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EXAMPLES OF HÖLDER-STABLE PHASE RETRIEVAL

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Abstract. Examples are constructed of infinite-dimensional subspaces $V \subset L^2(\mu)$ with the property that for any $f, g \in V$, if $|f|$ is approximately equal to $|g|$ with respect to the L^2 norm, then there exists a unimodular scalar z such that f is approximately equal to zg.

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let V be a closed subspace of the (real or complex) Hilbert space $L^2 = L^2(\mu)$. Calderbank, Daubechies, Freeman, and Freeman [3] have studied real subspaces of real-valued L^2 for which there exists $C < \infty$ satisfying

$$\min\big(\|f-g\|_{L^2}, \, \|f+g\|_{L^2}\big) \le C\big\| \, |f| - |g| \, \big\|_{L^2} \; \forall \, f, g \in V, \tag{1}$$

and have constructed the first examples of such infinite-dimensional subspaces. In this situation, if $|f|$ is known then f is uniquely determined almost everywhere up to an unavoidably arbitrary global phase factor of ± 1; if $|f|$ is known within a small tolerance in norm then up to such a global phase factor, f is determined within a correspondingly small tolerance. This issue arises for instance in crystallography, where one seeks to recover an unknown function $F \in L^2(\mathbb{R})$ from the absolute value of its Fourier transform $\widehat{F}$. Upon substituting $f = \widehat{F}$ and $g = \widehat{G}$, then invoking Plancherel's theorem to express $\|F \pm G\|_{L^2} = \|\widehat{F} \pm \widehat{G}\|_{L^2} = \|f \pm g\|_{L^2}$ and $\big\||\widehat{F}| - |\widehat{G}|\big\|_{L^2} = \big\||f| - |g|\big\|_{L^2}$, the inequality (1) expresses a desirable stability in the recovery of F from $|\widehat{F}|$.

There is an extensive literature concerning phase retrieval, that is, determination of f up to unavoidable ambiguity from $|f|$ under supplementary conditions, and concerning phase retrieval for finite-dimensional subspaces. Phase recovery for infinite-dimensional subspaces has been shown to be unstable in general by Cahill, Casazza, and Daubechies [2] and by Alaifari and Grohs [1]. We refer to Grohs et. al. [4] for an expository article on phase recovery, and to Calderbank et. al. [3] for an introduction to the specific topic of stability for infinite-dimensional subspaces. The present note develops simple and natural examples of infinite-dimensional subspaces in which versions of stable phase retrieval hold. These examples include certain variants of Rademacher series and lacunary Fourier series.

For complex-valued functions, the natural quantity on the left-hand side of the inequality (1) becomes $\min_{|z|=1} \|f - zg\|_{L^2}$, with the minimum taken over all complex numbers z of modulus 1. Stable phase retrieval for complex-valued functions is defined in [3] as

$$\min_{|z|=1} \|f - zg\|_{L^2} \le C\big\| \, |f| - |g| \, \big\|_{L^2} \; \forall \, f, g \in V. \tag{2}$$

We generalize the stable phase retrieval inequality in the following way.

Definition 1. *Let V be an infinite-dimensional subspace of the complex Hilbert space $L^2(\mu)$ for some measure μ. Let $p \in [1, \infty]$. We say that V satisfies L^p-Hölder-stable phase retrieval*

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if $V \subset L^p(\mu)$ and there exist parameters $\gamma \in (0,1]$ and $C < \infty$ such that

$$\min_{|z|=1} \|f - zg\|_{L^p} \le C\big\| \, |f| - |g| \, \big\|_{L^p}^{\gamma} \cdot (\|f\|_{L^2} + \|g\|_{L^2})^{1-\gamma} \quad \forall \, f, g \in V. \tag{3}$$

We say that V satisfies L^p-Lipschitz-stable phase retrieval if (3) *holds with $\gamma = 1$.*

We will write L^p-Hölder-SPR and L^p-Lipschitz-SPR. For real Hilbert spaces $L^2(\mu, \mathbb{R})$, this definition is modified by replacing $\{z \in \mathbb{C} : |z| = 1\}$ by $\{\pm 1\}$.

Only the exponents $p = 2, 4$ arise in the examples below.

