathbf{H}^s})(E^k)^{\frac{1}{2}} + \|K\|_{H^{n+1}(\Omega_t)}.$$

To estimate K in $H^{n+1}(\Omega_t)$, the starting point is similar (but slightly more technical) than the estimate for F in H^n from above. The idea is to do a quadrilinear frequency decomposition for K and study each summand individually. The relevant terms correspond to terms essentially of the form $(\Phi_l \mathcal{H} n_{\Gamma_t} \cdot \nabla \Phi_{\leq l} a_0)(\Phi_{\leq l} \mathcal{H} n_{\Gamma_t} \cdot \nabla \Phi_{\leq l} a_m)$ and $(\Phi_{\leq l} \mathcal{H} n_{\Gamma_t} \cdot \nabla \Phi_l a_0)(\Phi_{\leq l} \mathcal{H} n_{\Gamma_t} \cdot \nabla \Phi_{\leq l} a_m)$ and $(\Phi_{\leq l} \mathcal{H} n_{\Gamma_t} \cdot \nabla \Phi_{\leq l} a_0)(\Phi_l \mathcal{H} n_{\Gamma_t} \cdot \nabla \Phi_{\leq l} a_m)$ and $(\Phi_{\leq l} \mathcal{H} n_{\Gamma_t} \cdot \nabla \Phi_{\leq l} a_0)(\Phi_{\leq l} \mathcal{H} n_{\Gamma_t} \cdot \nabla \Phi_l a_m)$. The second and fourth terms can be handled almost identically to the estimate for F in H^n (as by the maximum principle, one can dispense with the factors of $\mathcal{H} n_{\Gamma_t}$). The first and third terms are handled similarly by decomposing a_0 and a_m into low and high frequency parts as in (7.47) and using Proposition 5.21 when $\Phi_l \mathcal{H} n_{\Gamma_t}$ is at high frequency compared to the other factors. One then obtains the desired estimate similarly to the estimate for F in H^n above. We omit the remaining details. □

We now turn to the estimate for the final source term, $\mathcal{R} = [D_t, \mathcal{N}^{k-1}]a$ in $L^2(\Gamma_t)$. To control this term, we first write

$$[D_t, \mathcal{N}^{k-1}]a = [D_t, \mathcal{N}]\mathcal{N}^{k-2}a + \mathcal{N}[D_t, \mathcal{N}^{k-2}]a.$$

For the latter term, we have by Lemma 5.24,

$$\|\mathcal{N}[D_t, \mathcal{N}^{k-2}]a\|_{L^2(\Gamma_t)} \lesssim_A \|[D_t, \mathcal{N}^{k-2}]a\|_{H^1(\Gamma_t)}.$$

Then using Proposition 5.33 and the coercivity bound, we estimate

$$\begin{aligned}\|[D_t, \mathcal{N}^{k-2}]a\|_{H^1(\Gamma_t)} &\lesssim_A \|v\|_{W^{1,\infty}(\Omega_t)}\|a\|_{H^{k-1}(\Gamma_t)} + \|a\|_{C^{\frac{1}{2}}(\Gamma_t)}(\|\Gamma\|_{H^k} + \|v\|_{H^k(\Omega_t)}) + \|a\|_{L^\infty(\Gamma_t)}\|v\|_{W^{1,\infty}(\Omega_t)}\|\Gamma\|_{H^k} \\ &\lesssim_A B(E^k)^{\frac{1}{2}}.\end{aligned}$$

To conclude the proof of Theorem 7.1, it remains to estimate $[D_t, \mathcal{N}]\mathcal{N}^{k-2}a$ in $L^2(\Gamma)$. This term is rather delicate due to the lack of a trace estimate in $L^2(\Gamma)$. To deal with this term, we have the following proposition.

Proposition 7.14. *Let $s \in \mathbb{R}$ with $s > \frac{d}{2} + 1$. Then we have,*

$$\|[\mathcal{N}, D_t]f\|_{L^2(\Gamma)} \lesssim_A B \log(1 + \|(v, \Gamma)\|_{\mathbf{H}^s})\|f\|_{H^1(\Gamma)}. \tag{7.50}$$

Our proof requires the following short lemma which is essentially a consequence of Proposition 5.18.

Lemma 7.15. *For each $l = 1, ..., d$, we have*

$$\|n_\Gamma \cdot (\nabla \Delta^{-1} \partial_l - e_l)\|_{H^{\frac{1}{2}}(\Omega) \to L^2(\Gamma)} \lesssim_A 1. \tag{7.51}$$

Proof. This will follow by interpolation if we can prove

$$\|n_\Gamma \cdot (\nabla \Delta^{-1} \partial_l - e_l)\|_{L^2(\Omega) \to H^{-\frac{1}{2}}(\Gamma)} + \|n_\Gamma \cdot (\nabla \Delta^{-1} \partial_l - e_l)\|_{H^{\frac{1}{2}+\delta}(\Omega) \to H^\delta(\Gamma)} \lesssim_A 1, \tag{7.52}$$

for some $0 < \delta < \epsilon$. The $H^{\frac{1}{2}+\delta} \to H^\delta$ bound follows easily from the trace inequality, the bound $\|n_{\Gamma_t}\|_{C^\epsilon(\Gamma_t)} \lesssim_A 1$ and Proposition 5.18. For the $L^2 \to H^{-\frac{1}{2}}$ bound we use duality. Indeed, let $f \in L^2(\Omega)$. Since $(\nabla \Delta^{-1} \partial_l - e_l)f$ is divergence free, we have

$$\int_\Gamma g n_\Gamma \cdot (\nabla \Delta^{-1} \partial_l - e_l) f \, dS = \int_\Omega \nabla \mathcal{H} g \cdot (\nabla \Delta^{-1} \partial_l - e_l) f \, dx \lesssim_A \|g\|_{H^{\frac{1}{2}}(\Gamma)} \|f\|_{L^2(\Omega)},$$

for every $g \in H^{\frac{1}{2}}(\Gamma)$. Therefore, we obtain (7.52) and thus also (7.51). □

Proof of Proposition 7.14. Now, returning to the proposition, we expand using (5.40),

$$[D_t, \mathcal{N}]f = D_t n_\Gamma \cdot \nabla \mathcal{H}f - n_\Gamma \cdot ((\nabla v)^*(\nabla \mathcal{H}f)) + n_\Gamma \cdot \nabla \Delta^{-1}\Delta(v \cdot \nabla \mathcal{H}f).$$

The first two terms on the right can easily be estimated in L^2 by the right-hand side of (7.50) by using (5.35) and Lemma 5.24. Now, we turn to the latter term. We write for simplicity $u := \mathcal{H}f$. We then split u as

$$u = \sum_{l \le l_0} \Phi_l u + \Phi_{>l_0} u =: \sum_{l \le l_0} u_l + u_{\ge l_0},$$

where l_0 is a parameter to be chosen. Note that u_l is not harmonic anymore, but it is to leading order. As usual, we also write the corresponding divergence free regularizations for v as $v_l := \Psi_l v$, $v_{<l} := \Psi_{<l} v$ and so forth.

The following lemma shows that we have a suitable estimate when u is replaced by a single dyadic regularization u_l.

Lemma 7.16. *For each $l \in \mathbb{N}_0$, we have*

$$\|\nabla_n \Delta^{-1}\Delta(v \cdot \nabla u_l)\|_{L^2(\Gamma)} \lesssim_A B\|f\|_{H^1(\Gamma)},$$

where the implicit constant does not depend on l.

Proof. We write

$$\nabla_n \Delta^{-1}\Delta(v \cdot \nabla u_l) = \nabla_n \Delta^{-1}\Delta(v_{<l} \cdot \nabla u_l) + \nabla_n \Delta^{-1}\Delta(v_{\ge l} \cdot \nabla u_l). \tag{7.53}$$

For the second term, where v is at high frequency, we use the identity $\Delta^{-1}\Delta = I - \mathcal{H}$ and the $H^1 \to L^2$ bound for $\mathcal{N}$ to estimate

$$\|\nabla_n \Delta^{-1}\Delta(v_{\ge l} \cdot \nabla u_l)\|_{L^2(\Gamma)} \lesssim_A \|\nabla(v_{\ge l} \cdot \nabla u_l)\|_{L^2(\Gamma)} + \|v_{\ge l} \cdot \nabla u_l\|_{H^1(\Gamma)}. \tag{7.54}$$

For the first term in (7.54), we distribute the derivative to obtain

$$\|\nabla(v_{\ge l} \cdot \nabla u_l)\|_{L^2(\Gamma)} \lesssim B\|\nabla u_l\|_{L^2(\Gamma)} + \|v_{\ge l} \cdot \nabla^2 u_l\|_{L^2(\Gamma)}. \tag{7.55}$$

For the first term in (7.55), we use the variant of the trace theorem leading to (4.8) and the fact that u_l is frequency localized to obtain

$$\|\nabla u_l\|_{L^2(\Gamma)} \lesssim \|\nabla u_l\|_{H^1(\Omega)}^{\frac{1}{2}} \|\nabla u_l\|_{L^2(\Omega)}^{\frac{1}{2}} \lesssim \|u\|_{H^{\frac{3}{2}}(\Omega)} \lesssim_A \|f\|_{H^1(\Gamma)}$$

where in the last estimate we used Proposition 5.18. For the second term in (7.55), we again use the trace theorem and the fact that $v_{\ge l}$ is higher frequency to obtain

$$\|v_{\ge l} \cdot \nabla^2 u_l\|_{L^2(\Gamma)} \lesssim \|v_{\ge l} \cdot \nabla^2 u_l\|_{L^2(\Omega)}^{\frac{1}{2}} \|v_{\ge l} \cdot \nabla^2 u_l\|_{H^1(\Omega)}^{\frac{1}{2}} \lesssim B\|u\|_{H^{\frac{3}{2}}(\Omega)} \lesssim B\|f\|_{H^1(\Gamma)}.$$

The term $\|v_{\ge l} \cdot \nabla u_l\|_{H^1(\Gamma)}$ in (7.54) is similarly estimated. For this, we only need to estimate $\|\nabla^\top(v_{\ge l} \cdot \nabla u_l)\|_{L^2(\Gamma)}$, and this is handled by an almost identical strategy to the above.

Now, to estimate the term in (7.53) where v is at low frequency, we distribute the Laplacian and use that $v_{<l}$ is divergence free to write $\nabla_n \Delta^{-1}\Delta(v_{<l} \cdot \nabla u_l)$ as a sum of terms of the form

$$\nabla_n \Delta^{-1}\partial_j(Dv_{<l} D u_l) + \nabla_n \Delta^{-1}\partial_j(v_{<l} C_l u),$$

where $C_l u := [\Delta, \Phi_l]u$. Using Lemma 7.15 we can then estimate

$$\|\nabla_n \Delta^{-1}\Delta(v_{<l} \cdot \nabla u_l)\|_{L^2(\Gamma)} \lesssim_A \|Dv_{<l}Du_l\|_{L^2(\Gamma)\cap H^{\frac{1}{2}}(\Omega)} + \|v_{<l}C_l u\|_{L^2(\Gamma)\cap H^{\frac{1}{2}}(\Omega)} =: J_1 + J_2.$$

Using that v is at low frequency, we can estimate similarly to the above,

$$J_1 \lesssim_A B\|f\|_{H^1(\Gamma)}.$$

For J_2, we note that C_l is an operator of order 1 and still retains essentially the frequency localization scale of 2^l. Therefore, we can estimate J_2 similarly. This completes the proof of the lemma. □

Returning to the proof of Proposition 7.14, we now estimate using Lemma 7.16,

$$\|\nabla_n \Delta^{-1}\Delta(v \cdot \nabla u)\|_{L^2(\Gamma)} \lesssim_A l_0 B\|f\|_{H^1(\Gamma)} + \|\nabla_n \Delta^{-1}\Delta(v \cdot \nabla u_{\geq l_0})\|_{L^2(\Gamma)}.$$

Again, using that v is divergence free, we can (as above) expand $\nabla_n \Delta^{-1}\Delta(v \cdot \nabla u_{\geq l_0})$ as a sum of terms of the form

$$\nabla_n \Delta^{-1}\partial_j(DvDu_{\geq l_0}) + \nabla_n \Delta^{-1}\partial_j(vC_{\leq l_0}u),$$

where $C_{\leq l_0}u = [\Delta, \Phi_{\leq l_0}]u$. For the latter term, we can simply estimate as above (since v is undifferentiated),

$$\|\nabla_n \Delta^{-1}\partial_j(vC_{\leq l_0}u)\|_{L^2(\Gamma)} \leq \sum_{l \leq l_0} \|\nabla_n \Delta^{-1}\partial_j(vC_l u)\|_{L^2(\Gamma)} \lesssim_A l_0 B\|f\|_{H^1(\Gamma)}.$$

For the other term, we use Lemma 7.15 to obtain

$$\|\nabla_n \Delta^{-1}\partial_j(DvDu_{\geq l_0})\|_{L^2(\Gamma)} \lesssim_A B\|Du_{\geq l_0}\|_{L^2(\Gamma)} + \|DvDu_{\geq l_0}\|_{H^{\frac{1}{2}}(\Omega)}.$$

Since u is harmonic we have

$$B\|Du_{\geq l_0}\|_{L^2(\Gamma)} \lesssim_A B\|f\|_{H^1(\Gamma)} + B\|Du_{<l_0}\|_{L^2(\Gamma)}.$$

Then expanding $u_{<l_0} = \sum_{l<l_0} u_l$ and using the trace theorem leading to (4.8) for each term as above, we get

$$B\|Du_{\geq l_0}\|_{L^2(\Gamma)} \lesssim_A Bl_0\|f\|_{H^1(\Gamma)}.$$

Finally, by product estimates and Sobolev embedding, it is easy to bound

$$\|DvDu_{\geq l_0}\|_{H^{\frac{1}{2}}(\Omega)} \lesssim_A B\|f\|_{H^1(\Gamma)} + \|Dv_{\geq l_0}\|_{H^{\frac{d}{2}+\epsilon}(\Omega)}\|f\|_{H^1(\Gamma)} \lesssim_A (B + 2^{-l_0\delta}\|(v,\Gamma)\|_{\mathbf{H}^s})\|f\|_{H^1(\Gamma)}$$

for some $\delta > 0$. Then choosing $l_0 \approx_\delta \log(1 + \|(v,\Gamma)\|_{\mathbf{H}^s})$, we conclude the proof of the proposition. □

Finally, we conclude the proof of Theorem 7.1 by observing first from the above proposition that we have

$$\|[D_t, \mathcal{N}]\mathcal{N}^{k-2}a\|_{L^2(\Gamma)} \lesssim_A B\log(1 + \|(v,\Gamma)\|_{\mathbf{H}^s})\|\mathcal{N}^{k-2}a\|_{H^1(\Gamma)}.$$

Then, using Proposition 5.30, Lemma 7.5, Lemma 7.7 and (7.15), we have

$$\|\mathcal{N}^{k-2}a\|_{H^1(\Gamma)} \lesssim_A (E^k)^{\frac{1}{2}}.$$

This finally concludes the proof of Theorem 7.1.

8. Construction of regular solutions

In this section, we give a new, direct method for constructing solutions to the free boundary Euler equations in the high regularity regime. Solutions at low regularity will be obtained in the next section as unique limits of these regular solutions.

Previous approaches to constructing solutions to free boundary fluid equations include using Lagrangian coordinates, Nash Moser iteration or taking the zero surface tension limit in the capillary problem. A more recent approach in the case of a laterally infinite ocean with flat bottom can be found in [47]. The article [47] uses a paralinearization of the Dirichlet-to-Neumann operator and a complicated iteration scheme to construct solutions. In contrast, we propose a new, geometric approach, implemented fully within the Eulerian coordinates.

Our novel approach is roughly inspired by nonlinear semigroup theory, where one constructs an approximate solution by discretizing the problem in time. To execute this approach successfully, one needs to show that the energy bounds are uniformly preserved throughout the time steps. In our setting, a classical semigroup approach would require one to solve an elliptic free boundary problem with very precise estimates. However, on the other end of the spectrum, one could try to view our equation as an ODE and use an Euler type iteration. Of course, a naïve Euler method cannot work because it loses derivatives. A partial fix to this would be to combine the Euler method with a transport part, which would reduce but not eliminate the loss of derivatives.

Our goal is to retain the simplicity of the Euler plus transport method, while ameliorating the derivative loss by an initial regularization of each iterate in our discretization. In short, we will split the time step into two main pieces:

(i) Regularization.
(ii) Euler plus transport.

To ensure that the uniform energy bounds survive, the regularization step needs to be done carefully. For this, we will take a modular approach and try to decouple this process into two steps, where we regularize individually the domain and the velocity. We believe that this modular approach will serve as a recipe for a new and relatively simple method for constructing solutions to various free boundary problems.

The overarching scheme we employ in this section was carried out in the case of a compressible gas in [27]. While we follow the same rough roadmap here, we stress that the main difficulties in the incompressible liquid case are quite different than for the gas. One obvious reason for this is that the surface of a liquid carries a non-trivial energy. Also, we introduce another new idea here, which is to begin the iteration with a regularized version of the initial data, and then to partially propagate these regularized bounds through the iteration.

8.1. **Basic setup and simplifications.** We begin by fixing a smooth reference hypersurface Γ_* and a collar neighborhood $\Lambda_* := \Lambda(\Gamma_*, \epsilon_0, \delta)$. Here, as usual, ϵ_0 and δ are some small but fixed positive constants. Given $k > \frac{d}{2} + 1$ sufficiently large and an initial state $(v_0, \Gamma_0) \in \mathbf{H}^k$, our aim is to construct a local solution $(v(t), \Gamma_t) \in \mathbf{H}^k$ whose lifespan depends only on the size of $\|(v_0, \Gamma_0)\|_{\mathbf{H}^k}$, the lower bound in the Taylor sign condition and the collar neighborhood Λ_*. We recall from Theorem 7.1 that we have the coercivity

$$1+\|(v,\Gamma)\|_{\mathbf{H}^k}^2 \approx_A E^k(v,\Gamma)$$

for any state $(v,\Gamma)\in\mathbf{H}^k$. For technical convenience, we will work with the slightly modified energy,

$$\mathcal{E}^k(v,\Gamma) := \|\nabla\mathcal{H}\mathcal{N}^{k-2}(a^{-1}D_t a)\|_{L^2(\Omega)}^2 + \|a^{-\frac{1}{2}}\mathcal{N}^{k-1}a\|_{L^2(\Gamma)}^2 + \|\omega\|_{H^{k-1}(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2 + 1. \tag{8.1}$$

This new energy is readily seen to be equivalent to the old one in the sense that

$$\mathcal{E}^k(v,\Gamma) \approx_A E^k(v,\Gamma). \tag{8.2}$$

The primary reason we modify the energy is that it will allow for cleaner cancellations in the energy when we later regularize the velocity.

Now, fix $M>0$. Given a small time step $\epsilon>0$ and a suitable pair of initial data $(v_0,\Gamma_0)\in\mathbf{H}^k$ with $\|(v_0,\Gamma_0)\|_{\mathbf{H}^k}\le M$, we aim to construct a sequence $(v_\epsilon(j\epsilon),\Gamma_\epsilon(j\epsilon))\in\mathbf{H}^k$ satisfying the following properties:

(i) (Norm bound). There is a uniform constant $c_0>0$ depending only on Λ_*, M and the lower bound in the Taylor sign condition such that if j is an integer with $0\le j\le c_0\epsilon^{-1}$, then

$$\|(v_\epsilon(j\epsilon),\Gamma_\epsilon(j\epsilon))\|_{\mathbf{H}^k}\le C(M),$$

where $C(M)>0$ is some constant depending on M.

(ii) (Approximate solution).

$$\begin{cases} v_\epsilon((j+1)\epsilon) = v_\epsilon(j\epsilon) - \epsilon(v_\epsilon(j\epsilon)\cdot\nabla v_\epsilon(j\epsilon) + \nabla p_\epsilon(j\epsilon) + ge_d) + \mathcal{O}_{C^1}(\epsilon^2) \quad \text{on } \Omega_\epsilon((j+1)\epsilon)\cap\Omega_\epsilon(j\epsilon), \\ \nabla\cdot v_\epsilon((j+1)\epsilon) = 0 \quad \text{on } \Omega_\epsilon((j+1)\epsilon), \\ \Omega_\epsilon((j+1)\epsilon) = (I+\epsilon v_\epsilon(j\epsilon))(\Omega_\epsilon(j\epsilon)) + \mathcal{O}_{C^1}(\epsilon^2). \end{cases}$$

We will not have to concern ourselves too much with the Taylor sign condition in this section as we are working at high regularity and this is a pointwise property. In particular, we will suppress the lower bound in the Taylor sign condition from our notation. A nice feature about the above iteration scheme is that it suffices to only carry out a single step. For this, we have the following theorem.

Theorem 8.1. *Let k be a sufficiently large even integer and $M>0$. Consider an initial data $(v_0,\Gamma_0)\in\mathbf{H}^k$ so that $\|(v_0,\Gamma_0)\|_{\mathbf{H}^k}\le M$ and v_0 and ω_0 satisfy the initial regularization bounds*

$$\|v_0\|_{H^{k+1}(\Omega_0)}\le K(M)\epsilon^{-1}, \qquad \|\omega_0\|_{H^{k+n}(\Omega_0)}\le K'(M)\epsilon^{-1-n}, \tag{8.3}$$

for $n=0,1$, where $K(M)$, $K'(M)>0$ are constants, possibly much larger than M, such that $K'(M)\ll K(M)$. Then there exists a one step iterate $(v_0,\Gamma_0)\mapsto(v_1,\Gamma_1)$ with the following properties:

(i) (Energy monotonicity).

$$\mathcal{E}^k(v_1,\Gamma_1)\le(1+C(M)\epsilon)\mathcal{E}^k(v_0,\Gamma_0). \tag{8.4}$$

(ii) (Good pointwise approximation).

$$\begin{cases} v_1 = v_0 - \epsilon(v_0\cdot\nabla v_0 + \nabla p_0 + ge_d) + \mathcal{O}_{C^1}(\epsilon^2) \quad \text{on } \Omega_1\cap\Omega_0, \\ \nabla\cdot v_1 = 0 \quad \text{on } \Omega_1, \\ \Omega_1 = (I+\epsilon v_0)(\Omega_0) + \mathcal{O}_{C^1}(\epsilon^2). \end{cases} \tag{8.5}$$

(iii) (Persistence of the regularization bounds). v_1 *satisfies the regularization bounds*

$$\|v_1\|_{H^{k+1}(\Omega_1)} \leq K(M)\epsilon^{-1}, \qquad \|\omega_1\|_{H^{k+n}(\Omega_1)} \leq (K'(M) + C(M)\epsilon)\epsilon^{-1-n}, \tag{8.6}$$

for $n = 0, 1$.

Remark 8.2. Property (8.6) ensures that v_1 retains the H^{k+1} regularization bound with the same constant compared to the first iterate, and ω_1 has a regularization bound which can only grow by an amount comparable to ϵ times the initial regularization bound, which is acceptable over $\approx_M \epsilon^{-1}$ iterations. The energy monotonicity property, along with the energy coercivity bound from Theorem 7.1 will ensure that the resulting sequence $(v_\epsilon(j\epsilon), \Gamma_\epsilon(j\epsilon))$ of approximate solutions we construct remains uniformly bounded in $\mathbf{H}^k$ for $j \ll_M \epsilon^{-1}$. The second property in Theorem 8.1 will ensure that $(v_\epsilon(j\epsilon), \Gamma_\epsilon(j\epsilon))$ converges in a weaker topology to a solution of the equation.

The assumption (8.3) for v_0 is for technical convenience. In the regularization step of the argument, it will allow us to decouple the process of regularizing the domain and regularizing the velocity into separate arguments (see Lemma 8.4 in the next section). The condition (8.6) ensures that (8.3) can be propagated from one iterate to the next. Assuming that the initial iterate satisfies (8.3) is harmless in practice. Indeed, by the regularization properties of $\Psi_{\leq \epsilon^{-1}}$, we can replace the first iterate in the resulting sequence $(v_\epsilon(j\epsilon), \Gamma_\epsilon(j\epsilon))$ with a suitable ϵ^{-1} scale regularization so that the base case is satisfied. We note crucially that such a regularization is only done once - on the initial iterate - as we only know that this regularization is bounded on $\mathbf{H}^k$ (it does not necessarily satisfy the more delicate energy monotonicity). In contrast, we require the much stricter energy monotonicity bound (8.4) for all other iterations as in the above theorem. The condition on the vorticity in (8.3) can also be harmlessly assumed for the initial iterate. When we later regularize the velocity, we will not regularize the vorticity, but rather only the irrotational component. This is why, in contrast to the H^{k+1} bound for v_1, the constant for ω_1 in (8.6) gets slightly worse. Nonetheless, the careful tracking of its bound in (8.6) ensures that it only grows by an acceptable amount in each iteration. The heuristic reason why the regularization bound on ω_1 is expected is because the vorticity should be essentially transported by the flow, and therefore should not suffer the derivative loss of the full velocity in the iteration step.

Outline of the argument. We now give a brief overview of the section. The first step is selecting a suitable regularization scale. To motivate this, we recall that the evolution of the domain and the irrotational component of the velocity is essentially governed by the following approximate equation for a:

$$D_t^2 a \approx -a\mathcal{N}a. \tag{8.7}$$

Therefore, heuristically, D_t behaves roughly as a "spatial" derivative of order $\frac{1}{2}$. To control quadratic errors in the energy monotonicity bound in the Euler plus transport iteration later, it is therefore natural to attempt to regularize the domain and the irrotational part of the velocity on the ϵ^{-1} scale, as we do in Theorem 8.1. As the vorticity is essentially transported by the flow, we are able to leave the rotational part of the velocity alone, and instead track its growth as in (8.6).

With the above discussion in mind, we begin our analysis in earnest in Section 8.2 by regularizing the domain on the ϵ^{-1} scale. More specifically, given $(v_0, \Gamma_0) \in \mathbf{H}^k$ with v_0 satisfying (8.3), we construct for each $0 < \epsilon \ll 1$ a domain $\Omega_\epsilon \subseteq \Omega_0$ whose boundary is within $\mathcal{O}_{C^1}(\epsilon^2)$ of Γ_0 and which satisfies the regularization bound $\|\Gamma_\epsilon\|_{H^{k+\alpha}} \lesssim_{M,\alpha} \epsilon^{-\alpha}$ for all $\alpha \geq 0$. This is achieved by performing a parabolic regularization of the graph parameterization η_0 on Γ_*, together with a slight contraction of the domain. We then define our new velocity $\tilde{v}_0 = \tilde{v}_0(\epsilon)$ by restricting the old velocity v_0 to the new domain Ω_ϵ. As will be the case in every step of the argument, the main difficulty is to carefully track the effect of the regularization on the energy growth. The main point in this part of the argument is to show that the parabolic regularization of η_0 induces a corresponding parabolic gain in the surface component of the energy $\|a^{-\frac{1}{2}}\mathcal{N}^{k-1}a\|^2_{L^2(\Gamma)}$, allowing us to control all of the resulting errors.

With the domain now regularized, we move on to regularizing the velocity in Section 8.3, which is step 2 of the argument. In this step, we leave the domain and rotational part of the velocity alone, and regularize the irrotational part of the velocity on the ϵ^{-1} scale. The way we execute this is by using the functional calculus for the Dirichlet-to-Neumann operator. The main difficulty in this step of the argument is in tracking the effect of this regularization on the $\|\nabla\mathcal{H}\mathcal{N}^{k-2}(a^{-1}D_t a)\|^2_{L^2(\Omega)}$ portion of the energy, which at leading order controls the irrotational component of the velocity. An additional objective in this step of the argument is to improve the constant in (8.3) so that we can ultimately close the bootstrap in the upcoming Euler plus transport phase of the argument.

The final step in our construction is to use an Euler plus transport iteration to flow the regularized variables $(v_\epsilon, \Gamma_\epsilon)$ along a discrete version of the Euler evolution. It is in this step of the argument that we expect to observe a $\frac{1}{2}$ derivative loss (see the equation (8.7) for $D_t^2 a$, for instance), which is why the above regularization procedure is imperative. The Euler plus transport argument we employ is carried out in Section 8.4. Control of the resulting energy growth is shown by carefully relating the good variables a, $D_t a$ and ω for the new iterate to the corresponding good variables for the regularized data. Then, with the energy uniformly bounded and the variables appropriately iterated, in Section 8.5 we conclude that our scheme converges in a weaker topology, completing the construction of solutions.

8.2. **Step 1: Domain regularization.** We begin with the domain regularization step. For this, we have the following proposition.

Proposition 8.3. *Given* $(v_0, \Gamma_0) \in \mathbf{H}^k$ *with* v_0 *satisfying* (8.3)*, there exists a domain* Ω_ϵ *contained in* Ω_0 *with boundary* $\Gamma_\epsilon \in \Lambda_*$ *such that the pair* $(v_{0|\Omega_\epsilon}, \Gamma_\epsilon)$ *satisfies*

(i) (Energy monotonicity).

$$\mathcal{E}^k(v_{0|\Omega_\epsilon}, \Gamma_\epsilon) \leq (1 + C(M)\epsilon)\mathcal{E}^k(v_0, \Gamma_0). \tag{8.8}$$

(ii) (Good pointwise approximation).

$$\eta_\epsilon = \eta_0 + \mathcal{O}_{C^1}(\epsilon^2) \quad \text{on} \quad \Gamma_*. \tag{8.9}$$

(iii) (Domain regularization bound). For every $\alpha \geq 0$*, there holds,*

$$\|\Gamma_\epsilon\|_{H^{k+\alpha}} \lesssim_{M,\alpha} \epsilon^{-\alpha}. \tag{8.10}$$

Proof. In the sequel, we will use $\tilde{v}_0$ as a shorthand for $v_{0|\Omega_\epsilon}$. To regularize Γ_0, we begin with the preliminary parabolic regularization of η_0 given by

$$\tilde{\eta}_\epsilon = e^{\epsilon^2 \Delta_{\Gamma_*}} \eta_0,$$

where Δ_{Γ_*} is the Laplace-Beltrami operator for Γ_*. The rationale for using the operator $e^{\epsilon^2\Delta_{\Gamma_*}}$ instead of, for instance, the operator $e^{-\epsilon|D|}$ is to ensure that when k is large enough, we have $\|\partial_\epsilon \tilde{\eta}_\epsilon\|_{H^{k-2}(\Gamma_*)} \lesssim_M \epsilon$. This ensures that the hypersurface parameterized by $\tilde{\eta}_\epsilon$ in collar coordinates is at a distance on the order of no more than $\mathcal{O}_M(\epsilon^2)$ from Γ_0 in the H^{k-2} topology (and thus the C^1 topology if k is large enough). We would also like to additionally guarantee that Ω_ϵ is contained in Ω_0, so that we can use the restriction of the velocity v_0 to Ω_ϵ as the velocity on the new domain. Therefore, we slightly correct the above parabolic regularization by defining our regularized hypersurface Γ_ϵ through the collar parameterization

$$\eta_\epsilon = \tilde{\eta}_\epsilon - C\epsilon^2,$$

where C is some positive constant depending on M only, imposed to ensure that the domain Ω_ϵ associated to Γ_ϵ is contained in Ω_0. Clearly, η_ϵ satisfies (8.10) and the required pointwise approximation property in (8.9). The main bulk of the work in this step of the argument will therefore be in understanding how the above parabolic regularization of the surface (and also the restriction of the velocity to Ω_ϵ) affects the energy.

Given $(\tilde{v}_0, \Gamma_\epsilon)$ as above, we define the associated quantities $\tilde{\omega}_0 := \nabla \times \tilde{v}_0$ and $\tilde{p}_0$, $D_t\tilde{p}_0$, $\tilde{a}_0$ and $D_t\tilde{a}_0$ on Ω_ϵ and Γ_ϵ by using the relevant Poisson equations, as in Section 7.1. We will use the notation $\mathcal{N}_\epsilon$ to refer to the Dirichlet-to-Neumann operator for Γ_ϵ. Before proceeding to the proof of energy monotonicity, we note that the above construction gives rise to a flow velocity V_ϵ in the parameter ϵ for the family of hypersurfaces Γ_ϵ by composing $\partial_\epsilon\eta_\epsilon\nu$ with the inverse of the collar coordinate parameterization $x \mapsto x + \eta_\epsilon(x)\nu(x)$. We may harmlessly assume that V_ϵ is defined on Ω_ϵ by harmonically extending it to Ω_ϵ. We use $D_\epsilon := \partial_\epsilon + V_\epsilon \cdot \nabla$ to denote the associated material derivative, which will be tangent to the family of hypersurfaces Γ_ϵ.

We also importantly make note of the fact that for every $s \in \mathbb{R}$, we have

$$\|\tilde{\omega}_0\|_{H^s(\Omega_\epsilon)} \leq \|\omega_0\|_{H^s(\Omega)}, \qquad \|\tilde{v}_0\|_{H^s(\Omega_\epsilon)} \leq \|v_0\|_{H^s(\Omega)}. \tag{8.11}$$

Therefore, the bounds in (8.3) are retained from the initial data and, moreover, the rotational component of the energy does not increase.

Now we turn to the energy monotonicity bound (8.8). We will need the following two lemmas.

Lemma 8.4 (Material derivative bounds)**.** *The following bound holds uniformly in ϵ:*

$$\|D_\epsilon \nabla \tilde{v}_0\|_{H^{k-1}(\Omega_\epsilon)} \lesssim_M 1. \tag{8.12}$$

Lemma 8.5 (Variation of the surface energy)**.** *Let k be a sufficiently large even integer. Then we have the following estimate for the $\tilde{a}_0$ component of the energy:*

$$\frac{d}{d\epsilon}\|\tilde{a}_0^{-\frac{1}{2}}\mathcal{N}_\epsilon^{k-1}\tilde{a}_0\|^2_{L^2(\Gamma_\epsilon)} \lesssim_M -\epsilon\|\Gamma_\epsilon\|^2_{H^{k+1}} + \mathcal{O}_M(1).$$

Lemma 8.4 will allow us to essentially ignore any contributions to the energy coming from the restriction $\tilde{v}_0$, while Lemma 8.5 will help in controlling the variation in ϵ of the irrotational components of the energy.

Before proving the above lemmas, let us see how they imply the energy monotonicity bound (8.8). Thanks to Lemma 8.5 and (8.11), we only need to study the $D_t a$ component of the energy. For this, we recall from the Laplace equation (7.4) that we have

$$\tilde{a}_0^{-1}D_t\tilde{a}_0 = \tilde{a}_0^{-1}n_{\Gamma_\epsilon} \cdot \nabla\tilde{v}_0 \cdot \nabla\tilde{p}_0 - \tilde{a}_0^{-1}n_{\Gamma_\epsilon} \cdot \nabla\Delta^{-1}_{\Omega_\epsilon}(\Delta\tilde{v}_0 \cdot \nabla\tilde{p}_0 + 4\mathrm{tr}(\nabla^2\tilde{p}_0 \cdot \nabla\tilde{v}_0) + 2\mathrm{tr}(\nabla\tilde{v}_0)^3) \quad \text{on } \Gamma_\epsilon. \tag{8.13}$$

We apply $D_\epsilon \nabla \mathcal{H}_\epsilon \mathcal{N}_\epsilon^{k-2}$ to (8.13) and distribute derivatives. We first dispense with the commutator. Using the standard $H^{\frac{1}{2}}(\Gamma_\epsilon) \to H^1(\Omega_\epsilon)$ bound for $\mathcal{H}_\epsilon$, the $H^{k-\frac{3}{2}}(\Gamma_\epsilon)$ to $H^{\frac{1}{2}}(\Gamma_\epsilon)$ bound for $\mathcal{N}_\epsilon^{k-2}$ from Proposition 5.30 and the $H^{\frac{1}{2}}(\Gamma_\epsilon) \to H^1(\Omega_\epsilon)$ bound for $[D_\epsilon, \mathcal{H}_\epsilon]$ from (5.39), we have

$$\|[D_\epsilon, \nabla \mathcal{H}_\epsilon \mathcal{N}_\epsilon^{k-2}](\tilde{a}_0^{-1} D_t \tilde{a}_0)\|_{L^2(\Omega_\epsilon)} \lesssim_M \|[D_\epsilon, \mathcal{N}_\epsilon^{k-2}](\tilde{a}_0^{-1} D_t \tilde{a}_0)\|_{H^{\frac{1}{2}}(\Gamma_\epsilon)} + \|\tilde{a}_0^{-1} D_t \tilde{a}_0\|_{H^{k-\frac{3}{2}}(\Gamma_\epsilon)}.$$

Then, using the formula (5.43) and the elliptic estimates in Section 5 as well as the bound $\|V_\epsilon\|_{H^{k-1}(\Gamma_\epsilon)} \lesssim_M 1$, it is straightforward to verify the commutator bound

$$\|[D_\epsilon, \mathcal{N}_\epsilon^{k-2}]\|_{H^{k-\frac{3}{2}}(\Gamma_\epsilon) \to H^{\frac{1}{2}}(\Gamma_\epsilon)} \lesssim_M 1.$$

By elliptic regularity, $\|\tilde{a}_0^{-1} D_t \tilde{a}_0\|_{H^{k-\frac{3}{2}}(\Gamma_\epsilon)}$ is $\mathcal{O}_M(1)$. Hence, we obtain

$$\|[D_\epsilon, \nabla \mathcal{H}_\epsilon \mathcal{N}_\epsilon^{k-2}](\tilde{a}_0^{-1} D_t \tilde{a}_0)\|_{L^2(\Omega_\epsilon)} \lesssim_M 1.$$

Using that

$$\|\nabla \mathcal{H}_\epsilon \mathcal{N}_\epsilon^{k-2} D_\epsilon(\tilde{a}_0^{-1} D_t \tilde{a}_0)\|_{L^2(\Omega_\epsilon)} \lesssim_M \|D_\epsilon(\tilde{a}_0^{-1} D_t \tilde{a}_0)\|_{H^{k-\frac{3}{2}}(\Gamma_\epsilon)},$$

it remains now to estimate $\|D_\epsilon(\tilde{a}_0^{-1} D_t \tilde{a}_0)\|_{H^{k-\frac{3}{2}}(\Gamma_\epsilon)}$. For this, we distribute the operator D_ϵ onto the various terms in (8.13). To expedite this process, we collect a few useful bounds. First, using Lemma 8.4, the trace theorem ensures that we have the bound

$$\|D_\epsilon \nabla \tilde{v}_0\|_{H^{k-\frac{3}{2}}(\Gamma_\epsilon)} + \|D_\epsilon \nabla \tilde{v}_0\|_{H^{k-1}(\Omega_\epsilon)} \lesssim_M 1.$$

Using the identities for $[\Delta_{\Omega_\epsilon}^{-1}, D_\epsilon]$ and $D_\epsilon n_{\Gamma_\epsilon}$ in Section 5.6, the Laplace equation for p_ϵ, and the fact that V_ϵ is harmonic, we also readily verify the bounds

$$\|D_\epsilon \tilde{p}_0\|_{H^{k+\frac{1}{2}}(\Omega_\epsilon)} + \|D_\epsilon n_{\Gamma_\epsilon}\|_{H^{k-2}(\Gamma_\epsilon)} + \|D_\epsilon \tilde{a}_0\|_{H^{k-2}(\Gamma_\epsilon)} \lesssim_M 1 \tag{8.14}$$

and

$$\|[D_\epsilon, \nabla]\|_{H^k(\Omega_\epsilon) \to H^{k-1}(\Omega_\epsilon)} + \|D_\epsilon n_{\Gamma_\epsilon}\|_{H^{k-\frac{3}{2}}(\Gamma_\epsilon)} + \|D_\epsilon \tilde{a}_0\|_{H^{k-\frac{3}{2}}(\Gamma_\epsilon)} \lesssim_M 1 + \|V_\epsilon\|_{H^{k-\frac{1}{2}}(\Gamma_\epsilon)}.$$

From the above bounds and (8.13), we obtain the estimate

$$\|D_\epsilon(\tilde{a}_0^{-1} D_t \tilde{a}_0)\|_{H^{k-\frac{3}{2}}(\Gamma_\epsilon)} \lesssim_M 1 + \|V_\epsilon\|_{H^{k-\frac{1}{2}}(\Gamma_\epsilon)}.$$

The term $\|V_\epsilon\|_{H^{k-\frac{1}{2}}(\Gamma_\epsilon)}$ does not contribute an $\mathcal{O}_M(1)$ error, as it "loses" half a derivative. However, from the definition and regularization properties of V_ϵ, we have

$$\|V_\epsilon\|_{H^{k-\frac{1}{2}}(\Gamma_\epsilon)} \lesssim_M 1 + \epsilon^{\frac{1}{2}} \|\eta_\epsilon\|_{H^{k+1}(\Gamma_*)}.$$

Hence, using Proposition 2.3 and Cauchy-Schwarz, we obtain

$$\frac{d}{d\epsilon} \|\nabla \mathcal{H}_\epsilon \mathcal{N}_\epsilon^{k-2}(\tilde{a}_0^{-1} D_t \tilde{a}_0)\|_{L^2(\Omega_\epsilon)}^2 \lesssim_M 1 + \delta_0 \epsilon \|\Gamma_\epsilon\|_{H^{k+1}}^2,$$

where $\delta_0 > 0$ is some sufficiently small constant. Using the parabolic gain from Lemma 8.5, we notice that the latter term on the right-hand side is harmless as long as $\delta_0 = \delta_0(M)$ is small enough.

It remains now to establish the two lemmas. We begin with Lemma 8.4, which is quite simple.

Proof. Since $\partial_\epsilon \tilde{v}_0 = 0$, we have

$$D_\epsilon \nabla \tilde{v}_0 = V_\epsilon \cdot \nabla\nabla \tilde{v}_0.$$

Then we use $\|V_\epsilon\|_{H^{k-\frac{3}{2}}(\Omega_\epsilon)} \lesssim_M \epsilon$ and $\|V_\epsilon\|_{H^{k-\frac{1}{2}}(\Omega_\epsilon)} \lesssim_M 1$ together with the inductive bound for v_0 from (8.3); namely, $\|v_0\|_{H^{k+1}(\Omega_0)} \leq K(M)\epsilon^{-1}$, to estimate

$$\|D_\epsilon \nabla \tilde{v}_0\|_{H^{k-1}(\Omega_\epsilon)} \lesssim_M \|V_\epsilon\|_{H^{k-\frac{3}{2}}(\Omega_\epsilon)} \|\tilde{v}_0\|_{H^{k+1}(\Omega_\epsilon)} + \|V_\epsilon\|_{H^{k-1}(\Omega_\epsilon)} \|\tilde{v}_0\|_{H^k(\Omega_\epsilon)} \lesssim_M 1.$$

This completes the proof of Lemma 8.4. □

Finally, we come to establishing Lemma 8.5, which is where the bulk of the work will be. We begin by establishing the following representation formula for the good variable $\mathcal{N}_\epsilon^{k-1}\tilde{a}_0$:

$$\mathcal{N}_\epsilon^{k-1}\tilde{a}_0 = (-1)^m \tilde{a}_0 \Delta_{\Gamma_\epsilon}^m \kappa_\epsilon + R_\epsilon, \tag{8.15}$$

where κ_ϵ is the mean curvature for Γ_ϵ, $2m = k-2$ and R_ϵ is a remainder term satisfying the bounds

$$\|R_\epsilon\|_{H^{\frac{1}{2}}(\Gamma_\epsilon)} + \epsilon^{\frac{1}{2}} \|R_\epsilon\|_{H^1(\Gamma_\epsilon)} + \|D_\epsilon R_\epsilon\|_{L^2(\Gamma_\epsilon)} \lesssim_M 1. \tag{8.16}$$

The importance of (8.15) will be clear later. Roughly speaking, (8.15) states that to leading order $\mathcal{N}_\epsilon^{k-1}\tilde{a}_0$ has a convenient local expression. Such an observation will facilitate the use of local formulas later on, consistent with our choice of domain regularization. Observe also that in (8.16), we have $D_\epsilon R_\epsilon = \mathcal{O}_{L^2(\Gamma_\epsilon)}(1)$. This is stronger than the expected bound $D_\epsilon R_\epsilon = \mathcal{O}_{H^{-\frac{1}{2}}(\Gamma_\epsilon)}(1)$. The reason for this improvement is the bound (8.12) for $D_\epsilon \nabla \tilde{v}_0$; this term would have had to have been treated more carefully if we had attempted to regularize the velocity in this step of the argument.

Proof of (8.15). In the following analysis, R_ϵ will generically denote a remainder term satisfying (8.16) which is allowed to change from line to line. Likewise, $\tilde{R}_\epsilon$ will denote an analogous remainder term but with

$$\tilde{R}_\epsilon = \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\epsilon)}(1), \qquad \epsilon^{\frac{1}{2}} \tilde{R}_\epsilon = \mathcal{O}_{H^{k-1}(\Gamma_\epsilon)}(1), \qquad D_\epsilon \tilde{R}_\epsilon = \mathcal{O}_{H^{k-2}(\Gamma_\epsilon)}(1). \tag{8.17}$$

To establish (8.15), we begin by relating $\mathcal{N}_\epsilon \tilde{a}_0$ to the mean curvature. Indeed, from $\Delta_{\Gamma_\epsilon} \tilde{p}_0 = 0$ and the formula

$$\Delta \tilde{p}_0|_{\Gamma_\epsilon} = \Delta_{\Gamma_\epsilon} \tilde{p}_0 - \kappa_\epsilon n_{\Gamma_\epsilon} \cdot \nabla \tilde{p}_0 + D^2 \tilde{p}_0(n_{\Gamma_\epsilon}, n_{\Gamma_\epsilon}),$$

we have

$$\begin{aligned} \tilde{a}_0 \kappa_\epsilon &= -n_i n_j \partial_i \partial_j \tilde{p}_0 + \Delta \tilde{p}_0 \\ &= -n_i n_j \partial_i \partial_j \tilde{p}_0 - \mathrm{tr}(\nabla \tilde{v}_0)^2 \\ &= -n_i n_j \partial_i \partial_j \tilde{p}_0 + \tilde{R}_\epsilon, \end{aligned}$$

where in the last line, we used Lemma 8.4 to check the remainder property for $D_\epsilon \tilde{R}_\epsilon$ and the inductive assumption (8.3) and interpolation to control $\epsilon^{\frac{1}{2}} \tilde{R}_\epsilon$ in $H^{k-1}(\Gamma_\epsilon)$. We now further expand using the Laplace equation for $\tilde{p}_0$,

$$\begin{aligned} -n_i n_j \partial_i \partial_j \tilde{p}_0 &= n_j \mathcal{N}_\epsilon(n_j \tilde{a}_0) + n_j n_{\Gamma_\epsilon} \cdot \nabla \Delta_{\Omega_\epsilon}^{-1} \partial_j \mathrm{tr}(\nabla \tilde{v}_0)^2 \\ &= n_j \mathcal{N}_\epsilon(n_j \tilde{a}_0) + \tilde{R}_\epsilon. \end{aligned}$$

Next, we expand

$$\begin{aligned} n_j \mathcal{N}_\epsilon(n_j \tilde{a}_0) &= \mathcal{N}_\epsilon \tilde{a}_0 + \tilde{a}_0 n_j \mathcal{N}_\epsilon n_j - 2 n_j n_{\Gamma_\epsilon} \cdot \nabla \Delta_{\Omega_\epsilon}^{-1} (\nabla \mathcal{H}_\epsilon n_j \cdot \nabla \mathcal{H}_\epsilon \tilde{a}_0) \\ &= \mathcal{N}_\epsilon \tilde{a}_0 + \tilde{a}_0 n_j \mathcal{N}_\epsilon n_j + \tilde{R}_\epsilon \\ &= \mathcal{N}_\epsilon \tilde{a}_0 + \tilde{R}_\epsilon, \end{aligned}$$

where in the first equality, we used the Leibniz rule (5.36) for $\mathcal{N}_\epsilon$. From the second to the third line, we used the Leibniz rule again, and that $\mathcal{N}_\epsilon(n_j n_j) = 0$. In summary, what we have so far is the identity

$$\mathcal{N}_\epsilon \tilde{a}_0 = \tilde{a}_0 \kappa_\epsilon + \tilde{R}_\epsilon. \tag{8.18}$$

The next step is to obtain the leading order identity,

$$\mathcal{N}_\epsilon^{k-1}\tilde{a}_0 = \tilde{a}_0 \mathcal{N}_\epsilon^{k-2}(\tilde{a}_0^{-1}\mathcal{N}_\epsilon \tilde{a}_0) + R_\epsilon \tag{8.19}$$

by applying $\mathcal{N}_\epsilon^{k-2}$ to $\mathcal{N}_\epsilon \tilde{a}_0$ and then commuting $\tilde{a}_0^{-1}$ with $\mathcal{N}_\epsilon^{k-2}$. Here, R_ϵ can be seen to satisfy the required bounds through the use of the various commutator identities for D_ϵ listed in Section 5.6 as well as the Leibniz rule (5.36), the elliptic estimates in Section 5 for $\mathcal{N}_\epsilon$ and the estimates in (8.14).