Let μ be a probability measure. Consider an orthonormal subset $\{r_j : j \in \mathbb{N}\}$ of the complex Hilbert space $L^2 = L^2(\mu) = L^2(\mu, \mathbb{C})$. Let $V \subset L^2$ be the closure of the span of $\{r_j\}$ over $\mathbb{C}$. Let $\mathbf{1}$ be the function $\mathbf{1}(x) \equiv 1$. Define associated functions

$$s_j = |r_j|^2 - \mathbf{1}. \tag{4}$$

In the case of $L^2(\mu, \mathbb{C})$, we consider closed subspaces spanned by orthogonal sets $\{r_j : j \in \mathbb{N}\}$ satisfying the following three hypotheses:

$$\big\{\mathbf{1}, \, s_i, \, r_j \overline{r_k} : i, j, k \in \mathbb{N} \text{ and } j \ne k\big\} \text{ is an orthogonal set.} \tag{5}$$

$$\sup_j \|r_j\|_{L^4} < \infty. \tag{6}$$

$$\text{There exists } \delta > 0 \text{ such that } \inf_i \|r_i\|_4^4 \ge 1 + \delta \text{ and } \inf_{j \ne k} \|r_j \overline{r_k}\|_2^2 \ge \delta. \tag{7}$$

Since $\|s_i\|_2^2 = \|r_i\|_4^4 - 2\|r_i\|_2^2 + 1 = \|r_i\|_4^4 - 1$ by the hypothesis that $\|r_i\|_2 = 1$, the first part of hypothesis (7) can be equivalently restated as $\|s_i\|_2^2 \ge \delta > 0$.

A consequence of these hypotheses is that $V \subset L^4$ and there exists $C < \infty$ such that

$$\|f\|_{L^4} \le C\|f\|_{L^2} \ \forall \, f \in V. \tag{8}$$

Indeed, if $f = \sum_k a_k r_k$ with $(a_k : k \in \mathbb{N}) \in \ell^2$ then $|f|^2$ is represented as the pairwise orthogonal sum

$$|f|^2 = \sum_{i \ne j} a_i \overline{a_j} r_i \overline{r_j} + \sum_k |a_k|^2 s_k + \|f\|_2^2 \cdot \mathbf{1}. \tag{9}$$

The L^4 norm bound follows using orthogonality and the Cauchy-Schwarz inequality, since $\|r_i \overline{r_j}\|_2 \le \|r_i\|_4 \|r_j\|_4$ and $\|s_k\|_2 \le 1 + \|r_k^2\|_2 \le 1 + \|r_k\|_4^2$ are uniformly bounded by (6). The inequality (8), and a similar L^6 norm inequality that holds under stronger hypotheses, are pillars of our reasoning.

We begin by observing that under the hypotheses of Proposition 1, $|f|$ determines f uniquely, up to multiplication by a unimodular complex scalar, for each $f \in V$. Indeed, $|f|$ certainly determines f if $|f| = 0$ almost everywhere. Consider next any $0 \ne f \in V$. Expand $f = \sum_k a_k r_k$, with $a \in \ell^2$. Then $|f|^2 \in L^2$, and has expansion (9). The terms of this sum are mutually orthogonal, and the series converges in L^2 norm. Therefore $|f|^2$ determines each of the coefficients in this expansion; it determines each $|a_n|^2$ and each product $a_i \overline{a_j}$. Choose some n_0 satisfying $a_{n_0} \ne 0$. Writing $a_n = |a_n| e^{i \arg(a_n)}$, $\arg(a_n) - \arg(a_{n_0})$ is determined modulo $2\pi\mathbb{Z}$ by $|a_n|^2$, $|a_{n_0}|^2$, and $a_n \overline{a_{n_0}}$. Therefore $|f|^2$ and $\arg(a_{n_0})$ together determine all coefficients a_n, and hence determine f, up to multiplication by $z = e^{i \arg(a_{n_0})}$.

Note that this reconstruction of f from $|f|$ is not stable in the sense desired, since it requires division by $|a_{n_0}|$, for which no *a priori* positive lower bound is available. Note also that it exploits only the coefficients of s_k and of $r_n \overline{r}_{n_0}$.

The next result asserts that under these same hypotheses, the reconstruction of f from $|f|$ can be done stably.

Proposition 1. *Let μ be a probability measure. Let $\{r_j\} \subset L^2(\mu, \mathbb{C})$ be an orthonormal set of complex-valued functions satisfying hypotheses* (5),(6),(7)*. Then V satisfies L^4-Lipschitz-SPR.*

Under a supplementary hypothesis, Proposition 1 has an immediate implication for L^2-Hölder-stable phase retrieval.