Before proceeding further, we recall the formula

$$-(\Delta_{\Gamma_\epsilon} + \mathcal{N}_\epsilon^2)f = \kappa_\epsilon \mathcal{N}_\epsilon f - 2n_{\Gamma_\epsilon}\cdot\nabla(-\Delta_{\Omega_\epsilon})^{-1}(\nabla\mathcal{H}_\epsilon n_{\Gamma_\epsilon}\cdot\nabla^2\mathcal{H}_\epsilon f) - \mathcal{N}_\epsilon n_{\Gamma_\epsilon}\cdot(\mathcal{N}_\epsilon f n_{\Gamma_\epsilon} + \nabla^\top f) \tag{8.20}$$

from [41, Equation A.13]. Also, we recall from (4.23) of [41] the commutator estimate

$$\|[\Delta_{\Gamma_\epsilon}, D_\epsilon]\|_{H^s(\Gamma_\epsilon)\to H^{s-2}(\Gamma_\epsilon)} \lesssim_M \|V_\epsilon\|_{H^{k-\frac{1}{2}}(\Omega_\epsilon)} \lesssim_M 1, \qquad 1 \le s \le k-1. \tag{8.21}$$

Then, given that $k-2 = 2m$ is even, applying (8.18), (8.19) and iterating (8.20) m times, we have

$$\mathcal{N}_\epsilon^{k-1}\tilde{a}_0 = \tilde{a}_0\mathcal{N}_\epsilon^{k-2}(\tilde{a}_0^{-1}\mathcal{N}_\epsilon\tilde{a}_0) + R_\epsilon = (-1)^m\tilde{a}_0\Delta_{\Gamma_\epsilon}^m(\tilde{a}_0^{-1}\mathcal{N}_\epsilon\tilde{a}_0) + R_\epsilon = (-1)^m\tilde{a}_0\Delta_{\Gamma_\epsilon}^m\kappa_\epsilon + R_\epsilon,$$

where by straightforward (but slightly tedious) computation we verify that the remainder term R_ϵ has the needed bounds through the use of the various commutator identities for D_ϵ listed in Section 5.6 as well as the above estimates (8.18)-(8.21), the relevant elliptic estimates in Section 5 and (8.14). □

Now, we are ready to establish the differential inequality in Lemma 8.5. For the sake of clarity, let us begin by assuming that the reference hypersurface is given by $\{x_d = 0\}$ and that Γ_ϵ is literally given by $x_d = \eta_\epsilon(x_1, ..., x_{d-1})$. Then the mean curvature and Laplace-Beltrami operator take the form

$$\kappa_\epsilon = -\frac{\Delta\eta_\epsilon}{(1+|\nabla\eta_\epsilon|^2)^{\frac{1}{2}}} + \frac{\partial_i\eta_\epsilon\partial_j\eta_\epsilon\partial_i\partial_j\eta_\epsilon}{(1+|\nabla\eta_\epsilon|^2)^{\frac{3}{2}}},$$

and

$$\Delta_{\Gamma_\epsilon} f = \frac{1}{\sqrt{1+|\nabla\eta_\epsilon|^2}}\partial_i(g_\epsilon^{ij}\sqrt{1+|\nabla\eta_\epsilon|^2}\partial_j f), \tag{8.22}$$

where $(g_\epsilon^{ij}) = (\delta_{ij} + \partial_i\eta_\epsilon\partial_j\eta_\epsilon)^{-1}$. Observe that g_ϵ^{ij} and $\nabla\eta_\epsilon$ are one derivative more regular than κ_ϵ. Therefore, by making use of the identity $\partial_\epsilon\eta_\epsilon = 2\epsilon\Delta_{\Gamma_*}\eta_\epsilon$ and the regularization bound (8.10), we can differentiate in ϵ and commute $2\epsilon\Delta_{\Gamma_*}$ with these coefficients to obtain,

$$(D_\epsilon(\mathcal{N}_\epsilon^{k-1}\tilde{a}_0))_* = 2(-1)^m\epsilon\Delta_{\Gamma_*}(\tilde{a}_0\Delta_{\Gamma_\epsilon}^m\kappa_\epsilon)_* + \mathcal{O}_{L^2(\Gamma_*)}(1), \tag{8.23}$$

where we define $f_*(x) := f(x + \eta_\epsilon(x)\nu(x))$ for a function f defined on Γ_ϵ. Moreover, by an exercise in local coordinates, the reader may check that (8.23), as written, is valid for general reference hypersurfaces Γ_*. Now, using (5.41), the bounds for R_ϵ, and Cauchy-Schwarz, it follows that

$$\frac{d}{d\epsilon}\|\tilde{a}_0^{-\frac{1}{2}}\mathcal{N}_\epsilon^{k-1}\tilde{a}_0\|_{L^2(\Gamma_\epsilon)}^2 \lesssim_M 1 - \epsilon\||D|_{\Gamma_*}(\Delta_{\Gamma_\epsilon}^m\kappa_\epsilon)_*\|_{L^2(\Gamma_*)}^2,$$

where $|D|_{\Gamma_*} = (-\Delta_{\Gamma_*})^{\frac{1}{2}}$. To conclude, we now only need to show the coercivity type bound

$$\|\eta_\epsilon\|_{H^{k+1}(\Gamma_*)} \lesssim_M 1 + \||D|_{\Gamma_*}(\Delta_{\Gamma_\epsilon}^m\kappa_\epsilon)_*\|_{L^2(\Gamma_*)}.$$

For this, we begin with Proposition 5.22 which yields

$$\|\eta_\epsilon\|_{H^{k+1}(\Gamma_*)} \lesssim_M 1 + \|\kappa_\epsilon\|_{H^{k-1}(\Gamma_\epsilon)}.$$

Then, using (8.22) and the fact that $2m = k-2$ (this being relevant for ensuring domain dependent implicit constants are at most $\mathcal{O}_M(1)$ in size), one can easily verify the ellipticity bound

$$\|\kappa_\epsilon\|_{H^{k-1}(\Gamma_\epsilon)} \lesssim_M 1 + \|\Delta^m_{\Gamma_\epsilon}\kappa_\epsilon\|_{H^1(\Gamma_\epsilon)} \lesssim_M 1 + \||D|_{\Gamma_*}(\Delta^m_{\Gamma_\epsilon}\kappa)_*\|_{L^2(\Gamma_*)}.$$

This concludes the proof. □

8.3. **Step 2: Velocity regularization.** Now, we aim to regularize the velocity $\tilde{v}_0$ on the ϵ^{-1} scale, which will help us to improve the regularization constant in (8.3). This will be needed to compensate for the losses in this constant in the upcoming transport step of the argument. Thanks to the previous step, we are reduced to the situation of regularizing on a fixed domain which has boundary regularized at the ϵ^{-1} scale. To perform this step of the regularization, we decompose the velocity $\tilde{v}_0$ into a rotational component which is tangent to the boundary and an irrotational component. Roughly speaking, we will then regularize the irrotational component of $\tilde{v}_0$ and leave the rotational component alone. We will then reconstruct the regularized velocity using the regularized irrotational part and the original (not regularized) rotational part of $\tilde{v}_0$. The precise procedure for doing this will come with some slight technical subtleties due to the fact that the normal to the surface is half a derivative less regular than the trace of the velocity on the boundary. We will outline these nuances in more detail shortly. Heuristically, the reason it is unnecessary to regularize the rotational part of $\tilde{v}_0$ in this construction is because the vorticity will not lose derivatives in the transport step of our argument later. In other words, the vorticity bound in (8.3) is expected to only worsen by an $\mathcal{O}_M(1)$ error when measured in H^k and an $\mathcal{O}_M(\epsilon^{-1})$ error when measured in H^{k+1}, which is acceptable.

Proposition 8.6. *Given the pair $(\tilde{v}_0, \Gamma_\epsilon)$ from the previous step, there exists a regularization $\tilde{v}_0 \mapsto v_\epsilon$ defined on Ω_ϵ which satisfies:*

(i) (Energy monotonicity).

$$\mathcal{E}^k(v_\epsilon, \Gamma_\epsilon) \le (1 + C(M)\epsilon)\mathcal{E}^k(\tilde{v}_0, \Gamma_\epsilon).$$

(ii) (Good pointwise approximation).

$$\begin{cases} v_\epsilon = \tilde{v}_0 + \mathcal{O}_{C^1}(\epsilon^2), \\ \nabla \cdot v_\epsilon = 0. \end{cases} \tag{8.24}$$

(iii) (Regularization bounds). For each $n = 1, 2$ and $K(M)$ large enough, there holds

$$\|v_\epsilon\|_{H^{k+n}(\Omega_\epsilon)} \le \frac{1}{4}K(M)\epsilon^{-n}. \tag{8.25}$$

Remark 8.7. The bound in (8.25) with $n = 1$ ensures that the constant in (8.3) is improved at this stage. The H^{k+2} bound will be needed to close the bootstrap in the final Euler plus transport step of the iteration in the next section because this step loses derivatives for the velocity.

Proof. We begin by recalling the rotational/irrotational decomposition of $\tilde{v}_0$ from Appendix A of [41]:

$$\tilde{v}_0 := \tilde{v}_0^{rot} + \tilde{v}_0^{ir},$$

where for a divergence free function v, we have $v^{ir} := \nabla \mathcal{H}_\epsilon \mathcal{N}_\epsilon^{-1}(v \cdot n_{\Gamma_\epsilon})$. Naïvely, we would like to directly regularize the irrotational part of $\tilde{v}_0$. However, this does not quite work because the normal n_{Γ_ϵ} is half a derivative less regular than the trace of $\tilde{v}_0$ on Γ_ϵ. To get around this, we will regularize the irrotational part of a suitable high frequency component of $\tilde{v}_0$. More precisely, let us consider a subregularization v_- of $\tilde{v}_0$, defined by $v_- := \Psi_{\leq \epsilon^{-\frac{1}{2}}}\tilde{v}_0$, which lives on an $\epsilon^{\frac{1}{2}}$ enlargement of Ω_ϵ. We then define $w := \tilde{v}_0 - v_-$. Loosely speaking, we think of w as the portion of $\tilde{v}_0$ with frequency greater than $\epsilon^{-\frac{1}{2}}$. In contrast to the full irrotational part of $\tilde{v}_0$, it is safe to regularize the irrotational part of w. The heuristic reason for this is that at leading order the term $w \cdot n_{\Gamma_\epsilon}$ can be interpreted as a high-low paraproduct. That is, the contribution of the portion where n_{Γ_ϵ} is at comparable or higher frequency compared to w is lower order as there is still a nontrivial high frequency component of w to compensate for the $\frac{1}{2}$ derivative discrepancy between the trace of w and n_{Γ_ϵ}.

For the irrotational part of w, the regularization we choose has to respect the energy monotonicity bound. We will see below that the spectral multiplier $\mathcal{P}_{\leq \epsilon^{-1}}(\mathcal{N}_\epsilon) := 1_{[-\epsilon^{-1},\epsilon^{-1}]}(\mathcal{N}_\epsilon)$ is very convenient for this purpose. We therefore define the irrotational component of our regularization v_ϵ of $\tilde{v}_0$ by removing the high frequency part of $w \cdot n_{\Gamma_\epsilon}$ as follows:

$$\begin{aligned} v_\epsilon^{ir} &:= \tilde{v}_0^{ir} - \nabla \mathcal{H}_\epsilon \mathcal{N}_\epsilon^{-1} \mathcal{P}_{>\epsilon^{-1}}(w \cdot n_{\Gamma_\epsilon}) \\ &= v_-^{ir} + \nabla \mathcal{H}_\epsilon \mathcal{N}_\epsilon^{-1} \mathcal{P}_{\leq \epsilon^{-1}}(w \cdot n_{\Gamma_\epsilon}). \end{aligned}$$

For simplicity, let us write

$$w_\epsilon^{ir} := \nabla \mathcal{H}_\epsilon \mathcal{N}_\epsilon^{-1} \mathcal{P}_{\leq \epsilon^{-1}}(w \cdot n_{\Gamma_\epsilon}).$$

We define the full regularization v_ϵ of $\tilde{v}_0$ by

$$v_\epsilon := \tilde{v}_0^{rot} + v_\epsilon^{ir}.$$

If k is large enough, the combination of Sobolev embedding, ellipticity of $\mathcal{N}$ and spectral calculus allows us to easily establish the pointwise approximation property (8.24). Next, we establish the regularization bound (8.25) for v_ϵ. We begin by writing

$$v_\epsilon = v_- + w_\epsilon^{ir} + w^{rot},$$

where w^{rot} is the rotational part of w. We then estimate piece by piece. It is first of all clear that the corresponding bound holds for v_-. So, we turn to estimating w_ϵ^{ir}. For this, we note the following preliminary bound for $\mathcal{N}_\epsilon^{-1}$ on the space $\dot{H}^s(\Gamma_\epsilon) := \{f \in H^s(\Gamma_\epsilon) : \int_{\Gamma_\epsilon} f = 0\}$ from Proposition A.5 in [41]:

$$\|\mathcal{N}_\epsilon^{-1} f\|_{\dot{H}^s(\Gamma_\epsilon)} \lesssim_M \|f\|_{H^{s-1}(\Gamma_\epsilon)}, \qquad 0 \leq s \leq 1. \tag{8.26}$$

From this and the functional calculus for $\mathcal{N}_\epsilon$, we deduce in particular the low regularity bound

$$\|\mathcal{P}_{\leq \epsilon^{-1}} \mathcal{N}_\epsilon^{-1}(w \cdot n_{\Gamma_\epsilon})\|_{L^2(\Gamma_\epsilon)} \lesssim_M \|w \cdot n_{\Gamma_\epsilon}\|_{H^{-1}(\Gamma_\epsilon)}. \tag{8.27}$$

This will be useful for handling the low frequency errors in the estimate for w_ϵ^{ir}. Next we check that (8.26) and (8.27), in conjunction with Proposition 5.9, Proposition 5.21, Proposition 5.26 and the regularization bounds for n_{Γ_ϵ} and w, yield

$$\|w_\epsilon^{ir}\|_{H^{k+n}(\Omega_\epsilon)} \lesssim_{M,n} \|\Gamma_\epsilon\|_{H^{k+\frac{1}{2}+n}} \|w \cdot n_{\Gamma_\epsilon}\|_{H^{k-2}(\Gamma_\epsilon)} + \|\mathcal{P}_{\leq \epsilon^{-1}}(w \cdot n_{\Gamma_\epsilon})\|_{H^{k-\frac{1}{2}+n}(\Gamma_\epsilon)} \lesssim_M \epsilon^{-n},$$

where the implicit constant can be taken to be much smaller than $K(M)$ since $K(M) \gg M$. Note that in the above estimate, we used the paraproduct structure of $w \cdot n_{\Gamma_\epsilon}$. More specifically, in the case when $k - \frac{1}{2}$ derivatives fall on n_{Γ_ϵ}, we compensated the half derivative loss by an $\epsilon^{\frac{1}{2}}$ gain from w.

Finally, we move on to showing the regularization bound for w^{rot}. Here, we use Proposition 5.27 to obtain

$$\begin{aligned}\|w^{rot}\|_{H^{k+n}(\Omega_\epsilon)} &\lesssim_M \|w^{rot}\|_{L^2(\Omega_\epsilon)} + \|\nabla\times w\|_{H^{k+n-1}(\Omega_\epsilon)} + \|\Gamma_\epsilon\|_{H^{k+n-\frac{1}{2}}} + \|\nabla^\top w^{rot}\cdot n_{\Gamma_\epsilon}\|_{H^{k+n-\frac{3}{2}}(\Gamma_\epsilon)} \\ &\lesssim_{M,K'(M)} \epsilon^{-n} + \|w^{rot}\|_{L^2(\Omega_\epsilon)} + \|\nabla^\top w^{rot}\cdot n_{\Gamma_\epsilon}\|_{H^{k+n-\frac{3}{2}}(\Gamma_\epsilon)},\end{aligned}$$

where we used (8.3) for $\tilde{\omega}_0$. Again, the implicit constant can be taken to be much smaller than $K(M)$ if $K'(M)$ in (8.3) is small enough compared to $K(M)$. To estimate $\|w^{rot}\|_{L^2(\Omega_\epsilon)}$, we simply use (8.26), the identity $w^{rot} = w - w^{ir}$ and the $H^{\frac{1}{2}}(\Gamma_\epsilon) \to H^1(\Omega_\epsilon)$ bound for $\mathcal{H}_\epsilon$ to crudely estimate

$$\|w^{rot}\|_{L^2(\Omega_\epsilon)} \lesssim_M \|w\|_{H^1(\Omega_\epsilon)}. \tag{8.28}$$

Then, using

$$\nabla^\top w^{rot}\cdot n_{\Gamma_\epsilon} = -w^{rot}\cdot\nabla^\top n_{\Gamma_\epsilon},$$

Proposition 5.9, Proposition 5.11 and the regularization bounds for Γ_ϵ, we have (if k is large enough)

$$\|w^{rot}\|_{H^{k+n}(\Omega_\epsilon)} \lesssim_M \epsilon^{-n} + \|w^{rot}\|_{H^{k-1+n}(\Omega_\epsilon)} + \epsilon^{-\frac{1}{2}-n}\|w^{rot}\|_{H^{k-2}(\Omega_\epsilon)},$$

which implies by interpolation and (8.28) that

$$\|w^{rot}\|_{H^{k+n}(\Omega_\epsilon)} \lesssim_{M,n} \epsilon^{-n} + \epsilon^{-\frac{1}{2}-n}\|w^{rot}\|_{H^{k-2}(\Omega_\epsilon)}.$$

From Proposition 5.27, the inequality (8.28) and the fact that w is localized to frequency $\geq \epsilon^{-\frac{1}{2}}$, we easily obtain

$$\|w^{rot}\|_{H^{k-2}(\Omega_\epsilon)} \lesssim_M \|w\|_{H^{k-2}(\Omega_\epsilon)} \lesssim_M \epsilon.$$

Therefore, we have

$$\|w^{rot}\|_{H^{k+n}(\Omega_\epsilon)} \lesssim_{M,n} \epsilon^{-n},$$

with implicit constant much smaller than $K(M)$. This yields the desired regularization bounds for v_ϵ.

Next, we turn to the energy monotonicity. The domain is fixed in this step, so it is advantageous to compare the difference between $\mathcal{E}^k(v_\epsilon, \Gamma_\epsilon)$ and $\mathcal{E}^k(\tilde{v}_0, \Gamma_\epsilon)$ directly. It will also be convenient to write the first term in $\mathcal{E}^k(v,\Gamma)$ as a surface integral:

$$\|\nabla\mathcal{H}\mathcal{N}^{k-2}(a^{-1}D_t a)\|^2_{L^2(\Omega)} = \|\mathcal{N}^{k-\frac{3}{2}}(a^{-1}D_t a)\|^2_{L^2(\Gamma)},$$

using integration by parts and the functional calculus for $\mathcal{N}$. Moreover, since the vorticity ω_ϵ is the same as $\tilde{\omega}_0$, we may restrict our attention to the two surface components of the energy in this step of the argument.

We begin with a simple algebraic identity for the a_ϵ component of the surface energy:

$$\begin{aligned}\int_{\Gamma_\epsilon} a_\epsilon^{-1}|\mathcal{N}_\epsilon^{k-1}a_\epsilon|^2\, dS = \int_{\Gamma_\epsilon} \tilde{a}_0^{-1}|\mathcal{N}_\epsilon^{k-1}\tilde{a}_0|^2\, dS + 2\int_{\Gamma_\epsilon} a_\epsilon^{-1}\mathcal{N}_\epsilon^{k-1}a_\epsilon\mathcal{N}_\epsilon^{k-1}(a_\epsilon - \tilde{a}_0)\, dS \\ - \|a_\epsilon^{-\frac{1}{2}}\mathcal{N}_\epsilon^{k-1}(a_\epsilon - \tilde{a}_0)\|^2_{L^2(\Gamma_\epsilon)} + \mathcal{O}_M(\epsilon).\end{aligned}$$

To derive an analogous relation for the other portion of the surface energy, we note that from the integer bounds for $\mathcal{N}$ in Section 5 and the identity $\|\mathcal{N}^{k-\frac{3}{2}}f\|_{L^2(\Gamma)} = \|\nabla\mathcal{H}\mathcal{N}^{k-2}f\|_{L^2(\Omega)}$, we have the estimate $\|\mathcal{N}_\epsilon^{k-\frac{3}{2}}\|_{H^{k-\frac{3}{2}}(\Gamma_\epsilon)\to L^2(\Gamma_\epsilon)} \lesssim_M 1$. On the other hand, we have the elliptic regularity estimate

$$\|\tilde{a}_0 - a_\epsilon\|_{H^{k-\frac{3}{2}}(\Gamma_\epsilon)} \lesssim_M \|\tilde{p}_0 - p_\epsilon\|_{H^k(\Omega_\epsilon)} \lesssim_M \|\tilde{v}_0 - v_\epsilon\|_{H^{k-1}(\Omega_\epsilon)} \lesssim_M \epsilon.$$

Together, these imply that

$$\int_{\Gamma_\epsilon} |\mathcal{N}_\epsilon^{k-\frac{3}{2}}(a_\epsilon^{-1}D_t a_\epsilon)|^2\, dS = \int_{\Gamma_\epsilon} |\mathcal{N}_\epsilon^{k-\frac{3}{2}}(\tilde{a}_0^{-1}D_t\tilde{a}_0)|^2\, dS + 2\int_{\Gamma_\epsilon} \mathcal{N}_\epsilon^{k-\frac{3}{2}}(a_\epsilon^{-1}D_t a_\epsilon)\mathcal{N}_\epsilon^{k-\frac{3}{2}}(a_\epsilon^{-1}(D_t a_\epsilon - D_t\tilde{a}_0))\, dS$$
$$- \|\mathcal{N}_\epsilon^{k-\frac{3}{2}}(a_\epsilon^{-1}(D_t a_\epsilon - D_t\tilde{a}_0))\|_{L^2(\Gamma_\epsilon)}^2 + \mathcal{O}_M(\epsilon).$$

Motivated by the identities above, let us define the "energy" corresponding to $\tilde{v}_0 - v_\epsilon$ by

$$\mathcal{E}^k(\tilde{v}_0 - v_\epsilon) := \|\mathcal{N}_\epsilon^{k-\frac{3}{2}}(a_\epsilon^{-1}(D_t\tilde{a}_0 - D_t a_\epsilon))\|_{L^2(\Gamma_\epsilon)}^2 + \|a_\epsilon^{-\frac{1}{2}}\mathcal{N}_\epsilon^{k-1}(\tilde{a}_0 - a_\epsilon)\|_{L^2(\Gamma_\epsilon)}^2.$$

In light of the above identities, it suffices to show that

$$2\int_{\Gamma_\epsilon} \mathcal{N}_\epsilon^{k-\frac{3}{2}}(a_\epsilon^{-1}D_t a_\epsilon)\mathcal{N}_\epsilon^{k-\frac{3}{2}}(a_\epsilon^{-1}(D_t a_\epsilon - D_t\tilde{a}_0))\, dS + 2\int_{\Gamma_\epsilon} a_\epsilon^{-1}\mathcal{N}_\epsilon^{k-1} a_\epsilon \mathcal{N}_\epsilon^{k-1}(a_\epsilon - \tilde{a}_0)\, dS$$
$$\leq C(M)\epsilon + \mathcal{E}^k(\tilde{v}_0 - v_\epsilon).$$

Our starting point is to observe the leading order relation given in the following lemma.