Corollary 2. *Let $\{r_n\}$ satisfy the hypotheses of Proposition 1. Assume also that there exist $q > 4$ and $C < \infty$ such that $V \subset L^q(\mu)$ and*

$$\|f\|_{L^q} \le C\|f\|_{L^2} \ \forall\, f \in V. \tag{10}$$

Then V satisfies L^2-Hölder-SPR with exponent $\gamma = \frac{q-4}{2q-4}$.

As is well known, for any even integer $q \ge 6$, the inequality (10) holds whenever the functions r_j are independent random variables, have uniformly bounded L^q norms, and satisfy $r_j \perp \mathbf{1}$. Indeed, consider the case $q = 6$. If $\|r_n\|_6 \le A < \infty$ for all n then

$$\begin{aligned}\|\sum_n a_n r_n\|_6^6 &= \sum_{i_1,i_2,i_3} \sum_{j_1,j_2,j_3} \prod_{k=1}^3 a_{i_k} \prod_{l=1}^3 \overline{a_{j_l}} \left\langle r_{i_1} r_{i_2} r_{i_3},\, r_{j_1} r_{j_2} r_{j_3} \right\rangle \\ &\le \sum_n |a_n|^6 A^6 + \binom{6}{2} A^6 \sum_m \sum_n |a_m|^4 |a_n|^2 + \binom{6}{3} A^6 \sum_m \sum_n |a_m|^3 |a_n|^3\end{aligned}$$

since $\left\langle r_{i_1} r_{i_2} r_{i_3},\, r_{j_1} r_{j_2} r_{j_3} \right\rangle = 0$ unless each of the six indices that appear in the inner product, appears at least twice. The same reasoning applies for arbitrary even integers $q \ge 8$.

We next present a class of examples of Hölder-phase retrieval, based on Proposition 1 and Corollary 2. It involves sums of independent random variables, and may be contrasted with a more elaborate construction in [3], which combines independent summands with summands having pairwise disjoint supports.

Example 1. *Let μ be a probability measure. Let r_n be independent identically distributed complex-valued random variables in $L^6(\mu)$ satisfying $\|r_n\|_{L^2} = 1$. Assume that*

$$r_n \perp \mathbf{1} \text{ and } r_n^2 \perp \mathbf{1} \tag{11}$$

$$\mu(\{x : |r_n(x)| \ne 1\}) > 0. \tag{12}$$

Then $\{r_n\}$ satisfies the hypotheses of Proposition 1 and of Corollary 2 with $q = 6$. Therefore the closure of its span in $L^2(\mu)$ satisfies L^4-Lipschitz-SPR, and satisfies L^2-Hölder-SPR with exponent $\gamma = \frac{1}{4}$. If $r_n \in L^q$ for every $q < \infty$ then L^2-Hölder-SPR holds with any exponent $\gamma < \frac{1}{2}$.

Example 1 does not apply to Rademacher series, for which $r_n = \pm 1$ each with probability $\frac{1}{2}$, violating hypothesis (12). Nor do Rademacher series satisfy phase retrieval, since $|r_m| \equiv |r_n|$ for all m, n.

In the formulation of Example 1, the hypothesis $r_n^2 \perp \mathbf{1}$, together with independence, ensure that $r_i \overline{r_j} \perp r_j \overline{r_i}$ whenever $i \ne j$, since

$$\langle r_i \overline{r_j},\, r_j \overline{r_i} \rangle = \int r_i^2\, \overline{r_j}^2 \, d\mu = \int r_i^2 \, d\mu \cdot \overline{\int r_j^2 \, d\mu} = \langle r_i^2, \mathbf{1} \rangle \cdot \overline{\langle r_j^2, \mathbf{1} \rangle} = 0.$$

The hypothesis that $|r_n|$ is not equal almost everywhere to 1 ensures that $\|s_n\|_2 \ne 0$. The other hypotheses of Proposition 1, and the embedding of V into L^6, are consequences of independence, identical distribution, and the assumption that $r_n \perp \mathbf{1}$. Details of the verifications are left to the reader. □

Before indicating other classes of examples with stable phase retrieval, we prove Corollary 2 and Proposition 1.