Lemma 8.8. *We have the following relation between $D_t a_\epsilon - D_t\tilde{a}_0$ and $(v_\epsilon - \tilde{v}_0)\cdot n_{\Gamma_\epsilon}$:*

$$a_\epsilon^{-1}(D_t a_\epsilon - D_t\tilde{a}_0) = -\mathcal{N}_\epsilon((v_\epsilon - \tilde{v}_0)\cdot n_{\Gamma_\epsilon}) + \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\epsilon)}(\epsilon). \tag{8.29}$$

Proof. We begin by noting the bound

$$\|\tilde{v}_0 - v_\epsilon\|_{H^{k-1}(\Omega_\epsilon)} \lesssim_M \epsilon \tag{8.30}$$

and the elliptic regularity estimate

$$\|\tilde{p}_0 - p_\epsilon\|_{H^k(\Omega_\epsilon)} \lesssim_M \|\tilde{v}_0 - v_\epsilon\|_{H^{k-1}(\Omega_\epsilon)} \lesssim_M \epsilon.$$

Using the equation for $D_t p$ from (7.4) we may therefore write

$$D_t a_\epsilon - D_t\tilde{a}_0 = n_{\Gamma_\epsilon}\cdot\nabla(v_\epsilon - \tilde{v}_0)\cdot\nabla p_\epsilon - n_{\Gamma_\epsilon}\cdot\nabla\Delta^{-1}(\Delta(v_\epsilon - \tilde{v}_0)\cdot\nabla p_\epsilon) + \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\epsilon)}(\epsilon).$$

Then, using the standard identity $\mathcal{N}f_{|\Gamma} = n\cdot\nabla f - n\cdot\nabla\Delta^{-1}\Delta f$ and commuting $n_{\Gamma_\epsilon}\cdot\nabla$ in the first term and Δ in the second term above, we can verify, from (8.30),

$$D_t a_\epsilon - D_t\tilde{a}_0 = -\mathcal{N}_\epsilon(a_\epsilon(v_\epsilon - \tilde{v}_0)\cdot n_{\Gamma_\epsilon}) + \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\epsilon)}(\epsilon).$$

The conclusion then follows by commuting $\mathcal{N}_\epsilon$ with a_ϵ using the Leibniz rule for $\mathcal{N}_\epsilon$ and (8.30). In the case when everything falls on a_ϵ, we also compensate with the surface regularization bound (8.10) and the associated improvement in the bound for $\tilde{v}_0 - v_\epsilon$ when measured in lower regularity Sobolev norms. □

We now turn to the a_ϵ component of the energy, which is straightforward. Indeed, by elliptic regularity,

$$2\int_{\Gamma_\epsilon} a_\epsilon^{-1}\mathcal{N}_\epsilon^{k-1}a_\epsilon\mathcal{N}_\epsilon^{k-1}(a_\epsilon - \tilde{a}_0)\, dS \lesssim_M \|a_\epsilon - \tilde{a}_0\|_{H^{k-1}(\Gamma_\epsilon)} \lesssim_M \|v_\epsilon - \tilde{v}_0\|_{H^{k-\frac{1}{2}}(\Omega_\epsilon)}.$$

To estimate $v_\epsilon - \tilde{v}_0$, we observe the identity $v_\epsilon - \tilde{v}_0 = \nabla\mathcal{H}_\epsilon\mathcal{N}_\epsilon^{-1}\mathcal{P}_{>\epsilon^{-1}}((v_\epsilon - \tilde{v}_0)\cdot n_{\Gamma_\epsilon})$, which follows from the idempotence $\mathcal{P}_{>\epsilon^{-1}} = \mathcal{P}_{>\epsilon^{-1}}^2$. Using this, Lemma 8.8 and ellipticity of $\mathcal{N}_\epsilon$, we have

$$\|v_\epsilon - \tilde{v}_0\|_{H^{k-\frac{1}{2}}(\Omega_\epsilon)} \lesssim_M \epsilon^{\frac{1}{2}}\|(v_\epsilon - \tilde{v}_0)\cdot n_{\Gamma_\epsilon}\|_{H^{k-\frac{1}{2}}(\Gamma_\epsilon)} \lesssim_M \epsilon^{\frac{1}{2}}(\mathcal{E}^k(\tilde{v}_0 - v_\epsilon))^{\frac{1}{2}} + C(M)\epsilon,$$

which suffices by Cauchy-Schwarz.

Next, we move to the more difficult portion of the energy which involves $D_t a_\epsilon$. We start by combining Lemma 8.8 with $(v_\epsilon - \tilde{v}_0)\cdot n_{\Gamma_\epsilon} = -\mathcal{P}_{>\epsilon^{-1}}(w\cdot n_{\Gamma_\epsilon})$ to obtain the relation

$$\int_{\Gamma_\epsilon} \mathcal{N}_\epsilon^{k-\frac{3}{2}}(a_\epsilon^{-1}D_t a_\epsilon)\mathcal{N}_\epsilon^{k-\frac{3}{2}}(a_\epsilon^{-1}(D_t a_\epsilon - D_t\tilde{a}_0))\,dS = \int_{\Gamma_\epsilon} \mathcal{N}_\epsilon^{k-\frac{3}{2}}(a_\epsilon^{-1}D_t a_\epsilon)\mathcal{N}_\epsilon^{k-\frac{1}{2}}\mathcal{P}_{>\epsilon^{-1}}(w\cdot n_{\Gamma_\epsilon})\,dS + \mathcal{O}_M(\epsilon).$$

Define p_- and $D_t a_-$ in the usual way using the relevant Laplace equations. We split the above integral into the two components,

(8.31)
$$\begin{aligned}\int_{\Gamma_\epsilon} \mathcal{N}_\epsilon^{k-\frac{3}{2}}(a_\epsilon^{-1}D_t a_\epsilon)\mathcal{N}_\epsilon^{k-\frac{1}{2}}\mathcal{P}_{>\epsilon^{-1}}(w\cdot n_{\Gamma_\epsilon})\,dS &= \int_{\Gamma_\epsilon} \mathcal{N}_\epsilon^{k-\frac{3}{2}}(a_\epsilon^{-1}D_t a_-)\mathcal{N}_\epsilon^{k-\frac{1}{2}}\mathcal{P}_{>\epsilon^{-1}}(w\cdot n_{\Gamma_\epsilon})\,dS \\ &+ \int_{\Gamma_\epsilon} \mathcal{N}_\epsilon^{k-\frac{3}{2}}(a_\epsilon^{-1}(D_t a_\epsilon - D_t a_-))\mathcal{N}_\epsilon^{k-\frac{1}{2}}\mathcal{P}_{>\epsilon^{-1}}(w\cdot n_{\Gamma_\epsilon})\,dS.\end{aligned}$$

We begin by studying the first term in (8.31). By self-adjointness of $\mathcal{N}_\epsilon$, we have

$$\begin{aligned}\int_{\Gamma_\epsilon} \mathcal{N}_\epsilon^{k-\frac{3}{2}}(a_\epsilon^{-1}D_t a_-)\mathcal{N}_\epsilon^{k-\frac{1}{2}}\mathcal{P}_{>\epsilon^{-1}}(w\cdot n_{\Gamma_\epsilon})\,dS &= \int_{\Gamma_\epsilon} \mathcal{N}_\epsilon^{k-\frac{1}{2}}(a_\epsilon^{-1}D_t a_-)\mathcal{N}_\epsilon^{k-\frac{3}{2}}\mathcal{P}_{>\epsilon^{-1}}(w\cdot n_{\Gamma_\epsilon})\,dS \\ &\lesssim_M \epsilon\|a_\epsilon^{-1}D_t a_-\|_{H^{k-\frac{1}{2}}(\Gamma_\epsilon)}\|\mathcal{P}_{>\epsilon^{-1}}\mathcal{N}_\epsilon^{k-\frac{1}{2}}(w\cdot n_{\Gamma_\epsilon})\|_{L^2(\Gamma_\epsilon)} + \mathcal{O}_M(\epsilon),\end{aligned}$$

where we used the multiplier $\mathcal{P}_{>\epsilon^{-1}}$ to recover a power of $\mathcal{N}_\epsilon$ in the high frequency term. Next, we show that $\|a_\epsilon^{-1}D_t a_-\|_{H^{k-\frac{1}{2}}(\Gamma_\epsilon)} \lesssim_M \epsilon^{-\frac{1}{2}}$. By Sobolev product estimates and the fact that $\|a_\epsilon^{-1}\|_{H^{k-\frac{1}{2}}(\Gamma_\epsilon)} \lesssim_M \epsilon^{-\frac{1}{2}}$, it suffices to show the same estimate for $\|D_t a_-\|_{H^{k-\frac{1}{2}}(\Gamma_\epsilon)}$. To see this, recall that, by definition,

$$D_t a_- = n_{\Gamma_\epsilon}\cdot\nabla v_-\cdot\nabla p_- - n_{\Gamma_\epsilon}\cdot\nabla D_t p_-.$$

Note then that by Proposition 5.11 we have the estimate $\|\nabla v_-\|_{H^{k-\frac{1}{2}}(\Gamma_\epsilon)} \lesssim_M \epsilon^{-\frac{1}{2}}$, since v_- is regularized at the $\epsilon^{-\frac{1}{2}}$ scale. Moreover, as $n_{\Gamma_\epsilon} = \mathcal{O}_{H^{k-1}}(1)$ and Γ_ϵ is regularized at the ϵ^{-1} scale, we have $\|n_{\Gamma_\epsilon}\|_{H^{k-\frac{1}{2}}(\Gamma_\epsilon)} \lesssim_M \epsilon^{-\frac{1}{2}}$. By Proposition 5.11 and Proposition 5.19, we also have $\|\nabla p_-\|_{H^{k-\frac{1}{2}}(\Gamma_\epsilon)} \lesssim_M \epsilon^{-\frac{1}{2}}$. Therefore, by Proposition 5.9, we have $\|n_{\Gamma_\epsilon}\cdot\nabla v_-\cdot\nabla p_-\|_{H^{k-\frac{1}{2}}(\Gamma_\epsilon)} \lesssim_M \epsilon^{-\frac{1}{2}}$.

Using Proposition 5.19 and the fact that the pressure terms in the Laplace equation for $D_t p_-$ always appear to one half derivative lower than top order, a similar analysis yields $\|n_{\Gamma_\epsilon}\cdot\nabla D_t p_-\|_{H^{k-\frac{1}{2}}(\Gamma_\epsilon)} \lesssim_M \epsilon^{-\frac{1}{2}}$. Therefore, we obtain from Lemma 8.8 the bound,

$$\begin{aligned}\int_{\Gamma_\epsilon} \mathcal{N}_\epsilon^{k-\frac{3}{2}}(a_\epsilon^{-1}D_t a_-)\mathcal{N}_\epsilon^{k-\frac{1}{2}}\mathcal{P}_{>\epsilon^{-1}}(w\cdot n_{\Gamma_\epsilon})\,dS &\lesssim_M \epsilon^{\frac{1}{2}}\|\mathcal{P}_{>\epsilon^{-1}}\mathcal{N}_\epsilon^{k-\frac{1}{2}}(w\cdot n_{\Gamma_\epsilon})\|_{L^2(\Gamma_\epsilon)} + \mathcal{O}_M(\epsilon) \\ &\lesssim_M \epsilon^{\frac{1}{2}}(\mathcal{E}^k(\tilde{v}_0 - v_\epsilon))^{\frac{1}{2}} + \mathcal{O}_M(\epsilon),\end{aligned}$$

as desired. It remains to deal with the other term in (8.31). For this, we need to expand $D_t a_\epsilon - D_t a_-$. As a first reduction, we note that we can replace every appearance of p_- with p_ϵ in the definition of $D_t a_-$ if we allow for $\mathcal{O}_M(\epsilon^{\frac{1}{2}})$ errors. This is because $\|p_\epsilon - p_-\|_{H^k(\Omega_\epsilon)} \lesssim_M \|v_\epsilon - v_-\|_{H^{k-1}(\Omega_\epsilon)} \lesssim_M \epsilon^{\frac{1}{2}}$. Hence, we have

$$D_t a_- = n_{\Gamma_\epsilon}\cdot\nabla v_-\cdot\nabla p_\epsilon - n_{\Gamma_\epsilon}\cdot\nabla\Delta_{\Omega_\epsilon}^{-1}(4\mathrm{tr}(\nabla^2 p_\epsilon\cdot\nabla v_-) + 2\mathrm{tr}(\nabla v_-)^3 + \Delta v_-\cdot\nabla p_\epsilon) + \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\epsilon)}(\epsilon^{\frac{1}{2}}).$$

We may also replace the lower order terms involving v_- by v_ϵ. Arguing similarly to Lemma 8.8, we then obtain the key identity

$$\begin{aligned}D_t a_- - D_t a_\epsilon &= a_\epsilon\mathcal{N}_\epsilon((v_\epsilon - v_-)\cdot n_{\Gamma_\epsilon}) + \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\epsilon)}(\epsilon^{\frac{1}{2}}) \\ &= a_\epsilon\mathcal{P}_{\leq\epsilon^{-1}}\mathcal{N}_\epsilon(w\cdot n_{\Gamma_\epsilon}) + \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\epsilon)}(\epsilon^{\frac{1}{2}}).\end{aligned}$$

Hence, we have

$$\int_{\Gamma_\epsilon} \mathcal{N}_\epsilon^{k-\frac{3}{2}}(a_\epsilon^{-1}(D_t a_\epsilon - D_t a_-))\mathcal{N}_\epsilon^{k-\frac{1}{2}}\mathcal{P}_{>\epsilon^{-1}}(w\cdot n_{\Gamma_\epsilon})\,dS \lesssim_M -\int_{\Gamma_\epsilon}\mathcal{N}_\epsilon^{k-\frac{1}{2}}\mathcal{P}_{\leq\epsilon^{-1}}(w\cdot n_{\Gamma_\epsilon})\mathcal{N}_\epsilon^{k-\frac{1}{2}}\mathcal{P}_{>\epsilon^{-1}}(w\cdot n_{\Gamma_\epsilon})\,dS + \epsilon^{\frac{1}{2}}\|\mathcal{N}_\epsilon^{k-\frac{1}{2}}\mathcal{P}_{>\epsilon^{-1}}(w\cdot n_{\Gamma_\epsilon})\|_{L^2(\Gamma_\epsilon)}.$$

The first term on the right-hand side above vanishes by orthogonality (this term is the reason we reweighted the energy in the first place) and the latter term is controlled by $\epsilon^{\frac{1}{2}}(\mathcal{E}^k(\tilde{v}_0 - v_\epsilon))^{\frac{1}{2}}$. Therefore, we obtain the desired bound for the $D_t a_\epsilon$ portion of the energy. This completes the proof of Proposition 8.6. ☐

8.4. **Step 3: Euler plus transport iteration.** In this subsection, we construct the iterate (v_1, Γ_1) from the regularized data $(v_\epsilon, \Gamma_\epsilon)$. Intuitively, what remains to be done is to carry out something akin to the Euler iteration

$$v_1 := v_\epsilon - \epsilon(v_\epsilon \cdot \nabla v_\epsilon + \nabla p_\epsilon + g e_d)$$

and then the domain transport

$$x_1(x) := x + \epsilon v_\epsilon(x).$$

Unfortunately, performed individually, these steps lose a full derivative in each iteration. Therefore, it is important that these two steps be carried out together. This will reduce the derivative loss and allow us to exploit a discrete version of the energy cancellation seen in the energy estimates. We will then use the regularization bounds from the previous subsections to control any remaining errors in the iteration. To carry out this process, we have the following proposition.

Proposition 8.9. *Given $(v_\epsilon, \Gamma_\epsilon)$ as in the previous step, there exists an iteration $(v_\epsilon, \Gamma_\epsilon) \mapsto (v_1, \Gamma_1)$ such that the following properties hold:*

(i) (Approximate solution).

$$\begin{cases} v_1 = v_\epsilon - \epsilon(v_\epsilon\cdot\nabla v_\epsilon + \nabla p_\epsilon + g e_d) + \mathcal{O}_{C^1}(\epsilon^2) \quad \text{on } \Omega_1\cap\Omega_\epsilon, \\ \nabla\cdot v_1 = 0 \quad \text{on } \Omega_1, \\ \Omega_1 = (I + \epsilon v_\epsilon)\Omega_\epsilon. \end{cases}$$

(ii) (Energy monotonicity bound).

$$\mathcal{E}^k(v_1, \Gamma_1) \leq (1 + C(M)\epsilon)\mathcal{E}^k(v_\epsilon, \Gamma_\epsilon).$$

Moreover, v_1 and ω_1 satisfy the inductive bounds (8.6).

We define the change of coordinates $x_1(x) := x + \epsilon v_\epsilon(x)$ and the iterated domain Ω_1 by

$$\Omega_1 := (I + \epsilon v_\epsilon)\Omega_\epsilon.$$

To define v_1, we proceed in two steps. First, we define

$$\tilde{v}_1(x_1) := v_\epsilon - \epsilon(\nabla p_\epsilon + g e_d). \tag{8.32}$$

We note that $\tilde{v}_1$ is not divergence free, so we define the full iterate v_1 by correcting the divergence of $\tilde{v}_1$ by a gradient potential:

$$v_1 := \tilde{v}_1 - \nabla\Delta_{\Omega_1}^{-1}(\nabla\cdot\tilde{v}_1).$$

At this point, we can verify the inductive bound (8.6) for v_1 and ω_1. We start with v_1. We recall that we have to show that

$$\|v_1\|_{H^{k+1}(\Omega_1)} \leq K(M)\epsilon^{-1}.$$

As a first step, using the regularization bound (8.25) for v_ϵ from the previous section, we have from the definition of $\tilde{v}_1$, the regularization bounds (8.10) for Γ_ϵ and the balanced elliptic estimate Proposition 5.19,

$$\|\tilde{v}_1\|_{H^{k+n}(\Omega_1)} \leq \frac{1}{3}K(M)\epsilon^{-n}, \tag{8.33}$$

for $n = 0, 1, 2$. Next, we aim to control the error between v_1 and $\tilde{v}_1$ in $H^k(\Omega_1)$ and $H^{k+1}(\Omega_1)$ (but not $H^{k+2}(\Omega_1)$). We have for $n = 0, 1$ from the balanced elliptic estimate Proposition 5.19,

$$\begin{aligned}\|v_1 - \tilde{v}_1\|_{H^{k+n}(\Omega_1)} &\lesssim_M \|\Gamma_1\|_{H^{k+\frac{1}{2}+n}} \|\nabla \cdot \tilde{v}_1\|_{H^{k-2}(\Omega_1)} + \|\nabla \cdot \tilde{v}_1\|_{H^{k-1+n}(\Omega_1)} \\ &\lesssim_M \epsilon^{-\frac{1}{2}-n}\|\nabla \cdot \tilde{v}_1\|_{H^{k-2}(\Omega_1)} + \|\nabla \cdot \tilde{v}_1\|_{H^{k-1+n}(\Omega_1)}.\end{aligned}$$

Above, we used the H^{k+1} and H^{k+2} (depending on if n is 0 or 1) regularization bounds for v_ϵ, Moser estimates, the bounds for Γ_ϵ and the relation $\Gamma_1 = (I + \epsilon v_\epsilon)(\Gamma_\epsilon)$ to control $\|\Gamma_1\|_{H^{k+\frac{1}{2}+n}} \lesssim_M \epsilon^{-\frac{1}{2}-n}$. By using the definition of $\tilde{v}_1$ and the regularization bounds for v_ϵ, it is straightforward to see that the divergence, $\nabla\cdot\tilde{v}_1$, contributes an error of size $\mathcal{O}_{H^{k-1+n}(\Omega_1)}(\epsilon^{\frac{3}{2}-n})$ and also $\mathcal{O}_{H^{k-2}(\Omega_1)}(\epsilon^2)$. Note that for this computation, one must use the cancellation between the velocity and the pressure in (8.32) in order to see the desired gain. Therefore, we have

$$\|\nabla\Delta_{\Omega_1}^{-1}(\nabla\cdot\tilde{v}_1)\|_{H^{k+n}(\Omega_1)} = \|v_1 - \tilde{v}_1\|_{H^{k+n}(\Omega_1)} \lesssim_M \epsilon^{\frac{3}{2}-n}.$$

From this and (8.33), we conclude the inductive bound

$$\|v_1\|_{H^{k+1}(\Omega_1)} \leq K(M)\epsilon^{-1},$$

and the leading order expansion for $v_1(x_1)$ in $H^k(\Omega_\epsilon)$,

$$v_1(x_1) = v_\epsilon - \epsilon(\nabla p_\epsilon + g e_d) + \mathcal{O}_{H^k(\Omega_\epsilon)}(\epsilon^{\frac{3}{2}}).$$

If k is large enough, then the leading order expansion (8.4) with $\mathcal{O}_{C^1}(\epsilon^2)$ error can be seen by slightly modifying the above argument. Now, we verify the inductive bound $\|\omega_1\|_{H^{k+n}(\Omega_1)} \leq \epsilon^{-1-n}(K'(M)+\epsilon C(M))$ for $n = 0, 1$. It suffices to establish this for $\tilde{\omega}_1$ since v_1 and $\tilde{v}_1$ agree up to a gradient. Taking curl in the definition of $\tilde{v}_1$ and using that $\omega_\epsilon = \tilde{\omega}_0$, we have

$$\|\nabla \times (\tilde{v}_1(x_1))\|_{H^{k+n}(\Omega_\epsilon)} \leq \|\tilde{\omega}_0\|_{H^{k+n}(\Omega_\epsilon)} \leq K'(M)\epsilon^{-1-n}. \tag{8.34}$$

By chain rule, using (8.33) and the regularization bounds for v_ϵ, we have

$$\|\tilde{\omega}_1(x_1)\|_{H^{k+n}(\Omega_\epsilon)} \leq \|\nabla \times (\tilde{v}_1(x_1))\|_{H^{k+n}(\Omega_\epsilon)} + C(M)\epsilon^{-n},$$

which by a change of variables and (8.34) yields

$$\|\tilde{\omega}_1\|_{H^{k+n}(\Omega_1)} \leq \epsilon^{-1-n}(K'(M) + \epsilon C(M)),$$

as desired. Note that in the above two lines, we treated $C(M)$ as an arbitrary constant, and relabelled it from line to line. Importantly, we did not do this for $K(M)$ and $K'(M)$.

Next, we work towards establishing the energy monotonicity bound for the transport part of the argument. As a first step, we aim to relate the good variables associated to the iterate v_1 to the good variables associated to v_ϵ at the regularity level of the energy. We have the following lemma.

Lemma 8.10 (Relations between the good variables)**.** *The following relations hold:*

(i) (Relation for ω_1).

$$\omega_1(x_1) = \omega_\epsilon + \mathcal{O}_{H^{k-1}(\Omega_\epsilon)}(\epsilon).$$

(ii) *(Relation for p_1).*

$$p_1(x_1) - p_\epsilon - \epsilon D_t p_\epsilon = \mathcal{O}_{H^{k+\frac{1}{2}}(\Omega_\epsilon)}(\epsilon). \tag{8.35}$$

(iii) *(Relation for a_1).*

$$a_1(x_1) = a_\epsilon + \epsilon D_t a_\epsilon + \mathcal{O}_{H^{k-1}(\Gamma_\epsilon)}(\epsilon). \tag{8.36}$$

(iv) *(Relation for $D_t a_1$).*

$$D_t a_1(x_1) = D_t a_\epsilon - \epsilon a_\epsilon \mathcal{N}_\epsilon a_\epsilon + \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\epsilon)}(\epsilon).$$

Proof. The relation for ω_1 is immediate. Next, we move to the relations for p_1 and a_1. By the chain rule and the Laplace equation (7.4) for $D_t p_\epsilon$, we have

$$\begin{aligned}
\Delta(p_1(x_1)) &= (\Delta p_1)(x_1) + \epsilon \Delta v_\epsilon \cdot (\nabla p_1)(x_1) + 2\epsilon \nabla v_\epsilon \cdot (\nabla^2 p_1)(x_1) + \mathcal{O}_{H^{k-\frac{3}{2}}(\Omega_\epsilon)}(\epsilon) \\
&= \Delta p_\epsilon + \epsilon \Delta D_t p_\epsilon + \epsilon \Delta v_\epsilon \cdot ((\nabla p_1)(x_1) - \nabla p_\epsilon) + \mathcal{O}_{H^{k-\frac{3}{2}}(\Omega_\epsilon)}(\epsilon) \\
&= \Delta p_\epsilon + \epsilon \Delta D_t p_\epsilon + \mathcal{O}_{H^{k-\frac{3}{2}}(\Omega_\epsilon)}(\epsilon),
\end{aligned}$$

where in the last line, we controlled $\epsilon \Delta v_\epsilon \cdot ((\nabla p_1)(x_1) - \nabla p_\epsilon) = \mathcal{O}_{H^{k-\frac{3}{2}}(\Omega_\epsilon)}(\epsilon)$ by using the regularization bounds for v_ϵ as well as the error bound $(\nabla p_1)(x_1) - \nabla p_\epsilon = \mathcal{O}_{L^\infty(\Omega_\epsilon)}(\epsilon)$, which is gotten by performing an $H^k(\Omega_\epsilon)$ elliptic estimate in the second line, using the fact that $p_1(x_1) - p_\epsilon$ vanishes on Γ_ϵ and that each of the source terms can be estimated directly in $H^{k-2}(\Omega_\epsilon)$ (but not in $H^{k-\frac{3}{2}}(\Omega_\epsilon)$). Therefore, since $p_1(x_1) - p_\epsilon - \epsilon D_t p_\epsilon$ vanishes on Γ_ϵ, we may now do a $H^{k+\frac{1}{2}}(\Omega_\epsilon)$ elliptic estimate to obtain the finer bound,

$$p_1(x_1) - p_\epsilon - \epsilon D_t p_\epsilon = \mathcal{O}_{H^{k+\frac{1}{2}}(\Omega_\epsilon)}(\epsilon), \tag{8.37}$$

which gives (8.35). We also deduce from this that

$$\begin{aligned}
(\nabla p_1)(x_1) &= \nabla p_\epsilon + \epsilon \nabla D_t p_\epsilon - \epsilon \nabla v_\epsilon \cdot (\nabla p_1)(x_1) + \mathcal{O}_{H^{k-\frac{1}{2}}(\Omega_\epsilon)}(\epsilon) \\
&= \nabla p_\epsilon + \epsilon D_t \nabla p_\epsilon + \mathcal{O}_{H^{k-\frac{1}{2}}(\Omega_\epsilon)}(\epsilon).
\end{aligned}$$

From this we see that

$$\begin{aligned}
a_1(x_1) &= a_\epsilon + \epsilon D_t a_\epsilon - (n_{\Gamma_1}(x_1) - n_{\Gamma_\epsilon}) \cdot (\nabla p_1)(x_1) + \mathcal{O}_{H^{k-1}(\Gamma_\epsilon)}(\epsilon) \\
&= a_\epsilon + \epsilon D_t a_\epsilon + \mathcal{O}_{H^{k-1}(\Gamma_\epsilon)}(\epsilon),
\end{aligned}$$

where in the last line we used

$$(n_{\Gamma_1}(x_1) - n_{\Gamma_\epsilon}) \cdot (\nabla p_1)(x_1) = -a_1(x_1)(n_{\Gamma_1}(x_1) - n_{\Gamma_\epsilon}) \cdot n_{\Gamma_1}(x_1) = -a_1(x_1)\frac{1}{2}|n_{\Gamma_1}(x_1) - n_{\Gamma_\epsilon}|^2 = \mathcal{O}_{H^{k-1}(\Gamma_\epsilon)}(\epsilon).$$

This gives the relation (8.36).

Next, we prove the relation for $D_t a_1$. First, we see that

$$\begin{aligned}
-(D_t \nabla p_1)(x_1) + D_t \nabla p_\epsilon &= ((\nabla v_1 \cdot \nabla p_1)(x_1) - \nabla v_\epsilon \cdot \nabla p_\epsilon) - ((\nabla D_t p_1)(x_1) - \nabla D_t p_\epsilon) \\
&= ((\nabla v_1)(x_1) - \nabla v_\epsilon) \cdot \nabla p_\epsilon - ((\nabla D_t p_1)(x_1) - \nabla D_t p_\epsilon) + \mathcal{O}_{H^{k-1}(\Omega_\epsilon)}(\epsilon).
\end{aligned} \tag{8.38}$$

To control the second term on the right-hand side above, we write out the Laplace equation for $D_t p_1(x_1)$:

$$\Delta(D_t p_1(x_1)) = (\Delta D_t p_1)(x_1) + \mathcal{O}_{H^{k-2}(\Omega_\epsilon)}(\epsilon).$$

By a similar analysis to the proof of (8.36) and the relation

$$(\Delta v_1)(x_1) = \Delta(v_1(x_1)) + \mathcal{O}_{H^{k-2}(\Omega_\epsilon)}(\epsilon) = \Delta v_\epsilon - \epsilon \nabla \Delta p_\epsilon + \mathcal{O}_{H^{k-2}(\Omega_\epsilon)}(\epsilon) = \Delta v_\epsilon + \mathcal{O}_{H^{k-2}(\Omega_\epsilon)}(\epsilon),$$

we obtain

$$\begin{aligned}(\Delta D_t p_1)(x_1) &= \Delta D_t p_\epsilon + (\Delta v_1 \cdot \nabla p_1)(x_1) - \Delta v_\epsilon \cdot \nabla p_\epsilon + 4\mathrm{tr}(\nabla v_1 \cdot \nabla^2 p_1)(x_1) - 4tr \nabla v_\epsilon \cdot \nabla^2 p_\epsilon + \mathcal{O}_{H^{k-2}(\Omega_\epsilon)}(\epsilon)\\ &= \Delta D_t p_\epsilon + 4tr\left(\nabla v_\epsilon \cdot ((\nabla^2 p_1)(x_1) - \nabla^2 p_\epsilon)\right) + \mathcal{O}_{H^{k-2}(\Omega_\epsilon)}(\epsilon)\\ &= \Delta D_t p_\epsilon + \mathcal{O}_{H^{k-2}(\Omega_\epsilon)}(\epsilon),\end{aligned}$$

where in the last line, we used (8.37) and that $\epsilon D_t p_\epsilon = \mathcal{O}_{H^k(\Omega_\epsilon)}(\epsilon)$. Combining the above with (8.38), one obtains by elliptic regularity,

$$-D_t \nabla p_1(x_1) + D_t \nabla p_\epsilon = ((\nabla v_1)(x_1) - \nabla v_\epsilon) \cdot \nabla p_\epsilon + \mathcal{O}_{H^{k-1}(\Omega_\epsilon)}(\epsilon).$$

Then, noting from (8.36) that

$$(D_t \nabla p_1)(x_1) \cdot (n_{\Gamma_1}(x_1) - n_{\Gamma_\epsilon}) = (D_t \nabla p_1)(x_1) \cdot (a_\epsilon^{-1} \nabla p_\epsilon - (a_1^{-1} \nabla p_1)(x_1)) = \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\epsilon)}(\epsilon)$$

and using the fact that Δp_ϵ is lower order, we obtain

$$\begin{aligned}D_t a_1(x_1) - D_t a_\epsilon &= -a_\epsilon n_{\Gamma_\epsilon} \cdot \nabla(v_1(x_1) - v_\epsilon) \cdot n_{\Gamma_\epsilon} - (D_t \nabla p_1)(x_1) \cdot (n_{\Gamma_1}(x_1) - n_{\Gamma_\epsilon}) + \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\epsilon)}(\epsilon)\\ &= \epsilon a_\epsilon n_{\Gamma_\epsilon} \cdot \nabla \nabla p_\epsilon \cdot n_{\Gamma_\epsilon} + \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\epsilon)}(\epsilon)\\ &= \epsilon a_\epsilon \mathcal{N}_\epsilon \nabla p_\epsilon \cdot n_{\Gamma_\epsilon} + \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\epsilon)}(\epsilon).\end{aligned}$$

Finally, noting that $\mathcal{N}_\epsilon n_{\Gamma_\epsilon} \cdot n_{\Gamma_\epsilon}$ is lower order, we have, thanks to the Leibniz rule for $\mathcal{N}_\epsilon$,

$$\epsilon a_\epsilon \mathcal{N}_\epsilon \nabla p_\epsilon \cdot n_{\Gamma_\epsilon} = -\epsilon a_\epsilon \mathcal{N}_\epsilon (n_{\Gamma_\epsilon} a_\epsilon) \cdot n_{\Gamma_\epsilon} = -\epsilon a_\epsilon \mathcal{N}_\epsilon a_\epsilon + \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\epsilon)}(\epsilon).$$

Therefore, we have the desired relation for $D_t a_1$. This completes the proof of the lemma. ☐

Energy monotonicity. To finish the proof of Proposition 8.9, it remains to establish energy monotonicity. The following lemma will allow us to more easily work with the relations in Lemma 8.10.

Lemma 8.11. *Define the "pulled-back" energy* $\mathcal{E}^k_*(v_1, \Gamma_1)$ *by*

$$\begin{aligned}\mathcal{E}^k_*(v_1, \Gamma_1) := 1 &+ \|\mathcal{N}_\epsilon^{k-\frac{3}{2}}(a_1^{-1}(x_1) D_t a_1(x_1))\|^2_{L^2(\Gamma_\epsilon)} + \|a_1^{-\frac{1}{2}}(x_1)\mathcal{N}_\epsilon^{k-1}(a_1(x_1))\|^2_{L^2(\Gamma_\epsilon)}\\ &+ \|\omega_1(x_1)\|^2_{H^{k-1}(\Omega_\epsilon)} + \|v_1(x_1)\|^2_{L^2(\Omega_\epsilon)}.\end{aligned}$$

Then we have the relation

$$\mathcal{E}^k(v_1, \Gamma_1) \le \mathcal{E}^k_*(v_1, \Gamma_1) + \mathcal{O}_M(\epsilon).$$

Before proving the above lemma, we show how it easily implies the desired energy monotonicity bound. In light of Lemma 8.11, it suffices to establish the bound

$$\mathcal{E}^k_*(v_1, \Gamma_1) \le (1 + C(M)\epsilon)\mathcal{E}^k(v_\epsilon, \Gamma_\epsilon).$$

The monotonicity bound for the vorticity is immediate from Lemma 8.10. For the surface components of the energy, we first use Lemma 8.10, the fact that $\|\mathcal{N}_\epsilon^{k-\frac{3}{2}}\|_{H^{k-\frac{3}{2}}(\Gamma_\epsilon) \to L^2(\Gamma_\epsilon)} \lesssim_M 1$ and the regularization bounds for Γ_ϵ and v_ϵ to obtain

$$\begin{aligned}&\int_{\Gamma_\epsilon} |\mathcal{N}_\epsilon^{k-\frac{3}{2}}(a_1^{-1}(x_1) D_t a_1(x_1))|^2 \, dS - \int_{\Gamma_\epsilon} |\mathcal{N}_\epsilon^{k-\frac{3}{2}}(a_\epsilon^{-1} D_t a_\epsilon)|^2 \, dS\\ &= 2\int_{\Gamma_\epsilon} \mathcal{N}_\epsilon^{k-\frac{3}{2}}(a_\epsilon^{-1} D_t a_\epsilon) \mathcal{N}_\epsilon^{k-\frac{3}{2}}(a_\epsilon^{-1}((D_t a_1)(x_1) - D_t a_\epsilon)) \, dS + \mathcal{O}_M(\epsilon) \qquad (8.39)\\ &= -2\epsilon \int_{\Gamma_\epsilon} a_\epsilon^{-1} \mathcal{N}_\epsilon^{k-1} D_t a_\epsilon \mathcal{N}_\epsilon^{k-1} a_\epsilon \, dS + \mathcal{O}_M(\epsilon),\end{aligned}$$

where in the last line, we used the commutator estimate $\|[\mathcal{N}_\epsilon^{k-1}, a_\epsilon^{-1}]D_t a_\epsilon\|_{L^2(\Gamma_\epsilon)} \lesssim_M 1$ to shift a factor of $\mathcal{N}_\epsilon^{\frac{1}{2}}$ onto $\mathcal{N}_\epsilon^{k-\frac{3}{2}} D_t a_\epsilon$. We similarly observe the leading order relation for the other component of the energy by using (8.36) to obtain,

$$\int_{\Gamma_\epsilon} a_1^{-1}(x_1)|\mathcal{N}_\epsilon^{k-1}(a_1(x_1))|^2\, dS - \int_{\Gamma_\epsilon} a_\epsilon^{-1}|\mathcal{N}_\epsilon^{k-1}a_\epsilon|^2\, dS = 2\epsilon \int_{\Gamma_\epsilon} a_\epsilon^{-1}\mathcal{N}_\epsilon^{k-1}D_t a_\epsilon \mathcal{N}_\epsilon^{k-1}a_\epsilon\, dS + \mathcal{O}_M(\epsilon).$$

The first term on the right-hand side of the above relation cancels the main term on the right-hand side of (8.39). Combining everything together then gives

$$\mathcal{E}^k(v_1, \Gamma_1) \leq (1 + C(M)\epsilon)\mathcal{E}^k(v_\epsilon, \Gamma_\epsilon),$$

as desired. It remains now to establish Lemma 8.11.

Proof of Lemma 8.11. By a simple change of variables, it is clear that the difference between $\|\omega_1(x_1)\|^2_{H^{k-1}(\Omega_\epsilon)}$ and $\|\omega_1\|^2_{H^{k-1}(\Omega_1)}$ contributes only $\mathcal{O}_M(\epsilon)$ errors. This is likewise true for the L^2 component of the velocity. The main difficulty is in dealing with the surface components of the energy. For this, we need the following proposition.

Proposition 8.12. *Let $-\frac{1}{2} \leq s \leq k-2$ and let $f \in H^{s+1}(\Gamma_1)$. Then we have the following bound on Γ_ϵ:*

$$\|(\mathcal{N}_1 f)(x_1) - \mathcal{N}_\epsilon(f(x_1))\|_{H^s(\Gamma_\epsilon)} \lesssim_M \epsilon\|f\|_{H^{s+1}(\Gamma_1)}.$$

Proof. First, we handle the case $s = -\frac{1}{2}$. If $g \in C^\infty(\Gamma_\epsilon)$, we write $h = g(x_1^{-1})\mathcal{H}_1 J$ where J is the Jacobian corresponding to the change of variables $y = x_1(x)$. Then we have by the divergence theorem,

$$\begin{aligned}\int_{\Gamma_\epsilon} g((\mathcal{N}_1 f)(x_1) - \mathcal{N}_\epsilon(f(x_1)))\, dS &= \int_{\Gamma_1} h\mathcal{N}_1 f\, dS - \int_{\Gamma_\epsilon} g\mathcal{N}_\epsilon(f(x_1))\, dS \\ &= \int_{\Omega_1} \nabla\mathcal{H}_1 h \cdot \nabla\mathcal{H}_1 f\, dx - \int_{\Omega_\epsilon} \nabla\mathcal{H}_\epsilon g \cdot \nabla\mathcal{H}_\epsilon(f(x_1))\, dx.\end{aligned}$$

Using again the change of variables $x \mapsto x_1$ for the first term in the second line above, together with the estimates

$$\|\mathcal{H}_1 h\|_{H^1(\Omega_1)} \lesssim_M \|g\|_{H^{\frac{1}{2}}(\Gamma_\epsilon)} \quad \text{and} \quad \|(\nabla\mathcal{H}_1 f)(x_1)\|_{L^2(\Omega_\epsilon)} \lesssim_M \|f\|_{H^{\frac{1}{2}}(\Gamma_1)},$$

it is easy to verify

(8.40)
$$\begin{aligned}\int_{\Gamma_\epsilon} g((\mathcal{N}_1 f)(x_1) - \mathcal{N}_\epsilon(f(x_1)))\, dS \lesssim_M &\int_{\Omega_\epsilon} \nabla((\mathcal{H}_1 h)(x_1) - \mathcal{H}_\epsilon g) \cdot (\nabla\mathcal{H}_1 f)(x_1)\, dx \\ &+ \int_{\Omega_\epsilon} \nabla H_\epsilon g \cdot \nabla((\mathcal{H}_1 f)(x_1) - \mathcal{H}_\epsilon(f(x_1)))\, dx + \epsilon\|g\|_{H^{\frac{1}{2}}(\Gamma_\epsilon)}\|f\|_{H^{\frac{1}{2}}(\Gamma_1)}.\end{aligned}$$

We label the first and second terms on the right-hand side above by I_1 and I_2. For I_1, we use the fact that on Γ_ϵ we have

$$(\mathcal{H}_1 h)(x_1) - \mathcal{H}_\epsilon g = (J(x_1) - I)g$$

to obtain the following simple elliptic estimate

$$I_1 \lesssim_M \epsilon\|f\|_{H^{\frac{1}{2}}(\Gamma_1)}\|g\|_{H^{\frac{1}{2}}(\Gamma_\epsilon)} + \|f\|_{H^{\frac{1}{2}}(\Gamma_1)}\|\Delta((\mathcal{H}_1 h)(x_1))\|_{H^{-1}(\Omega_\epsilon)} \lesssim_M \epsilon\|f\|_{H^{\frac{1}{2}}(\Gamma_1)}\|g\|_{H^{\frac{1}{2}}(\Gamma_\epsilon)},$$

where we used the chain rule and that $\mathcal{H}_1 h$ is harmonic to estimate $\Delta((\mathcal{H}_1 h)(x_1))$. A similar elliptic estimate yields the same bound for I_2. This establishes the case $s = -\frac{1}{2}$. By interpolation, we only need to handle the remaining cases when $\frac{1}{2} \leq s \leq k-2$. As a starting point, we have from some simple manipulations with the chain rule and the trace inequality,

$$\|(\mathcal{N}_1 f)(x_1) - \mathcal{N}_\epsilon(f(x_1))\|_{H^s(\Gamma_\epsilon)} \lesssim \epsilon \|f\|_{H^{s+1}(\Gamma_1)} + \|(n_{\Gamma_1}(x_1) - n_{\Gamma_\epsilon}) \cdot (\nabla \mathcal{H}_1 f)(x_1)\|_{H^s(\Gamma_\epsilon)} + \|(\mathcal{H}_1 f)(x_1) - \mathcal{H}_\epsilon(f(x_1))\|_{H^{s+\frac{3}{2}}(\Omega_\epsilon)}.$$

By writing $n_{\Gamma_1}(x_1) - n_{\Gamma_\epsilon} = a_\epsilon^{-1} \nabla p_\epsilon - a_1^{-1}(x_1)(\nabla p_1)(x_1)$ and using the relations in Lemma 8.10 and that $s \leq k-2$, the second term on the right is straightforward to control by $\epsilon \|f\|_{H^{s+1}(\Gamma_\epsilon)}$. For the third term, we do an elliptic estimate analogous to the $s = -\frac{1}{2}$ case (using that $(\mathcal{H}_1 f)(x_1) - \mathcal{H}_\epsilon(f(x_1)) = 0$ on Γ_ϵ) to obtain

$$\|(\mathcal{H}_1 f)(x_1) - \mathcal{H}_\epsilon(f(x_1))\|_{H^{s+\frac{3}{2}}(\Omega_\epsilon)} \lesssim_M \|\Delta((\mathcal{H}_1 f)(x_1))\|_{H^{s-\frac{1}{2}}(\Omega_\epsilon)} \lesssim_M \epsilon \|f\|_{H^{s+1}(\Gamma_1)}.$$

This completes the proof. □

Now we return to the proof of Lemma 8.11. We note first that

$$\|(\mathcal{N}_1^{k-1} a_1)(x_1) - \mathcal{N}_\epsilon^{k-1}(a_1(x_1))\|_{L^2(\Gamma_\epsilon)} \lesssim \|\mathcal{N}_\epsilon(\mathcal{N}_1^{k-2} a_1)(x_1) - \mathcal{N}_\epsilon^{k-1}(a_1(x_1))\|_{L^2(\Gamma_\epsilon)} + \|\mathcal{N}_\epsilon(\mathcal{N}_1^{k-2} a_1)(x_1) - (\mathcal{N}_1^{k-1} a_1)(x_1)\|_{L^2(\Gamma_\epsilon)}.$$

Applying Proposition 8.12 to the term in the second line and using the $H^1 \to L^2$ bound for $\mathcal{N}$, we have

$$\|(\mathcal{N}_1^{k-1} a_1)(x_1) - \mathcal{N}_\epsilon^{k-1}(a_1(x_1))\|_{L^2(\Gamma_\epsilon)} \lesssim_M \|(\mathcal{N}_1^{k-2} a_1)(x_1) - \mathcal{N}_\epsilon^{k-2}(a_1(x_1))\|_{H^1(\Gamma_\epsilon)} + \mathcal{O}_M(\epsilon).$$

Iterating this procedure and applying Proposition 8.12 $k-2$ times, we see that we have

$$\|(\mathcal{N}_1^{k-1} a_1)(x_1) - \mathcal{N}_\epsilon^{k-1}(a_1(x_1))\|_{L^2(\Gamma_\epsilon)} \lesssim_M \epsilon.$$

It follows from the above and a change of variables that we have

$$\|a_1^{-\frac{1}{2}} \mathcal{N}_1^{k-1} a_1\|_{L^2(\Gamma_1)}^2 \leq \|a_1^{-\frac{1}{2}}(x_1) \mathcal{N}_\epsilon^{k-1}(a_1(x_1))\|_{L^2(\Gamma_\epsilon)}^2 + \mathcal{O}_M(\epsilon).$$

To conclude the proof of Lemma 8.11, we need to show that

$$\|\nabla \mathcal{H}_1(\mathcal{N}_1^{k-2}(a_1^{-1} D_t a_1))\|_{L^2(\Omega_1)}^2 \leq \|\nabla \mathcal{H}_\epsilon \mathcal{N}_\epsilon^{k-2}(a_1^{-1}(x_1) D_t a_1(x_1))\|_{L^2(\Omega_\epsilon)}^2 + \mathcal{O}_M(\epsilon).$$

From a change of variables, we see that

$$\|\nabla \mathcal{H}_1(\mathcal{N}_1^{k-2}(a_1^{-1} D_t a_1))\|_{L^2(\Omega_1)}^2 - \|\nabla \mathcal{H}_\epsilon \mathcal{N}_\epsilon^{k-2}(a_1^{-1}(x_1) D_t a_1(x_1))\|_{L^2(\Omega_\epsilon)}^2 \lesssim_M \mathcal{J} + \mathcal{O}_M(\epsilon),$$

where

$$\mathcal{J} := \|(\nabla \mathcal{H}_1 \mathcal{N}_1^{k-2}(a_1^{-1} D_t a_1))(x_1) - \nabla \mathcal{H}_\epsilon \mathcal{N}_\epsilon^{k-2}(a^{-1}(x_1) D_t a_1(x_1))\|_{L^2(\Omega_\epsilon)}.$$

By elliptic regularity, it is easy to verify the bound

$$\mathcal{J} \lesssim_M \|(\mathcal{N}_1^{k-2}(a_1^{-1} D_t a_1))(x_1) - \mathcal{N}_\epsilon^{k-2}(a_1^{-1}(x_1) D_t a_1(x_1))\|_{H^{\frac{1}{2}}(\Gamma_\epsilon)} + \mathcal{O}_M(\epsilon).$$

From here, we use Proposition 8.12 similarly to the other surface term in the energy to estimate

$$\|(\mathcal{N}_1^{k-2}(a_1^{-1} D_t a_1))(x_1) - \mathcal{N}_\epsilon^{k-2}(a_1^{-1}(x_1) D_t a_1(x_1))\|_{H^{\frac{1}{2}}(\Gamma_\epsilon)} \lesssim_M \epsilon.$$

This completes the proof. □

8.5. **Convergence of the iteration scheme.** We have now arrived at the final step of the existence proof, where we use our one step iteration result in Theorem 8.1 in order to prove the existence of regular solutions. Precisely, we aim to establish the following theorem.