Proof of Corollary 2. By Hölder's inequality,

$$\||f|-|g|\|_4 \le \||f|-|g|\|_2^\theta (\|f\|_q + \|g\|_q)^{1-\theta} \le C\||f|-|g|\|_2^\theta (\|f\|_2 + \|g\|_2)^{1-\theta}$$

where $\theta \in (0,1)$ is defined by the relation $\frac{1}{4} = \frac{\theta}{2} + \frac{1-\theta}{q}$. Therefore for any $f, g \in V$, by Hölder's inequality and Proposition 1,

$$\min_{|z|=1} \|f - zg\|_2 \le \min_{|z|=1} \|f - zg\|_4 \le C'\||f|-|g|\|_4 \le C''\||f|-|g|\|_2^\theta (\|f\|_2 + \|g\|_2)^{1-\theta}.$$

□

The proof of Proposition 1 relies on the following elementary inequality, which is proved below.

Lemma 3. *Let $\{r_j\}$ satisfy hypotheses* (5), (6), *and* (7). *For any $f, g \in V$,*

$$\||f|^2 - |g|^2\|_2^2 \ge \delta\Big[\|f\|_2^2\|g\|_2^2 - |\langle f, g\rangle|^2\Big] + (\|f\|_2^2 - \|g\|_2^2)^2. \tag{13}$$

Proof of Proposition 1. By multiplying by scalars and interchanging the roles of f, g if necessary, we may assume with no loss of generality that $\|f\|_2 \le \|g\|_2 = 1$. By Hölder's inequality,

$$\begin{aligned} \||f|^2 - |g|^2\|_2 &\le \big\| \, |f| + |g| \, \big\|_4 \cdot \big\| \, |f| - |g| \, \big\|_4 \\ &\le C(\|f\|_2 + \|g\|_2) \big\| \, |f| - |g| \, \big\|_4 \le 2C \big\| \, |f| - |g| \, \big\|_4. \end{aligned} \tag{14}$$

Write $f = re^{i\theta}g + h$ with $r \ge 0$, $\theta \in \mathbb{R}$, and $h \perp g$. Then $|\langle f, g\rangle|^2 = r^2$ and

$$\|f\|_2^2\|g\|_2^2 - |\langle f, g\rangle|^2 = (r^2 + \|h\|_2^2) - r^2 = \|h\|_2^2. \tag{15}$$

Let $\delta \in (0,1]$ be a parameter for which the conclusion (13) of Lemma 3 holds.

Inserting (15) into (13) gives

$$\delta\|h\|_2^2 + (1 - r^2 - \|h\|_2^2)^2 \le \||f|^2 - |g|^2\|_2^2 \le 4C^2\||f| - |g|\|_4^2.$$

Therefore since $0 < \delta \le 1$,

$$\delta\|h\|_2^2 + \tfrac{1}{4}\delta(1 - r^2 - \|h\|_2^2)^2 \le 4C^2\||f| - |g|\|_4^2.$$

The left-hand side is

$$(\delta - \tfrac{1}{2}\delta(1 - r^2))\|h\|_2^2 + \tfrac{1}{4}\delta(1 - r^2)^2 + \tfrac{1}{4}\delta\|h\|_2^4 \ge \tfrac{1}{2}\delta\|h\|_2^2 + \tfrac{1}{4}\delta(1 - r^2)^2$$

and therefore since $(1 - r) \le (1 - r^2)$,

$$\|h\|_2^2 + (1 - r)^2 \le 16C^2\delta^{-1}\||f| - |g|\|_4^2.$$

Defining $z = e^{i\theta}$, $\|f - zg\|_2^2 = \|h\|_2^2 + (1 - r)^2$ and therefore

$$\|f - zg\|_2^2 \le 16C^2\delta^{-1}\||f| - |g|\|_4^2.$$

Since $f - zg \in V$, its L^4 norm is majorized by a constant multiple of its L^2 norm. Thus $\|f - zg\|_4 \le C'\||f| - |g|\|_4$ for another finite constant C' which depends on δ. □

Proof of Lemma 3. Under the hypothesis that $\|r_j\|_2 = 1$, $\|s_j\|_2^2 = \|r_j\|_4^4 - 1$. Therefore the hypothesis $\inf_j \|r_j\|_4^4 \geq 1 + \delta$ is equivalent to $\inf_j \|s_j\|_2^2 \geq \delta$.