Theorem 8.13. *Let k be a sufficiently large even integer and $M > 0$. Let $(v_0, \Gamma_0) \in \mathbf{H}^k$ be an initial data set so that $\|(v_0, \Gamma_0)\|_{\mathbf{H}^k} \leq M$. Then there exists $T = T(M)$ and a solution (v, Γ) to the free boundary incompressible Euler equations on $[0, T]$ with this initial data and the following regularity properties:*

$$(v, \Gamma) \in L^\infty([0, T]; \mathbf{H}^k) \cap C([0, T]; \mathbf{H}^{k-1})$$

with the uniform bound

$$\|(v, \Gamma)(t)\|_{\mathbf{H}^k} \lesssim_M 1, \qquad t \in [0, T].$$

We remark that the solution we construct is unique by the result in Theorem 4.6. One missing piece here is the lack of continuity in $\mathbf{H}^k$, which does not follow from the proof below. However, this will be rectified in the next section. We now turn to the proof of the theorem.

Proof. Starting from the initial data $(v_0, \Gamma_0) \in \mathbf{H}^k$ with $\Gamma_0 \in \Lambda_* := \Lambda(\Gamma_*, \epsilon_0, \delta)$, for each small time scale ϵ we construct a discrete approximate solution $(v_\epsilon, \Gamma_\epsilon)$ which is defined at discrete times $t = 0, \epsilon, 2\epsilon, \ldots,$ as follows:

(i) We define $(v_\epsilon(0), \Gamma_\epsilon(0))$ by directly regularizing (v_0, Γ_0) at scale ϵ. Such a regularization is provided by Proposition 6.2 with $\epsilon = 2^{-j}$. In view of the higher regularity bound there, these regularized data will satisfy the hypothesis of our one step Theorem 8.1, with M replaced by $\tilde{M} = C(A)M$.

(ii) We inductively define the approximate solutions $(v_\epsilon(j\epsilon), \Gamma_\epsilon(j\epsilon))$ by repeatedly applying the iteration step in Theorem 8.1.

To control the growth of the $\mathbf{H}^k$ norms of $(v_\epsilon, \Gamma_\epsilon)$ we rely on the energy monotonicity relation, together with the coercivity property in Theorem 7.1 (and also the relation (8.2)). We use the energy coercivity in both ways. At time $t = 0$ we have

$$\mathcal{E}^k(v_\epsilon(0), \Gamma_\epsilon(0)) \leq C_1(A)M.$$

We let our iteration continue for as long as

$$\begin{aligned} &\mathcal{E}^k(v_\epsilon(j\epsilon), \Gamma_\epsilon(j\epsilon)) \leq 2C_1(A)M, \\ &\Gamma_\epsilon(j\epsilon) \in 2\Lambda_* := \Lambda(\Gamma_*, \epsilon_0, 2\delta). \end{aligned} \tag{8.41}$$

As long as this happens, using the coercivity in the other direction we get

$$\|(v_\epsilon(j\epsilon), \Gamma_\epsilon(j\epsilon))\|_{\mathbf{H}^k} \leq C_2(A)M.$$

Now by the energy monotonicity bound (8.4) we conclude that

$$\mathcal{E}^k(v_\epsilon(j\epsilon), \Gamma_\epsilon(j\epsilon)) \leq (1 + C(C_2(A)M)\epsilon)^j \mathcal{E}^k(v_\epsilon(0), \Gamma_\epsilon(0)) \leq e^{C(C_2(A)M)\epsilon j} \mathcal{E}^k(v_\epsilon(0), \Gamma_\epsilon(0)).$$

Hence we can reach the cutoff given by the first inequality in (8.41) no earlier than at time

$$t = \epsilon j < T(M) := C(C_2(A)M)^{-1},$$

which is a bound that does not depend on ϵ. Similarly, for the second requirement in (8.41), the relations (8.5) ensure that at each step the boundary only moves by $\mathcal{O}(\epsilon)$, so by step j it moves at most by $\mathcal{O}(j\epsilon)$. This leads to a similar constraint as above on the number of steps. Analogous reasoning shows that the vorticity growth in (8.6) is also harmless on this time scale.

To summarize, we have proved that the discrete approximate solutions $(v_\epsilon, \Gamma_\epsilon)$ are all defined up to the above time $T(M)$, and satisfy the uniform bound

$$\|(v_\epsilon, \Gamma_\epsilon)\|_{\mathbf{H}^k} \lesssim_M 1 \qquad \text{in } [0,T],$$

with $\Gamma_\epsilon \in 2\Lambda_*$. Since k is large enough, by Sobolev embeddings, this yields uniform bounds, say, in C^3,

$$\|v_\epsilon\|_{C^3} + \|\eta_\epsilon\|_{C^3} \lesssim_M 1 \qquad \text{in } [0,T], \tag{8.42}$$

where $\eta_\epsilon := \eta_{\Gamma_\epsilon}$ is the defining function for $\Gamma_\epsilon \in 2\Lambda_*$.

The other piece of information we have about v_ϵ comes from (8.5). However, this only tells us what happens over a single time step of size ϵ, so we need to iterate it over multiple steps. We begin with the first relation for the velocity in (8.5), which implies that

$$|v_\epsilon(t,x) - v_\epsilon(s,y)| + |\nabla v_\epsilon(t,x) - \nabla v_\epsilon(s,y)| \lesssim_M |t-s| + |x-y|, \qquad t-s=\epsilon.$$

Iterating this we arrive at

$$|v_\epsilon(t,x) - v_\epsilon(s,y)| + |\nabla v_\epsilon(t,x) - \nabla v_\epsilon(s,y)| \lesssim_M |t-s| + |x-y|, \qquad t,s \in \epsilon\mathbb{N} \cap [0,T]. \tag{8.43}$$

A similar reasoning based on the last part of (8.5) yields

$$\|\eta_\epsilon(t) - \eta_\epsilon(s)\|_{C^1} \lesssim_M |t-s|, \qquad t,s \in \epsilon\mathbb{N} \cap [0,T]. \tag{8.44}$$

Similarly, from (8.35) in Lemma 8.10 and the elliptic estimate $\|D_t p_\epsilon\|_{H^k} \lesssim_M 1$ for each time, we also get a difference bound for the pressure; namely,

$$|\nabla p_\epsilon(t,x) - \nabla p_\epsilon(s,y)| \lesssim_M |t-s| + |x-y|, \qquad t,s \in \epsilon\mathbb{N} \cap [0,T]. \tag{8.45}$$

Equipped with the last three Lipschitz bounds in time, we are now able to return to (8.5) and reiterate in order to obtain second order information. As above, we begin with the first relation in (8.5). Here we reiterate directly, using the bounds (8.43) and (8.45) in order to compare the expressions on the right at different times in the uniform norm. This yields

$$v_\epsilon(t) = v_\epsilon(s) - (t-s)(v_\epsilon(s)\cdot\nabla v_\epsilon(s) + \nabla p_\epsilon(s) + g e_d) + \mathcal{O}((t-s)^2), \qquad t,s \in \epsilon\mathbb{N} \cap [0,T]. \tag{8.46}$$

The same procedure applied to the last component of (8.5) yields

$$\Omega_\epsilon(t) = (I + (t-s)v_\epsilon(s))\Omega_\epsilon(s) + \mathcal{O}((t-s)^2), \qquad t,s \in \epsilon\mathbb{N} \cap [0,T]. \tag{8.47}$$

We now have enough information about our approximate solutions $(v_\epsilon, \Gamma_\epsilon)$, and we seek to obtain the desired solution (v, Γ) by taking the limit of $(v_\epsilon, \Gamma_\epsilon)$ on a subsequence as $\epsilon \to 0$. For this it is convenient to take ϵ of the form $\epsilon = 2^{-m}$, where we let $m \to \infty$. Then the time domains of the corresponding approximate solutions v_m are nested.

Starting from the Lipschitz bounds (8.43), (8.44) and (8.45), a careful application of the Arzela-Ascoli theorem yields uniformly convergent subsequences

$$\eta_m \to \eta, \qquad v_m \to v, \qquad \nabla v_m \to \nabla v, \qquad \nabla p_m \to \nabla p, \tag{8.48}$$

whose limits still satisfies the bounds (8.43), (8.44) and (8.45). It remains to show that (v, Γ) is the desired solution to the free boundary incompressible Euler equations, with Γ defined by η and p, where p is the associated pressure.

We begin by upgrading the spatial regularity of v and η. For this we observe that for $t \in 2^{-j}\mathbb{N} \cap [0,T]$ we can pass to the limit as $m \to \infty$ in (8.42) to obtain the uniform bound

$$\|v\|_{C^3} + \|\eta\|_{C^3} \lesssim_M 1.$$

Since both v and η are Lipschitz continuous in t, this extends easily to all $t \in [0,T]$. A similar argument applies to the $\mathbf{H}^k$ norm of (v,Γ).

Next we show that (v,Γ) solves the free boundary incompressible Euler equations, which we do in several steps:

i) The initial data. The fact that at the initial time we have $(v(0),\Gamma(0)) = (v_0,\Gamma_0)$ follows directly from the construction of $(v_\epsilon(0),\Gamma_\epsilon(0))$; namely, by Proposition 6.2.

ii) The pressure equation. To verify that p is the pressure associated to v and Γ we simply use the uniform convergence of ∇v_m, η_m and ∇p_m in order to pass to the limit in the pressure equation (1.5).

iii) The incompressible Euler equations. Here we directly use the uniform convergence (8.48) in order to pass to the limit in (8.46). This implies that v is differentiable in time, and that the incompressible Euler equations are verified.

iv) The kinematic boundary condition. Arguing as above, this time we directly use the uniform convergence (8.48) in order to pass to the limit in (8.47).

Finally, the $C(\mathbf{H}^{k-1})$ regularity of (v,Γ) follows directly from the incompressible Euler equations and the kinematic boundary condition. □

9. Rough solutions

In this section, we aim to construct solutions in the state space $\mathbf{H}^s$ as limits of regular solutions for $s > \frac{d}{2}+1$. The general procedure for executing this construction will be as follows.

(i) We regularize the initial data.
(ii) We prove uniform bounds for the corresponding regularized solutions.
(iii) We show convergence of the regularized solutions in a weaker topology.
(iv) We combine the difference estimates and the uniform $\mathbf{H}^s$ bounds from step (ii) to obtain convergence in the $\mathbf{H}^s$ topology.

As will be seen below, this procedure carries with it various subtleties since it involves comparing functions defined on different domains. In addition, we must carefully address the fact that our control parameters in the difference and energy estimates are not entirely consistent.

9.1. **Initial data regularization.** Let $(v_0,\Gamma_0) \in \mathbf{H}^s$ be an initial data. The first step is to place Γ_0 within a suitable collar $\Lambda_* = \Lambda(\Gamma_*,\epsilon,\delta)$ with $\delta \ll 1$. Since $\Gamma_0 \in H^s \subseteq C^{1,\epsilon+}$, Γ_* is easily obtained by regularizing Γ_0 on a small enough spatial scale. We remark that the price to pay for a small enough regularization scale is that the higher Sobolev norms H^k of Γ_* will be large; but this is acceptable, as explained in Remark 3.4.

Let $M := \|(v_0,\Gamma_0)\|_{\mathbf{H}^s}$ denote the data size measured relative to the collar Λ_*, and write c_0 for the lower bound on the Taylor term. We begin by constructing regularized data at each dyadic scale 2^j. For this, we define $\Gamma_{0,j}$ (along with $\Omega_{0,j}$) by regularizing the collar parameterization η_0. More specifically, we define $\eta_{0,j} := P_{\leq j}\eta_0$, where the meaning of $P_{\leq j}$ is as in Section 6. Then, we define the regularized velocity $v_{0,j} := \Psi_{\leq j} v_0$. Here, we recall that, as long as j is much larger than M, $v_{0,j}$ is defined on some 2^{-j} enlargement of both $\Omega_{0,j}$ and Ω_0. Indeed, by Sobolev embeddings, we have the distance bound

$$|\eta_{0,j}-\eta_0| \lesssim_M 2^{-\frac{3}{2}j}.$$

Moreover, for such j, we stay in the collar and have a uniform lower bound on the Taylor term.

9.2. **Uniform bounds and lifespan of regular solutions.** By Theorem 8.13, the regularized data $(v_{0,j},\Gamma_{0,j})$ from the previous step generate corresponding smooth solutions (v_j,Γ_j). Our goal now is to establish uniform bounds for these regular solutions and, in particular, show that they have a lifespan which depends only on the size of the initial data (v_0,Γ_0) in $\mathbf{H}^s$, Taylor sign and the collar. To do this, we carry out a bootstrap argument with the $\mathbf{H}^s$ norm of (v_j,Γ_j).

In the argument below, we will be working with the enlarged control parameter $\tilde{B}_j(t) := \|v_j\|_{W^{1,\infty}(\Omega_j)} + \|\Gamma_j\|_{C^{1,\frac{1}{2}}} + \|D_t p_j\|_{W^{1,\infty}(\Omega_j)}$ for the corresponding solution (v_j,Γ_j). Note that the reason we work for now with $\tilde{B}_j$ instead of just $B_j(t) := \|v_j\|_{W^{1,\infty}(\Omega_j)} + \|\Gamma_j\|_{C^{1,\frac{1}{2}}}$ is because we will make use of the difference estimates which require control of $D_t p_j$. By elliptic regularity and Sobolev embeddings, it is easy to see that $\tilde{B}_j$ is controlled by some polynomial in $\|(v_j,\Gamma_j)\|_{\mathbf{H}^s}$.

Fix some large parameters A_0 and B_0 depending only on the numerical constants for the data (M, c_0 and so forth) such that $A_0 \ll B_0$. As alluded to above, we make the bootstrap assumption

$$\|(v_j,\Gamma_j)(t)\|_{\mathbf{H}^s} \leq 2B_0, \quad A_j(t) \leq 2A_0, \quad a_j(t) \geq \frac{c_0}{2}, \quad \Gamma_j(t) \in 2\Lambda_*, \quad t\in[0,T], \quad j(M) =: j_0 \leq j \leq j_1,$$

with $j(M)$ sufficiently large depending on M, in a time interval $[0,T]$ where all the (v_j,Γ_j) are defined as smooth solutions with boundaries in the collar. Above, j_1 is some finite but arbitrarily large parameter, introduced for technical convenience to ensure that we run the bootstrap on only finitely many solutions at a time. Our aim will be to show that we can improve this bootstrap assumption as long as $T \leq T_0$ for some time $T_0 > 0$ which is independent of j_1.

For any large integer $k > s > \frac{d}{2}+1$ as in Theorem 8.13, we may consider the solutions (v_j,Γ_j) as solutions in $\mathbf{H}^k$. In light of Theorems 7.1 and 8.13, for each $j \geq j_0$, the solution (v_j,Γ_j) can be continued past time T in $\mathbf{H}^k$ (and therefore $\mathbf{H}^s$) as long as the bootstrap is satisfied. Morally speaking, our choice for T_0 will be

$$T_0 \ll \frac{1}{P(B_0)},$$

for some fixed polynomial P, though this is not entirely accurate, as T_0 will also depend on the collar and c_0. Thanks to the energy bound in Theorem 7.1, if the bootstrap could be extended to such a T_0, it would guarantee uniform $\mathbf{H}^k$ bounds for (v_j,Γ_j) for any integer $k > \frac{d}{2}+1$ in terms of its initial data in $\mathbf{H}^k$. The main difficulty we face is that, a priori, the $\mathbf{H}^s$ bounds for (v_j,Γ_j) do not necessarily propagate for noninteger s. The goal, therefore, is to establish $\mathbf{H}^s$ bounds for noninteger s. We will do this by working solely with the energy estimates for integer indices and the difference estimates.

We begin by letting c_j be the $\mathbf{H}^s$ admissible frequency envelope for the initial data (v_0, Γ_0) given by (6.3). We let $\alpha \geq 1$ be such that $k = s + \alpha$ is an integer. From Proposition 6.6 we know that the regularized data $(v_{0,j}, \Gamma_{0,j})$ satisfy the bounds

$$\|(v_{0,j}, \Gamma_{0,j})\|_{\mathbf{H}^{s+\alpha}} \lesssim_{A_0} 2^{\alpha j} c_j \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}. \tag{9.1}$$

From the energy bounds in Theorem 7.1 and the bootstrap hypothesis, we deduce from (9.1) and the definition of c_j that

$$\|(v_j, \Gamma_j)(t)\|_{\mathbf{H}^{s+\alpha}} \lesssim_{A_0} 2^{\alpha j} c_j (1 + \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}), \qquad t \in [0, T], \tag{9.2}$$

as long as $T \leq T_0 \ll \frac{1}{P(B_0)}$. One may think of this as a high frequency bound, which roughly speaking allows us to control frequencies $\gtrsim 2^j$ in (v_j, Γ_j). Note that in (9.2) we suppressed the implicit dependence on the Taylor term and the collar. We will do this throughout the subsection except when these terms are of primary importance, as it will be clear that our argument can handle these minor technicalities.

To estimate low frequencies we use the difference estimates. Precisely, at the initial time we claim that we have the difference bound

$$D((v_{0,j}, \Gamma_{0,j}), (v_{0,j+1}, \Gamma_{0,j+1})) \lesssim_{A_0} 2^{-2js} c_j^2 \|(v_0, \Gamma_0)\|^2_{\mathbf{H}^s}. \tag{9.3}$$

This bound is clear by Proposition 6.6 for the first term in (4.2). To see this for the surface integral, we use that on $\tilde{\Gamma}_{0,j} := \partial(\Omega_{0,j} \cap \Omega_{0,j+1})$, the pressure difference $p_{0,j} - p_{0,j+1}$ is proportional (with implicit constant depending on A_0) to the distance between $\Gamma_{0,j}$ and $\Gamma_{0,j+1}$, measured using the displacement function (4.1). Combining this with a change of variables, we have

$$\int_{\tilde{\Gamma}_{0,j}} |p_{0,j} - p_{0,j+1}|^2 \, dS \approx_{A_0} \|\eta_{0,j+1} - \eta_{0,j}\|^2_{L^2(\Gamma_*)} \lesssim_{A_0} 2^{-2js} c_j^2 \|(v_0, \Gamma_0)\|^2_{\mathbf{H}^s},$$

from which (9.3) follows. By Theorem 4.2, we can propagate the difference bound (9.3) to obtain

$$D((v_j, \Gamma_j)(t), (v_{j+1}, \Gamma_{j+1})(t)) \lesssim_{A_0} 2^{-2js} c_j^2 \|(v_0, \Gamma_0)\|^2_{\mathbf{H}^s}, \qquad t \in [0, T], \tag{9.4}$$

as long as $T \leq T_0 \ll \frac{1}{P(B_0)}$. In particular, this gives by a similar argument to the above,

$$\|v_{j+1} - v_j\|_{L^2(\Omega_j \cap \Omega_{j+1})}, \quad \|\eta_{j+1} - \eta_j\|_{L^2(\Gamma_*)} \lesssim_{A_0} 2^{-js} c_j \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}. \tag{9.5}$$

Now, the goal is to combine the high frequency bound (9.2) and the L^2 difference bound (9.5) in order to obtain a uniform $\mathbf{H}^s$ bound of the form

$$\|(v_j, \Gamma_j)\|_{\mathbf{H}^s} \lesssim_{A_0} 1 + \|(v_0, \Gamma_0)\|_{\mathbf{H}^s},$$

for $T \leq T_0$. To establish such a bound for Γ_j, we consider the telescoping series on Γ_* given by

$$\eta_j = \eta_{j_0} + \sum_{j_0 \leq l \leq j-1} (\eta_{l+1} - \eta_l). \tag{9.6}$$

From the higher energy bound (9.2), we have for each $j_0 \leq l \leq j - 1$,

$$\|\eta_{l+1} - \eta_l\|_{H^{s+\alpha}(\Gamma_*)} \lesssim_{A_0} 2^{l\alpha} c_l (1 + \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}). \tag{9.7}$$

Using the telescoping sum and interpolation, it is straightforward to verify from (9.5), (9.7) and an argument similar to Proposition 6.6 (see also [28]) that for each $k \geq 0$,

$$\|P_k \eta_j\|_{H^s(\Gamma_*)} \lesssim_{A_0} c_k (1 + \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}). \tag{9.8}$$

As a consequence, by almost orthogonality, we obtain the uniform bound

$$\|\Gamma_j\|_{H^s} \lesssim_{A_0} 1 + \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}. \tag{9.9}$$

Next, we turn to the bound for v_j. We first note that the analogous decomposition to (9.6) for v_j does not work because for each $l \leq j-1$, v_l and v_{l+1} are defined on different domains. However, we can compare v_l and v_{l+1} by first regularizing each function $v_l \mapsto \Psi_{\leq l} v_l$ which is defined on a 2^{-l} enlargement of Ω_l. For this comparison to work, we need to know that Γ_j and Γ_{j+1} are sufficiently close. By interpolating using (9.5) and (9.9) we have

$$\|\eta_{j+1} - \eta_j\|_{L^\infty(\Gamma_*)} \lesssim_{A_0} 2^{-\frac{3}{2}j}, \qquad \|\eta_{j+1} - \eta_j\|_{C^{1,\frac{1}{2}}(\Gamma_*)} \lesssim_{A_0} 2^{-\delta j}, \tag{9.10}$$

for some $\delta > 0$. Now, we return to the uniform bound for v_j. Thanks to (9.10), we can safely consider the decomposition on Ω_j,

$$v_j = \Psi_{\leq j_0} v_{j_0} + \sum_{j_0 \leq l \leq j-1} \Psi_{\leq l+1} v_{l+1} - \Psi_{\leq l} v_l + (I - \Psi_{\leq j}) v_j. \tag{9.11}$$

The first term in the telescoping decomposition is trivial to bound. We therefore focus our attention on the remaining terms. First, define for $l \geq j_0$

$$\tilde{\Omega}_l = \bigcap_{k=l}^{j} \Omega_k.$$

Thanks again to (9.10), for j_0 large enough (independent of j and only depending on the data parameters), we can arrange for the regularization operator $\Psi_{\leq l}$ to be bounded from $H^s(\tilde{\Omega}_l)$ to $H^s(\tilde{\Omega}'_l)$ where $\tilde{\Omega}'_l$ is some 2^{-l} enlargement of the union of all of the Ω_k for $k \geq l$. We will use this fact to establish the following lemma which will help us to estimate the intermediate terms in (9.11).