Express $f, g \in V$ as $f = \sum_k a_k r_k$ and $g = \sum_k b_k r_k$. By (9),

$$|f|^2 - |g|^2 = \sum_{i \neq j} (a_i \overline{a_j} - b_i \overline{b_j}) r_i \overline{r_j} + (\|f\|_2^2 - \|g\|_2^2)\mathbf{1} + \sum_k (|a_k|^2 - |b_k|^2) s_k \tag{16}$$

where $\mathbf{1}$ is the constant function 1. The functions $\mathbf{1}$, s_k, and $r_i\overline{r_j}$ with $i \neq j$ are pairwise orthogonal by hypothesis (5). Therefore

$$\begin{aligned} \||f|^2 - |g|^2\|_2^2 &= \sum_k ||a_k|^2 - |b_k|^2|^2 \|s_k\|_2^2 + (\|f\|_2^2 - \|g\|_2^2)^2 + \sum_{i \neq j} |a_i \overline{a_j} - b_i \overline{b_j}|^2 \|r_i \overline{r_j}\|_2^2 \\ &\geq \delta \sum_k ||a_k|^2 - |b_k|^2|^2 + (\|f\|_2^2 - \|g\|_2^2)^2 + \delta \sum_{i \neq j} |a_i \overline{a_j} - b_i \overline{b_j}|^2 \end{aligned} \tag{17}$$

by hypothesis (7).

Algebraic manipulation of the last term on the right-hand side gives

$$\begin{aligned} \sum_{i \neq j} |a_i \overline{a_j} - b_i \overline{b_j}|^2 &= (\sum_k |a_k|^2)^2 + (\sum_k |b_k|^2)^2 - 2|\sum_k a_k \overline{b_k}|^2 - \sum_k (|a_k|^2 - |b_k|^2)^2. \\ &= \|f\|_2^4 + \|g\|_2^4 - 2|\langle f, g\rangle|^2 - \sum_k (|a_k|^2 - |b_k|^2)^2 \\ &= 2\Big[\|f\|_2^2\|g\|_2^2 - |\langle f, g\rangle|^2\Big] + (\|f\|_2^2 - \|g\|_2^2)^2 - \sum_k (|a_k|^2 - |b_k|^2)^2. \end{aligned}$$

Substituting this expression into the preceding lower bound, two terms cancel, leaving

$$\begin{aligned} \||f|^2 - |g|^2\|_2^2 &\geq 2\delta\Big[\|f\|_2^2\|g\|_2^2 - |\langle f, g\rangle|^2\Big] + (1+\delta)(\|f\|_2^2 - \|g\|_2^2)^2 \\ &\geq 2\delta\Big[\|f\|_2^2\|g\|_2^2 - |\langle f, g\rangle|^2\Big] + (\|f\|_2^2 - \|g\|_2^2)^2. \end{aligned}$$

□

A well-known theme is the analogy between lacunary Fourier series and sums of independent random variables. Our next two examples express this theme.

Example 2. *Let $N \geq 2$ and let $P \in L^2([0,1], \mathbb{C})$ be a trigonometric polynomial*

$$P(x) = \sum_{k=1}^{N} \alpha_k e^{2\pi i k x}$$

with coefficients $\alpha_k \in \mathbb{C}$. Suppose that $|P|$ is not constant. Let $A \in \mathbb{N}$ satisfy $A > 3N$. Let $V \subset L^2([0,1], \mathbb{C})$ be the closure of the span of $\{P(A^n x) : n \in \mathbb{N}\}$. Then V satisfies L^4-Lipschitz-SPR, and satisfies L^2-Hölder-SPR with any exponent $\gamma < \frac{1}{2}$.

Example 2 is an instance of Corollary 2, with arbitrarily large $q < \infty$. Verification of the hypotheses of the corollary is left to the reader. The L^q norm inequality (10) holds since $\sum_{n=1}^{\infty} a_n \sum_{k=1}^{N} \alpha_k e^{2\pi i A^n k x}$ is a sum of N lacunary Fourier series, and since any lacunary series with ℓ^2 coefficients defines a function in L^q for all $q < \infty$.

Example 3. *The closure of the subspace of $L^2([0,1], \mathbb{R})$ spanned by $\{\sin(2\pi 4^n x) : n \in \mathbb{N}\}$ satisfies L^4-Lipschitz-SPR. It satisfies L^2-Hölder-SPR with any exponent $\gamma < \frac{1}{2}$.*

Example 5, below, is a more efficient version of Example 3.