Lemma 9.1. *Let $j_0 \leq l \leq j-1$, where j_0 is some universal parameter depending only on the numerical constants for the data. Then given the above decomposition for v_j, we have*

$$\|\Psi_{\leq l+1} v_{l+1} - \Psi_{\leq l} v_l\|_{L^2(\Omega_j)} \lesssim_{A_0} 2^{-ls} c_l (1 + \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}), \tag{9.12}$$

$$\|\Psi_{\leq l+1} v_{l+1} - \Psi_{\leq l} v_l\|_{H^{s+\alpha}(\Omega_j)} \lesssim_{A_0} 2^{l\alpha} c_l (1 + \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}). \tag{9.13}$$

By Sobolev embedding, a corollary of this lemma is the following pointwise bound at the C^1 regularity.

Corollary 9.2. *We have the estimate*

$$\|\Psi_{\leq l+1} v_{l+1} - \Psi_{\leq l} v_l\|_{C^1(\Omega_j)} \lesssim_{A_0} 2^{-l\delta} (1 + \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}), \qquad \delta > 0.$$

Proof. The latter bound (9.13) is clear from the $H^{s+\alpha}$ boundedness of $\Psi_{\leq l}$ and (9.2). For the first bound, we split

$$\Psi_{\leq l+1} v_{l+1} - \Psi_{\leq l} v_l = (\Psi_{\leq l+1} - \Psi_{\leq l}) v_{l+1} + \Psi_{\leq l}(v_{l+1} - v_l).$$

Using Proposition 6.2 and (9.2), we have

$$\|(\Psi_{\leq l+1} - \Psi_{\leq l}) v_{l+1}\|_{L^2(\Omega_j)} \lesssim_{A_0} 2^{-ls} c_l (1 + \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}).$$

For the remaining term, we use the difference bound and the L^2 boundedness of $\Psi_{\leq l}$ to obtain

$$\|\Psi_{\leq l}(v_{l+1} - v_l)\|_{L^2(\Omega_j)} \lesssim_{A_0} D((v_l, \Gamma_l), (v_{l+1}, \Gamma_{l+1}))^{\frac{1}{2}} \lesssim_{A_0} 2^{-ls} c_l \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}.$$

□

We also observe that the same bounds in Lemma 9.1 hold for the third term in (9.11) but with the parameter l replaced by j in the corresponding estimates. This is immediate for (9.13) and follows by telescopic summation from Proposition 6.6 in the case of (9.12).

We can use the above lemma (and the corresponding bounds for $(I-\Psi_{\leq j})v_j$) to estimate similarly to (9.8) that for each $k \geq 0$,

$$\|P_k v_j\|_{H^s(\mathbb{R}^d)} \lesssim_{A_0} c_k(1+\|(v_0,\Gamma_0)\|_{\mathbf{H}^s}),$$

where we carefully note here that for each $k \geq 0$, P_k should be interpreted as $P_k E_{\Omega_j}$ where E_{Ω_j} is the extension operator on Ω_j from Proposition 5.12. From this observation and almost orthogonality, we obtain the desired uniform bound,

$$\|(v_j,\Gamma_j)(t)\|_{\mathbf{H}^s} \lesssim_{A_0} 1+\|(v_0,\Gamma_0)\|_{\mathbf{H}^s},$$

for $t \in [0,T_0]$. In particular, if the constant B_0 is chosen to be sufficiently large relative to A_0 and the data size, this improves the bootstrap assumption for $\|(v_j,\Gamma_j)\|_{\mathbf{H}^s}$. It remains to improve the bootstrap assumption for A_j and at the same time the Taylor term and the collar neighborhood size. For this we rely on a computation similar to [17, 41] for the Lagrangian flow map $u_j(t,\cdot):\Omega_{0,j}\to\Omega_j(t)$, defined as the solution to the ODE

$$\partial_t u_j(t,y) = v_j(t,u_j(t,y)), \qquad y\in\Omega_{0,j}, \qquad u_j(0)=I.$$

Since $s > \frac{d}{2}+1$, if T_0 is small enough, then for any $0\leq t\leq T\leq T_0$ we have the bound

$$\begin{aligned}\|u_j(t,\cdot)-I\|_{H^s(\Omega_{0,j})} &\lesssim \int_0^t \|v_j(t',\cdot)\|_{H^s(\Omega_j(t'))}\|u_j(t',\cdot)\|^s_{H^s(\Omega_{0,j})}\,dt' \\ &\lesssim_{A_0} t\|(v_0,\Gamma_0)\|_{\mathbf{H}^s}.\end{aligned}$$

If A_0 is large enough relative to the data size, this easily implies simultaneously

$$\Gamma_j(t)\in\frac{3}{2}\Lambda_*, \qquad \|\Gamma_j(t)\|_{C^{1,\epsilon}} \ll A_0,$$

as long as T_0 is small enough. Doing a similar computation with u_t in place of u and using the equation

$$\partial_t^2 u_j(t,y) = \partial_t(v_j(t,u_j(t,y))) = -(\nabla p_j + ge_d)(t,u_j(t,y))$$

together with the elliptic estimates for the pressure, we obtain also

$$\|v_j(t)\|_{C^{\frac{1}{2}+\epsilon}(\Omega_j)} \ll A_0.$$

This improves the bootstrap assumption for A_j. Finally, a similar argument but instead with the pressure gradient and the H^s bound for $D_t p$ allows one to close the bootstrap for a_j as long as T_0 is sufficiently small depending on M and c_0.

9.3. **The limiting solution.** Here we show that for $T\leq T_0$,

$$(v,\Gamma) = \lim_{j\to\infty}(v_j,\Gamma_j) \quad \text{in} \quad C([0,T];\mathbf{H}^s).$$

First, we show domain convergence in H^s, which is more straightforward. Indeed, from (9.10) we see that the limiting domain Ω exists and has Lipschitz boundary Γ. Next, we let $j\geq j_0$ and consider the telescoping sum

$$\eta-\eta_j = \sum_{l=j}^{\infty}\eta_{l+1}-\eta_l.$$

An analysis similar to the previous subsection, using the difference bounds and the higher energy bounds, yields

$$\|\eta - \eta_j\|_{L^\infty(\Gamma_*)} \lesssim_{A_0} 2^{-\frac{3}{2}j} \tag{9.14}$$

and

$$\|\eta - \eta_j\|_{C([0,T];H^s(\Gamma_*))} \lesssim_{A_0} \|c_{\geq j}\|_{l^2}(1 + |(v_0, \Gamma_0)\|_{\mathbf{H}^s}),$$

which in particular shows convergence of $\Gamma_j \to \Gamma$ in $C([0,T]; H^s(\Gamma_*))$. Next, we turn to showing the convergence $v_j \to v$ in $C([0,T]; \mathbf{H}^s)$. We, formally, define v through the telescoping sum

$$v = \Psi_{\leq j_0} v_{j_0} + \sum_{l \geq j_0} \Psi_{\leq l+1} v_{l+1} - \Psi_{\leq l} v_l,$$

where, as usual, j_0 ensures that all the terms in the sum are defined on Ω. Thanks to (9.14), this is possible. We begin by showing that $\Psi_{\leq j} v_j \to v$ in $H^s(\Omega_t)$ uniformly in t (which is again unambiguous thanks to (9.14)). We have

$$v - \Psi_{\leq j} v_j = \sum_{l \geq j} \Psi_{\leq l+1} v_{l+1} - \Psi_{\leq l} v_l.$$

From this we see that

$$\|v - \Psi_{\leq j} v_j\|_{H^s(\Omega_t)} \lesssim_{A_0} \|c_{\geq j}\|_{l^2}(1 + \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}),$$

which establishes the desired uniform convergence in $H^s(\Omega_t)$. To show convergence of v_j in the sense of Definition 3.5, we consider the regularization $\tilde{v} = \Psi_{\leq m} v_m$. We then have as above,

$$\|v - \Psi_{\leq m} v_m\|_{H^s(\Omega)} \lesssim_{A_0} \|c_{\geq m}\|_{l^2}(1 + \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}),$$

which goes to 0 as $m \to \infty$. On the other hand, for $j > m$, we have

$$\begin{aligned}\|v_j - \Psi_{\leq m} v_m\|_{H^s(\Omega_j)} \lesssim_{A_0} &\|(1 - \Psi_{\leq j}) v_j\|_{H^s(\Omega_j)} + \|\Psi_{\leq j}(v_j - v)\|_{H^s(\Omega_j)} + \|\Psi_{\leq m}(v_m - v)\|_{H^s(\Omega_j)} \\ &+ \|\Psi_{\leq j} v - \Psi_{\leq m} v\|_{H^s(\Omega_j)}.\end{aligned}$$

Using (9.2) for the first term and the difference bounds for $D((v_j, \Gamma_j), (v, \Gamma))$, $D((v_m, \Gamma_m), (v, \Gamma))$ for the second and third terms, respectively, we obtain

$$\|v_j - \Psi_{\leq m} v_m\|_{H^s(\Omega_j)} \lesssim_{A_0} \|c_{\geq m}\|_{l^2}(1 + \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}) + \|\Psi_{\leq j} v - \Psi_{\leq m} v\|_{H^s(\Omega_j)}.$$

To estimate the last term above, we have

$$\begin{aligned}\|\Psi_{\leq j} v - \Psi_{\leq m} v\|_{H^s(\Omega_j)} &\lesssim_{A_0} \|(\Psi_{\leq j} - \Psi_{\leq m})(v - \Psi_{\leq m} v_m)\|_{H^s(\Omega_j)} + \|(\Psi_{\leq j} - \Psi_{\leq m})\Psi_{\leq m} v_m\|_{H^s(\Omega_j)} \\ &\lesssim_{A_0} \|v - \Psi_{\leq m} v_m\|_{H^s(\Omega)} + 2^{-m\alpha}\|v_m\|_{H^{s+\alpha}(\Omega_m)} \\ &\lesssim_{A_0} \|c_{\geq m}\|_{l^2}(1 + \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}),\end{aligned}$$

where we used (9.2) to estimate the second term in the last inequality. The combination of the above estimates establishes strong convergence in $\mathbf{H}^s$. A similar argument shows continuity of v with values in $\mathbf{H}^s$. Finally, one may also check that the limiting solution solves the free boundary Euler equations.

9.4. **Continuous dependence.** Given a sequence of initial data $(v_0^n, \Gamma_0^n) \in \mathbf{H}^s$ such that $(v_0^n, \Gamma_0^n) \to (v_0, \Gamma_0)$, we aim to show that we have the corresponding convergence of the solutions $(v^n, \Gamma^n) \to (v, \Gamma)$ in $C([0,T]; \mathbf{H}^s)$. First, we note that thanks to the data convergence, the corresponding solutions have a uniform in n lifespan in $\mathbf{H}^s$, and so, on some compact time interval $[0,T]$, we have $\|(v^n,\Gamma^n)\|_{\mathbf{H}^s} + \|(v,\Gamma)\|_{\mathbf{H}^s} \lesssim_M 1$. Let us denote by c_j^n and c_j the admissible frequency envelopes for the data (v_0^n, Γ_0^n) and (v_0, Γ_0), respectively. Now, let $\epsilon > 0$ and let $\delta = \delta(\epsilon) > 0$ be a small positive constant to be chosen. Moreover, let $n_0 = n_0(\epsilon)$ be some large integer to be chosen.

By definition of convergence in $\mathbf{H}^s$, there is a divergence free function $v_0^\delta \in H^s(\Omega_0^\delta)$ defined on some enlarged domain Ω_0^δ such that

$$\|v_0 - v_0^\delta\|_{H^s(\Omega_0)} + \limsup_{n\to\infty} \|v_0^n - v_0^\delta\|_{H^s(\Omega_0^n)} < \delta.$$

Moreover, for n large enough, depending only on δ, v_0^δ is defined on a neighborhood of Ω_0 and Ω_0^n. Moreover, we may also assume that v_0^δ belongs to $H^s(\mathbb{R}^d)$. Indeed, for some $\delta' \ll \delta$, v_0^δ is defined on the domain Ω_0' defined by taking $\eta_0' = \eta_0 + \delta'$. Then we can extend v_0^δ to $\mathbb{R}^d$ using Proposition 5.12. We note that v_0^δ is not necessarily divergence free on $\mathbb{R}^d$ but is on an enlargement of Ω_0 and Ω_0^n for n large enough. Now, let c_j^δ denote the admissible frequency envelope for (v_0^δ, Γ_0) (note that we are using the same domain Ω_0 as v_0 for the frequency envelope here; if δ is small enough, Taylor sign holds for this state) and denote by $(v^\delta, \Gamma^\delta)$ the corresponding $\mathbf{H}^s$ solution (which we note has lifespan comparable to v and v^n for n large enough). We begin by choosing $j = j(\epsilon)$ large enough so that

$$\|c_{\geq j}\|_{l^2} < \epsilon. \tag{9.15}$$

We next observe that we can choose $\delta(\epsilon)$ and then $n_0(\delta)$ so that

$$\|c^n_{\geq j}\|_{l^2} \lesssim_M \epsilon + \|c_{\geq j}\|_{l^2} \lesssim_M \epsilon, \tag{9.16}$$

for $n \geq n_0$. One can establish this by estimating the error when comparing terms in c_j^δ and c_j^n and then the error when comparing terms in c_j^δ and c_j by using (6.3) and square summing. The main error in the first comparison is essentially comprised of two terms. The first term to control involves the error between η_0^n and η_0. If δ is small enough and n is large enough, we have

$$\|\eta_0^n - \eta_0\|_{H^s(\Gamma_*)} < \delta < \epsilon.$$

The second source of error comes from the extensions of the velocity functions,

$$\|E_{\Omega_0^n} v_0^n - E_{\Omega_0} v_0^\delta\|_{H^s(\mathbb{R}^d)} \leq \|E_{\Omega_0^n} v_0^\delta - E_{\Omega_0} v_0^\delta\|_{H^s(\mathbb{R}^d)} + \|E_{\Omega_0^n}(v_0^n - v_0^\delta)\|_{H^s(\mathbb{R}^d)}.$$

If $\delta \ll_M \epsilon$, then the latter term is $\mathcal{O}(\epsilon)$ by (uniform in n) boundedness of $E_{\Omega_0^n}$ and the definition of v_0^δ. The first term is $\mathcal{O}(\epsilon)$ if n is large enough (relative to δ) thanks to the continuity property of the family $E_{\Omega_0^n}$ in Proposition 5.12. Then one establishes (9.16) by comparing c_j and c_j^δ which just involves controlling essentially the error term $\|E_{\Omega_0}(v_0^\delta - v_0)\|_{H^s(\mathbb{R}^d)}$.

Now that we have uniform smallness of the initial data frequency envelopes, the next step is to compare the corresponding solutions. First, thanks to the difference estimates, we observe that for large enough n, Γ^n and Γ^δ are within distance $\ll 2^{-j}$ as long as δ is chosen small enough relative to j (recall that j was chosen to ensure (9.15)). Indeed, by interpolating and using the uniform $\mathbf{H}^s$ bound, we have

$$\|\eta^n - \eta^\delta\|_{L^\infty(\Gamma_*)} \lesssim_M D((v^n,\Gamma^n),(v^\delta,\Gamma^\delta))^{\frac{3}{4s}} \lesssim_M \delta^{\frac{3}{2s}}.$$

This ensures that we may compare $\Psi_{\leq j} v^\delta$ to v^n. Denoting by (v_j^n, Γ_j^n) the regular solution corresponding to the regularized data $(v_{0,j}^n, \Gamma_{0,j}^n)$ (from the previous section), we have

$$\begin{aligned}\|\Psi_{\leq j} v^\delta - v^n\|_{H^s(\Omega^n)} &\lesssim \|\Psi_{\leq j}(v^\delta - v^n)\|_{H^s(\Omega^n)} + \|\Psi_{\leq j}(v^n - v_j^n)\|_{H^s(\Omega^n)} + \|v^n - \Psi_{\leq j} v_j^n\|_{H^s(\Omega^n)} \\ &\lesssim_M \|c_{\geq j}^n\|_{l^2} + 2^{js} D((v^n, \Gamma^n), (v_j^n, \Gamma_j^n))^{\frac{1}{2}} + 2^{js} D((v^n, \Gamma^n), (v^\delta, \Gamma^\delta))^{\frac{1}{2}} \\ &\lesssim_M \|c_{\geq j}^n\|_{l^2} + 2^{js} D((v^n, \Gamma^n), (v^\delta, \Gamma^\delta))^{\frac{1}{2}},\end{aligned}$$

which if δ is small enough gives

$$\|\Psi_{\leq j} v^\delta - v^n\|_{H^s(\Omega^n)} \lesssim_M \epsilon.$$

Similarly, we may obtain

$$\|\eta^n - \eta\|_{H^s(\Gamma_*)} \lesssim_M \epsilon$$

and

$$\|\Psi_{\leq j} v^\delta - v\|_{H^s(\Omega)} \lesssim_M \epsilon.$$

This establishes continuous dependence.

9.5. **Lifespan of rough solutions.** Here, we finally establish the continuation criterion from Theorem 1.7 for $\mathbf{H}^s$ solutions. We consider initial data $(v_0, \Gamma_0) \in \mathbf{H}^s$ and the corresponding solution (v, Γ) in a time interval $[0, T)$ which has the property that

$$\mathcal{C} := \sup_{0 \leq t < T} A(t) + \int_0^T B(t)\, dt < \infty, \qquad a(t) \geq c_0 > 0, \qquad t \in [0, T),$$

and whose domains Ω_t maintain a uniform thickness. Unlike with the construction of rough solutions, we now work with the weaker control parameter

$$B(t) = \|v\|_{W^{1,\infty}(\Omega_t)} + \|\Gamma_t\|_{C^{1,\frac{1}{2}}}.$$

One starting difficulty we face in this proof is that we do not a priori have a fixed reference collar neighborhood. However, the uniform bound on $A(t)$ guarantees that the free boundaries Γ_t are uniformly of class $C^{1,\epsilon}$, and the uniform bound on v guarantees that they move at most with velocity $\mathcal{O}(1)$. This implies that the limiting boundary $\Gamma_T = \lim_{t \to T} \Gamma_t$ exists in the uniform topology, and also belongs to $C^{1,\epsilon}$, with the corresponding domain Ω_T having positive thickness. Furthermore, by interpolation, it follows that

$$\lim_{t \to T} \Gamma_t = \Gamma_T \qquad \text{in } C^{1,\epsilon_1}, \qquad 0 < \epsilon_1 < \epsilon.$$

This allows us define the reference boundary Γ_* as a regularization of Γ_T, so that $\Gamma_T \in \Lambda(\Gamma_*, \epsilon/2, \delta/4)$ for an acceptable choice of δ ensuring that $\Lambda(\Gamma_*, \epsilon/2, \delta/2)$ is also a well-defined collar (cf. Remark 3.4). Then the above convergence implies that $\Gamma_t \in \Lambda_* := \Lambda(\Gamma_*, \epsilon/2, \delta/2)$ for t close to T.

Reinitializing the starting time close to T, we arrive at the case where we have the initial data $(v_0, \Gamma_0) \in \mathbf{H}^s$ and the corresponding solution (v, Γ) in a time interval $[0, T)$ with the property that

$$\Gamma_t \in \Lambda_*, \qquad t \in [0, T).$$

From the local well-posedness theorem, it suffices to show that

$$\|(v, \Gamma)\|_{L^\infty([0,T); \mathbf{H}^s)} < \infty. \tag{9.17}$$

Similarly to the previous subsections, the strategy we would like to employ will involve showing that the control parameters for a suitable family of regularized solutions (v_j, Γ_j) can be controlled to leading order by the control parameters for (v, Γ). The main difficulty is that v_j and v are defined on different domains.

As in the previous sections, as long as we can ensure that Γ_j and Γ are within distance $2^{-j(1+\delta)}$ of each other, we can compare v with $\Psi_{\leq j} v_j$. However, there is one added difficulty now. The difference bound, which ensured the closeness of domains in the previous sections, has a stronger control parameter involving the term $\|D_t p\|_{W^{1,\infty}(\Omega_t)}$ in addition to $B(t)$, which from Lemma 7.9 has size controlled by $B(t)$ and an additional logarithmic factor.

To overcome this, we will divide $[0,T)$ into two disjoint intervals $[0,\tilde{T}]$ and $[\tilde{T},T)$ where $0<\tilde{T}<T$ and $\tilde{T}$ has the property that

$$\int_{\tilde{T}}^{T} B(t)\,dt < \delta_0,$$

where δ_0 is some parameter to be chosen depending only on $\mathcal{C}$, c_0, the collar and the $\mathbf{H}^s$ norm of (v_0,Γ_0). Given such a $\tilde{T}$, we consider the regularized data $(v_{\tilde{T},j},\Gamma_{\tilde{T},j})$ of $(v(\tilde{T}),\Gamma_{\tilde{T}})$ and the corresponding solutions (v_j,Γ_j). We remark that $\tilde{T}$ and δ_0 need to be chosen carefully to not depend on j, but we postpone this choice for now. Their purpose is to guarantee that the stronger control parameter $D_t p$ in the difference bounds as well as the logarithmic factor in the energy bounds does not cause the distance between Γ_j and Γ to grow larger than $2^{-j(1+\delta)}$ for times $t<T$ where (v_j,Γ_j) is defined.

From the continuous dependence result, the above regularized solutions converge to (v,Γ) in $[\tilde{T},T)$ and their lifespans T_j satisfy

$$\liminf_{j\to\infty} T_j \geq T-\tilde{T}.$$

However, a priori, we do not have a uniform L^1_T bound on their corresponding control parameters B_j, nor a uniform L^∞_T bound on A_j, nor a uniform lower bound on the corresponding Taylor terms a_j. Arguing similarly to the previous subsections, if such bounds could be established, one could hope to use them to establish a uniform $\mathbf{H}^s$ bound on the regularized solutions (v_j,Γ_j) and hence extend their time of existence by an amount uniform in j. To establish such uniform control on these pointwise parameters, we will run a relatively simple bootstrap argument. From here on, we write $M:=\|(v_0,\Gamma_0)\|_{\mathbf{H}^s}$ and $M_{\tilde{T}}:=\|(v(\tilde{T}),\Gamma_{\tilde{T}})\|_{\mathbf{H}^s}$. To set up the bootstrap, we begin by noting that at time $\tilde{T}$, we have by Sobolev embedding and interpolation, the bound

$$\|\eta_j(\tilde{T})-\eta(\tilde{T})\|_{C^{1,\epsilon}(\Gamma_*)} \lesssim 2^{-\frac{j}{2}} M_{\tilde{T}}. \tag{9.18}$$

Moreover, by the properties of $\Psi_{\leq j}$, we have $\|v_j(\tilde{T})\|_{C^{\frac{1}{2}+\epsilon}} \lesssim_{\mathcal{C}} 1$. Hence, initially we have

$$A_j(\tilde{T}) \leq P(\mathcal{C}) + 2^{-\frac{j}{2}} M_{\tilde{T}} \tag{9.19}$$

where $P>1$ is some sufficiently large positive polynomial. As long as the choice of $\tilde{T}$ we make later on depends only on $\mathcal{C}$ and c_0 (but not on j), we can arrange by taking j large enough, the initial bound

$$A_j(\tilde{T}) \leq 2P(\mathcal{C}). \tag{9.20}$$

Finally, if j is large enough, and $\tilde{T}$ is as above, we also initially have (for instance),

$$a_j(\tilde{T}) \geq \frac{2}{3}c_0.$$

Now, we make the bootstrap assumption that on a time interval $[\tilde{T},T_0]$ with $\tilde{T}<T_0<T$ we have the bounds

$$\int_{\tilde{T}}^{T_0} B_j(t)\,dt < 4C_1(A)\delta_0, \qquad A_j(t)\leq 4P(\mathcal{C}), \qquad a_j(t)\geq \frac{1}{2}c_0, \qquad \Gamma_j(t)\in 2\Lambda_* \tag{9.21}$$

for $j \geq j_0(M, T_0)$ and some large universal constant $C_1 \gg 1$ depending only on $A := \sup_{t\in[0,T)} A(t)$. Our goal will be to show that the constant $4C_1\delta_0$ can be improved to $2C_1\delta_0$ and the constant $4P(\mathcal{C})$ can be improved to $2P(\mathcal{C})$, with similar improvements on the Taylor term and the collar. After we close this boostrap, we will give a separate argument which uses the uniform bounds on the control parameters to establish a uniform bound for (v_j, Γ_j) in $\mathbf{H}^s$, and hence permit us to continue the solution. To close the above bootstrap, we aim to establish the bounds

$$B_j \leq C_1(A)B + C_2 2^{-\delta j}, \quad A_j \leq P(\mathcal{C}) + C_2 2^{-\delta j}, \quad a_j \geq \frac{2}{3}c_0, \quad \Gamma_j(t) \in \frac{3}{2}\Lambda_*, \tag{9.22}$$

where $\delta > 0$ is some small positive constant and C_2 depends on the size of $M_{\tilde{T}}$ as well as the constant $\mathcal{C}$ above. The bootstrap can then be closed by choosing j_0 large enough to absorb the contribution of C_2.