A series $f(x) = \sum_n a_n \sin(2\pi 4^n x)$ is a sum of two lacunary series, hence satisfies $\|f\|_{L^q} \le C_q \|a\|_{\ell^2}$ for all $q < \infty$. Taking q arbitrarily large in Corollary 2 gives the second conclusion. The same reasoning applies in Example 2.

If complex rather than real linear combinations are allowed, then phase retrieval cannot hold in Example 3, nor in any example with two real-valued basis functions r, r'. Indeed, $f = r + ir'$ and $g = \overline{f} = r - ir'$ satisfy $|f| \equiv |g|$, but f is not a constant multiple of g.

Proposition 1 and Corollary 2 do not apply to Example 3 as stated, since with $r_n(x) = 2^{1/2}\sin(2\pi 4^n x)$ one has $r_i\overline{r_j} = r_j\overline{r_i}$ for all i,j. However, a small modification of the reasoning gives the following general result, Proposition 4, whose hypotheses are satisfied in Example 3.

For Hilbert spaces $L^2(\mu,\mathbb{R})$ of real-valued functions with orthonormal bases of real-valued functions r_n we modify the orthogonality hypothesis (5) as follows:

$$\big\{\mathbf{1},\, s_i,\, r_j r_k : i,j,k \in \mathbb{N} \text{ and } j < k\big\} \text{ is an orthogonal set.} \tag{18}$$

Proposition 4. *Let μ be a probability measure. Let $\{r_j\} \subset L^2(\mu)$ be an orthonormal set of real-valued functions satisfying hypotheses* (18),(6),(7). *Let $V \subset L^2(\mu,\mathbb{R})$ be the closure of the span of $\{r_j : j \in \mathbb{N}\}$ over $\mathbb{R}$. There exists $C < \infty$ such that*

$$\min(\|f-g\|_4,\, \|f+g\|_4) \le C\big\|\,|f|-|g|\,\big\|_4 \;\forall f,g \in V. \tag{19}$$

If there exist $q > 4$ and $C < \infty$ such that the L^q norm inequality (10) *holds for all functions in V then there exists $C' < \infty$ such that*

$$\min(\|f-g\|_2,\, \|f+g\|_2) \le C'\big\|\,|f|-|g|\,\big\|_2^{\gamma} \cdot (\|f\|_2 + \|g\|_2)^{1-\gamma} \;\forall f,g \in V \tag{20}$$

with $\gamma = \frac{q-4}{2q-4}$.

The only changes from the proof of Proposition 1 are that in (16), the first term becomes $2\sum_{i<j}(a_ia_j - b_ib_j)r_ir_j$ and consequently that on the right-hand side of (17), the last term is changed to

$$4\sum_{i<j}(a_ia_j - b_ib_j)^2\|r_ir_j\|_2^2 = 2\sum_{i\ne j}|a_ia_j - b_ib_j|^2\|r_i\overline{r_j}\|_2^2.$$

The corresponding quantity in the proof of Proposition 1 is $\sum_{i\ne j}|a_ia_j - b_ib_j|^2\|r_i\overline{r_j}\|_2^2$. The factor of 2 thus arising is favorable for our purpose. □

If 4^n is replaced by 3^n or 2^n in Example 3 then Proposition 4 no longer applies. Indeed, if 3^n is used the desired orthogonality between s_n and $r_{n+1}r_n$ fails to hold; $e^{2\pi i\cdot 2\cdot 3^n x}$ occurs with nonzero coefficient in the Fourier series for s_n, while $e^{2\pi i\cdot 3^{n+1}x}\cdot e^{-2\pi i\cdot 3^n x} = e^{2\pi i\cdot 2\cdot 3^n x}$ also occurs with nonzero coefficient in the Fourier series for $r_{n+1}r_n$. A similar issue arises for 2^n.

Another application of Proposition 4 is a real analogue of Example 1.

Example 4. *Let μ be a probability measure. Let $q > 4$ be an even integer. Let r_n be independent identically distributed real-valued random variables in $L^q(\mu)$ satisfying $\|r_n\|_{L^2} = 1$. Assume that*

$$\begin{cases} r_n \perp \mathbf{1} \\ \mu(\{x : |r_n(x)| \ne 1\}) > 0. \end{cases} \tag{21}$$

Then $\{r_n\}$ satisfies the hypotheses of Proposition 4, and consequently the closure of its span in $L^2(\mu,\mathbb{R})$ satisfies L^4-Lipschitz-SPR, and satisfies L^2-Hölder-SPR with $\gamma = \frac{q-4}{2q-4}$.