As mentioned above, the main difficulty in comparing B_j with B and A_j with A is, as usual, the fact that the corresponding domains Ω_j and Ω are different. Our starting point is to select the parameter δ_0 and the time $\tilde{T}(\delta_0)$ to ensure that Ω_j and Ω are close enough. As mentioned above, in order for our argument not to be circular, we need to ensure that the choice of δ_0 depends only on c_0 and $\mathcal{C}$. Our first aim is to obtain some preliminary bounds for $\eta_j - \eta_{j+1}$ in L^∞ and $C^{1,\frac{1}{2}}$. We let k be the smallest integer larger than s. First, by the double exponential bound in Theorem 7.1 and the bootstrap hypothesis, we have for each j,

$$\|(v_j, \Gamma_j)\|_{\mathbf{H}^k}^2 \lesssim_A \exp\left(\exp(K\delta_0)\log(K(1 + 2^{2j(k-s)}\|(v(\tilde{T}), \Gamma_{\tilde{T}})\|_{\mathbf{H}^s}^2)\right).$$

Above, K is some (possibly large) constant depending on $\mathcal{C}$ and c_0 which we will let change from line to line. In the above estimate, if we take $K\delta_0 \ll 1$ (in particular, δ_0 does not depend on j), then we can arrange for

$$\|(v_j, \Gamma_j)\|_{\mathbf{H}^k}^2 \lesssim K 2^{2j(k-s)} M_{\tilde{T}}^2 (M_{\tilde{T}} 2^j)^\delta \tag{9.23}$$

for some small constant $\delta > 0$, where we assumed without loss of generality that $M_{\tilde{T}} \geq 1$ to simplify notation. Note here that there is a slight loss compared to (9.2) coming from the double exponential bound in the energy estimate. On the other hand, the difference estimates, Lemma 7.9 and the energy coercivity ensures that by Grönwall and the bootstrap assumption, we have

$$D((v_j, \Gamma_j), (v_{j+1}, \Gamma_{j+1})) \lesssim 2^{-2js} K M_{\tilde{T}}^2 \exp\left(K\delta_0 \mathcal{I}_j\right),$$

where $\mathcal{I}_j = \sup_{\tilde{T}<t\leq T_0}(\log(K + KE^k(v_j, \Gamma_j)) + \log(K + KE^k(v_{j+1}, \Gamma_{j+1})))$ and k is, again, the smallest integer larger than s. By the higher energy bound and the bootstrap assumption, we have

$$\mathcal{I}_j \lesssim K \log(1 + 2^{2jk}\|(v(\tilde{T}), \Gamma_{\tilde{T}})\|_{L^2}^2) \lesssim_k Kj,$$

where we used the higher energy bound for the regularized solution to propagate $\log(1 + E^k(v_j, \Gamma_j))$ and control $\log(1+E^k(v_j, \Gamma_j))$ by $\log(1+2^{2kj}\|(v(\tilde{T}), \Gamma_{\tilde{T}})\|_{L^2}^2)$ as well as the fact that the volume of Ω_t is conserved and Hölder's inequality to estimate $\|(v(\tilde{T}), \Gamma_{\tilde{T}})\|_{L^2} \lesssim_A 1$. Again, we choose δ_0 small enough (and therefore $\tilde{T}$) depending only on $\mathcal{C}$ and c_0 so that

$$\exp(K\delta_0 \mathcal{I}_j) \leq 2^{j\delta},$$

for some sufficiently small $\delta > 0$ (depending only on s). Next, we pick j_0 depending on $M_{\tilde{T}}$, $\mathcal{C}$ and c_0 so that if $j \geq j_0$ (after possibly relabelling δ), we have

$$D((v_j, \Gamma_j), (v_{j+1}, \Gamma_{j+1})) \lesssim 2^{-2j(s-\delta)}, \qquad \|(v_j, \Gamma_j)\|_{\mathbf{H}^k}^2 \lesssim 2^{2j(k-s)} 2^{j\delta}$$

with universal implicit constant. The key point to observe here is that there is now a slight loss in the difference estimates and energy estimates compared to the previous subsections because of the stronger control parameter in the difference bounds and the logarithmic factor in the energy estimates. However, by using these estimates, we still obtain by Sobolev embedding and interpolating, the bounds (after possibly relabelling δ)

$$\|\eta_j - \eta_{j+1}\|_{C^{1,\frac{1}{2}}(\Gamma_*)} \lesssim 2^{-\delta j}, \quad \|\eta_j - \eta_{j+1}\|_{C^{1,\epsilon}(\Gamma_*)} \lesssim 2^{-\frac{1}{2}j}, \quad \|\eta_j - \eta_{j+1}\|_{L^\infty(\Gamma_*)} \lesssim 2^{-\frac{3}{2}j}, \tag{9.24}$$

all with universal implicit constant if j_0 is large enough. The first bound will give us control of $\|\Gamma_j\|_{C^{1,\frac{1}{2}}}$ in the first estimate in (9.22). The second bound above gives us control over $\|\Gamma_j\|_{C^{1,\epsilon}}$ for the second estimate in (9.22) and also shows that $\Gamma_j \in \frac{3}{2}\Lambda_*$. The third bound ensures that Γ_j and Γ_{j+1} are sufficiently close. With this closeness established, we now work towards closing the bootstrap (9.22) for the $\|v_j\|_{W^{1,\infty}(\Omega_j)}$ component of B_j and the $\|v_j\|_{C^{\frac{1}{2}+\epsilon}(\Omega_j)}$ component of A_j. We show the details for $\|v_j\|_{W^{1,\infty}(\Omega_j)}$ as the other component is very similar. We estimate in three steps. First, we observe that from the bounds for $\Psi_{\leq j}$, we have

$$\|\Psi_{\leq j} v\|_{W^{1,\infty}} \lesssim_A B. \tag{9.25}$$

We can ensure that the implicit constant in this estimate is less than $C_1(A)$ if $C_1(A)$ is initially chosen large enough. Then we compare $\Psi_{\leq j} v$ and $\Psi_{\leq j} v_j$ which is justified thanks to (9.24). We have

$$\Psi_{\leq j} v - \Psi_{\leq j} v_j = \sum_{l \geq j} \Psi_{\leq j} v_{l+1} - \Psi_{\leq j} v_l.$$

By Sobolev embedding and a similar argument to the $C^{1,\frac{1}{2}}$ bound for $\eta_{j+1} - \eta_j$, we see that

$$\|\Psi_{\leq j} v_{l+1} - \Psi_{\leq j} v_l\|_{W^{1,\infty}} \leq C_2 2^{-l\delta},$$

which gives by summation

$$\|\Psi_{\leq j} v - \Psi_{\leq j} v_j\|_{W^{1,\infty}} \leq C_2 2^{-j\delta}. \tag{9.26}$$

Using the error bound for $I - \Psi_{\leq j}$, Sobolev embedding and the higher energy bounds, we also have

$$\|\Psi_{\leq j} v_j - v_j\|_{W^{1,\infty}} \leq C_2 2^{-j\delta}. \tag{9.27}$$

Combining (9.25), (9.26) and (9.27) shows that

$$\|v_j\|_{W^{1,\infty}(\Omega_j)} \leq C_1(A)B + C_2 2^{-j\delta}.$$

Doing a similar estimate for $\|v_j\|_{C^{\frac{1}{2}+\epsilon}(\Omega_j)}$ and taking j large enough allows us to close the bootstrap for A_j.

It remains now to improve the bootstrap assumption for the Taylor term a_j. To do this, we need a suitable way of comparing the C^1 norms of the pressures p_j and p. We begin by defining the shrunken domain Ω' via $\eta' := \eta - 2^{-j_0}$. As Ω_j is within distance $\mathcal{O}(2^{-\frac{3}{2}j})$ of Ω for $j \geq j_0$, it follows that

$$\Omega' \subset \Omega \cap \bigcap_{j \geq j_0} \Omega_j.$$

We next note the following bound which holds on Ω' for any $0 < \delta < \frac{\epsilon}{2}$,

$$\|v_j - v\|_{C^{\frac{1}{2}+\delta}(\Omega')} \leq C_2 2^{-j_0\delta}. \tag{9.28}$$

This follows by similar reasoning to the above. Now, we establish the following C^1 estimate for $p - p_j$:

$$\|p - p_j\|_{C^1(\Omega')} \leq C_2 2^{-j_0\delta}. \tag{9.29}$$

We begin by splitting $p - p_j$ into an inhomogeneous part plus a harmonic part on Ω',

$$p - p_j = \Delta^{-1}\Delta(p - p_j) + \mathcal{H}(p - p_j).$$

Using Proposition 5.15, the dynamic boundary condition and the fact that the boundary of Ω' is within distance 2^{-j_0} of the boundaries of Ω and Ω_j, we have

$$\|\mathcal{H}(p-p_j)\|_{C^{1,\delta}(\Omega')} \lesssim_{\mathcal{C}} 2^{-j_0\delta}(\|p\|_{C^{1,\epsilon}(\Omega)} + \|p_j\|_{C^{1,\epsilon}(\Omega_j)}).$$

By Lemma 7.5 and the bootstrap assumption on A_j, this gives

$$\|\mathcal{H}(p-p_j)\|_{C^{1,\delta}(\Omega')} \lesssim_{\mathcal{C}} 2^{-j_0\delta}.$$

To estimate the inhomogeneous part, we can argue similarly to the proof of Lemma 7.5 using a bilinear frequency decomposition for $\Delta(p_j - p)$, to obtain

$$\|\Delta^{-1}\Delta(p-p_j)\|_{C^1(\Omega')} \lesssim_{\mathcal{C}} \|v - v_j\|_{C^{\frac{1}{2}+\delta}(\Omega')} \leq C_2 2^{-j_0\delta},$$

where in the second inequality we used (9.28). Finally, to close the bootstrap on the Taylor term a_j, we can work in collar coordinates on Γ_* to estimate

$$\inf_{x\in\Gamma_j} |\nabla p_j(x)| \geq \inf_{x\in\Gamma} |\nabla p(x)| - \|p_j - p\|_{C^1(\Omega')} - 2^{-j_0\delta}(\|p_j\|_{C^{1,\epsilon}(\Omega_j)} + \|p\|_{C^{1,\epsilon}(\Omega)}).$$

In the above, we first estimate the error between $\nabla p_j(x + \eta_j(x)\nu(x))$ and $\nabla p_j(x + \eta'(x)\nu(x))$ (and also $\nabla p(x + \eta'(x)\nu(x))$ and $\nabla p(x + \eta(x)\nu(x))$) using the $C^{1,\epsilon}$ Hölder regularity of p_j and p. Then, we estimate the difference between $\nabla p_j(x + \eta'(x)\nu(x))$ and $\nabla p(x + \eta'(x)\nu(x))$ on the common domain using our bounds for $\|p_j - p\|_{C^1(\Omega')}$.

Taking j_0 large enough and using (9.29) and Lemma 7.5, this gives

$$a_j \geq \frac{2}{3}c_0,$$

which closes the bootstrap for a_j.

From the above argument, we see that for $j \geq j_0$, the regular solutions (v_j, Γ_j) are defined on the interval $[\tilde{T}, T]$ and satisfy the assumptions (9.21). What we do not yet know is whether we have a uniform in j bound for the $\mathbf{H}^s$ norm of (v_j, Γ_j). Once we have this, (9.17) will follow from our continuous dependence result. From here on, we assume without loss of generality that $M_{\tilde{T}} \gg C(A)$. We let c_j denote the frequency envelope for the data at time $\tilde{T}$. Similarly to the above, on a time interval $[\tilde{T}, T_0]$, we make the bootstrap assumption that for finitely many $j \geq j_0$,

$$\|(v_j, \Gamma_j)\|_{\mathbf{H}^s} \leq M_{\tilde{T}}^2. \tag{9.30}$$

As in the previous subsection, we let $\alpha \geq 1$ be such that $s + \alpha$ is an integer. Then the higher energy bounds, (9.30) and (9.21) yield

$$\|(v_j, \Gamma_j)\|_{\mathbf{H}^{s+\alpha}} \lesssim 2^{j\alpha} c_j \exp(K\delta_0 \log(M_{\tilde{T}}^2)) M_{\tilde{T}}$$

where K is some constant depending on $\mathcal{C}$. As long as δ_0 is such that $K\delta_0 \ll 1$, we obtain

$$\|(v_j, \Gamma_j)\|_{\mathbf{H}^{s+\alpha}} \lesssim 2^{j\alpha} c_j M_{\tilde{T}}^{1+\delta} \tag{9.31}$$

for some positive constant $\delta \ll 1$. A similar argument with the difference bounds yields

$$D((v_j, \Gamma_j), (v_{j+1}, \Gamma_{j+1}))^{\frac{1}{2}} \lesssim 2^{-js} c_j M_{\tilde{T}}^{1+\delta}.$$

Arguing as in the local well-posedness result, we can use the above two bounds to estimate

$$\|(v_j, \Gamma_j)\|_{\mathbf{H}^s} \lesssim M_{\tilde{T}}^{1+\delta},$$

which improves the bootstrap. We are then able to finally conclude the bound (9.17) and thus the proof of Theorem 1.7.

References

[1] Albert Ai. Low regularity solutions for gravity water waves. *Water Waves*, 1(1):145–215, 2019.

[2] Albert Ai. Low regularity solutions for gravity water waves II: The 2D case. *Ann. PDE*, 6(1):Paper No. 4, 117, 2020.

[3] Albert Ai, Mihaela Ifrim, and Daniel Tataru. Two dimensional gravity waves at low regularity I: Energy estimates. *arXiv preprint arXiv:1910.05323*, 2019.

[4] T. Alazard, N. Burq, and C. Zuily. On the water-wave equations with surface tension. *Duke Math. J.*, 158(3):413–499, 2011.

[5] T. Alazard, N. Burq, and C. Zuily. On the Cauchy problem for gravity water waves. *Invent. Math.*, 198(1):71–163, 2014.

[6] Thomas Alazard and Jean-Marc Delort. Sobolev estimates for two dimensional gravity water waves. *Astérisque*, pages viii+241, 2015.

[7] Mustafa Sencer Aydin, Igor Kukavica, Wojciech S Ożański, and Amjad Tuffaha. Construction of the free-boundary 3D incompressible Euler flow under limited regularity. *arXiv preprint arXiv:2307.02201*, 2023.

[8] J. T. Beale, T. Kato, and A. Majda. Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Comm. Math. Phys.*, 94(1):61–66, 1984.

[9] Klaus Beyer and Matthias Günther. On the Cauchy problem for a capillary drop. I. Irrotational motion. *Math. Methods Appl. Sci.*, 21(12):1149–1183, 1998.

[10] Jean-Michel Bony. Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. *Ann. Sci. École Norm. Sup. (4)*, 14(2):209–246, 1981.

[11] Jean Bourgain and Dong Li. Strong ill-posedness of the incompressible Euler equation in borderline Sobolev spaces. *Invent. Math.*, 201(1):97–157, 2015.

[12] Jean Bourgain and Dong Li. Strong illposedness of the incompressible Euler equation in integer C^m spaces. *Geom. Funct. Anal.*, 25(1):1–86, 2015.

[13] Angel Castro and David Lannes. Well-posedness and shallow-water stability for a new Hamiltonian formulation of the water waves equations with vorticity. *Indiana Univ. Math. J.*, 64(4):1169–1270, 2015.

[14] Demetrios Christodoulou and Hans Lindblad. On the motion of the free surface of a liquid. *Comm. Pure Appl. Math.*, 53(12):1536–1602, 2000.

[15] Daniel Coutand and Steve Shkoller. Well-posedness of the free-surface incompressible Euler equations with or without surface tension. *J. Amer. Math. Soc.*, 20(3):829–930, 2007.

[16] Thibault De Poyferré. Blow-up conditions for gravity water-waves. *arXiv preprint arXiv:1407.6881*, 2014.

[17] Thibault de Poyferré. A priori estimates for water waves with emerging bottom. *Arch. Ration. Mech. Anal.*, 232(2):763–812, 2019.

[18] Marcelo M. Disconzi, Mihaela Ifrim, and Daniel Tataru. The relativistic Euler equations with a physical vacuum boundary: Hadamard local well-posedness, rough solutions, and continuation criterion. *Arch. Ration. Mech. Anal.*, 245(1):127–182, 2022.

[19] Marcelo M. Disconzi, Igor Kukavica, and Amjad Tuffaha. A Lagrangian interior regularity result for the incompressible free boundary Euler equation with surface tension. *SIAM J. Math. Anal.*, 51(5):3982–4022, 2019.

[20] G. Dziuk and C. M. Elliott. Finite elements on evolving surfaces. *IMA J. Numer. Anal.*, 27(2):262–292, 2007.

[21] David G. Ebin. The equations of motion of a perfect fluid with free boundary are not well posed. *Comm. Partial Differential Equations*, 12(10):1175–1201, 1987.

[22] Tarek M. Elgindi and Nader Masmoudi. L^∞ ill-posedness for a class of equations arising in hydrodynamics. *Arch. Ration. Mech. Anal.*, 235(3):1979–2025, 2020.

[23] E. B. Fabes, M. Jodeit, Jr., and N. M. Rivière. Potential techniques for boundary value problems on C^1-domains. *Acta Math.*, 141(3-4):165–186, 1978.

[24] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[25] Daniel Ginsberg. On the breakdown of solutions to the incompressible Euler equations with free surface boundary. *SIAM J. Math. Anal.*, 53(3):3366–3384, 2021.

[26] John K. Hunter, Mihaela Ifrim, and Daniel Tataru. Two dimensional water waves in holomorphic coordinates. *Comm. Math. Phys.*, 346(2):483–552, 2016.

[27] Mihaela Ifrim and Daniel Tataru. The compressible Euler equations in a physical vacuum: a comprehensive Eulerian approach. *arXiv preprint arXiv:2007.05668*, 2020.

[28] Mihaela Ifrim and Daniel Tataru. Local well-posedness for quasi-linear problems: a primer. *Bull. Amer. Math. Soc. (N.S.)*, 60(2):167–194, 2023.

[29] David Jerison and Carlos E. Kenig. The inhomogeneous Dirichlet problem in Lipschitz domains. *J. Funct. Anal.*, 130(1):161–219, 1995.

[30] V. Kreĭg and K. E. Veĭn. Mathematical aspects of surface waves on water. *Uspekhi Mat. Nauk*, 62(3(375)):95–116, 2007.

[31] Igor Kukavica and Amjad Tuffaha. A sharp regularity result for the Euler equation with a free interface. *Asymptot. Anal.*, 106(2):121–145, 2018.

[32] David Lannes. Well-posedness of the water-waves equations. *J. Amer. Math. Soc.*, 18(3):605–654, 2005.

[33] Hans Lindblad. Well-posedness for the motion of an incompressible liquid with free surface boundary. *Ann. of Math. (2)*, 162(1):109–194, 2005.

[34] Alessandra Lunardi. *Interpolation theory*, volume 16 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]*. Edizioni della Normale, Pisa, 2018. Third edition.

[35] William McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000.

[36] Guy Métivier. *Para-differential calculus and applications to the Cauchy problem for nonlinear systems*, volume 5 of *Centro di Ricerca Matematica Ennio De Giorgi (CRM) Series*. Edizioni della Normale, Pisa, 2008.

[37] Mei Ming and Chao Wang. Water waves problem with surface tension in a corner domain I: A priori estimates with constrained contact angle. *SIAM J. Math. Anal.*, 52(5):4861–4899, 2020.

[38] Mei Ming and Chao Wang. Local well-posedness of the capillary-gravity water waves with acute contact angles. *arXiv preprint arXiv:2112.14001*, 2021.

[39] Mei Ming and Chao Wang. Water-waves problem with surface tension in a corner domain II: the local well-posedness. *Comm. Pure Appl. Math.*, 74(2):225–285, 2021.

[40] Dorina Mitrea, Irina Mitrea, and Marius Mitrea. *Geometric Harmonic Analysis V: Fredholm Theory and Finer Estimates for Integral Operators, with Applications to Boundary Problems*. Springer, 2023.

[41] Jalal Shatah and Chongchun Zeng. Geometry and a priori estimates for free boundary problems of the Euler equation. *Comm. Pure Appl. Math.*, 61(5):698–744, 2008.

[42] Jalal Shatah and Chongchun Zeng. A priori estimates for fluid interface problems. *Comm. Pure Appl. Math.*, 61(6):848–876, 2008.

[43] Jalal Shatah and Chongchun Zeng. Local well-posedness for fluid interface problems. *Arch. Ration. Mech. Anal.*, 199(2):653–705, 2011.

[44] Elias M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[45] Gregory Verchota. Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains. *J. Funct. Anal.*, 59(3):572–611, 1984.

[46] Chao Wang and Zhifei Zhang. Break-down criterion for the water-wave equation. *Sci. China Math.*, 60(1):21–58, 2017.

[47] Chao Wang, Zhifei Zhang, Weiren Zhao, and Yunrui Zheng. Local well-posedness and break-down criterion of the incompressible Euler equations with free boundary. *Mem. Amer. Math. Soc.*, 270(1318):v + 119, 2021.

[48] Ian Wood. Maximal L^p-regularity for the Laplacian on Lipschitz domains. *Math. Z.*, 255(4):855–875, 2007.

[49] Sijue Wu. Well-posedness in Sobolev spaces of the full water wave problem in 2-D. *Invent. Math.*, 130(1):39–72, 1997.

[50] Sijue Wu. Well-posedness in Sobolev spaces of the full water wave problem in 3-D. *J. Amer. Math. Soc.*, 12(2):445–495, 1999.

[51] Ping Zhang and Zhifei Zhang. On the free boundary problem of three-dimensional incompressible Euler equations. *Comm. Pure Appl. Math.*, 61(7):877–940, 2008.

Department of Mathematics, University of Wisconsin - Madison
Email address: ifrim@wisc.edu

Department of Mathematics, University of California at Berkeley
Email address: bpineau@berkeley.edu

Department of Mathematics, University of California at Berkeley
Email address: tataru@math.berkeley.edu

Department of Mathematics, ETH Zürich
Email address: mitchell.taylor@math.ethz.ch