We conclude by lightly modifying a construction of Rudin [5] to create examples of trigonometric series that exhibit Hölder-stable phase retrieval, yet are rather far from being lacunary in nature. To simplify matters, we set this example in the ambient Hilbert space $L^2([0,1]\times[0,1])$, with respect to two-dimensional Lebesgue measure. Define r_ν to be

$$r_\nu(x,y) = 2^{1/2}\sin(2\pi\nu y)\, e^{2\pi i n_\nu x}, \tag{22}$$

where $(n_\nu : \nu\in\mathbb{N})$ is a subsequence of $\mathbb{N}$ to be specified.

To quantify the asymptotic density of a subsequence (n_ν) of $\mathbb{N}$, define $\alpha(N)$ to be the number of indices ν satisfying $n_\nu \le N$.

Example 5. *There exists a strictly increasing sequence $(n_\nu : \nu\in\mathbb{N})$, satisfying the asymptotic density lower bound $\limsup_{N\to\infty} N^{-1/2}\alpha(N) > 0$ such that the closed subspace V of $L^2([0,1]\times[0,1])$ spanned by the functions r_ν defined in (22) satisfies L^4-Lipschitz-SPR.*

There exists such a sequence satisfying $\limsup_{N\to\infty} N^{-1/3}\alpha(N) > 0$ such that V also satisfies L^2-Hölder-SPR with $\gamma = \frac14$.

Thus these sequences (n_ν) are denser than lacunary sequences.

Proof. In §4.7 of [5], Rudin constructs a sequence n_ν that satisfies $\limsup N^{-1/2}\alpha(N) > 0$ such that $n_i+n_j = n_k+n_l$ if and only if (i,j) is a permutation of (k,l), and deduces from this property the inequality $\|f\|_4 \le C\|f\|_2$ for all L^2 functions of the form $f(x) = \sum_\nu c_\nu e^{2\pi i n_\nu x}$. Let (n_ν) be any such sequence, and define $\{r_\nu\}$ by (22). Hypothesis (6), the uniform upper bound for $\|r_\nu\|_4$, certainly holds. The nonconstant factors $\sin(2\pi\nu y)$ ensure a uniform lower bound $\|r_\nu\|_4^4 \ge 1+\delta$, so (7) holds.

To verify hypothesis (5), first consider any inner product $\langle r_j\overline{r_k},\, r_l\overline{r_m}\rangle$ with $j\ne k$ and $l\ne m$. Calculation of this inner product produces a factor of $\int_0^1 e^{2\pi i(n_j-n_k-n_l+n_m)x}\,dx$, which vanishes unless $n_j-n_k-n_l+n_m = 0$. Equivalently, $n_j+n_m = n_l+n_k$. Therefore by Rudin's construction, (l,k) is a permutation of (j,m). If $j\ne k$, this implies that $(j,k) = (l,m)$. The associated functions $s_k(x,y) = 2\sin^2(2\pi ky)-1 = -\cos(4\pi ky)$ are independent of x, hence satisfy $s_k \perp r_i\overline{r_j}$ whenever $i\ne j$. Finally, if $k\ne l$ the $s_k\perp s_l$ since $\cos(4\pi ky)\perp\cos(4\pi ly)$ in $L^2([0,1])$.

Rudin [5] likewise constructs a sequence satisfying $\limsup N^{-1/3}\alpha(N) > 0$, satisfying the same conditions in the preceding paragraph, and satisfying $\|\sum_\nu b_\nu e^{2\pi i n_\nu x}\|_6 \le C\|b\|_{\ell^2}$ for all coefficient sequences $b\in\ell^2$. Consequently for any function $f(x,y)$ of the form $\sum_\nu a_\nu \sin(2\pi\nu y)e^{2\pi i n_\nu x}$ with $a\in\ell^2$,

$$\begin{aligned}\int_{[0,1]}\int_{[0,1]} \Big|\sum_\nu a_\nu\sin(2\pi\nu y)e^{2\pi i n_\nu x}\Big|^6\,dx\,dy &\le C\int_{[0,1]}\Big(\sum_\nu |a_\nu\sin(2\pi\nu y)|^2\Big)^{6/2}\,dy\\ &\le C\int_{[0,1]}\Big(\sum_\nu |a_\nu|^2\Big)^3\,dy = C\|a\|_{\ell^2}^6 \le 8C\|f\|_{L^2}^6.\end{aligned}$$

In each of these two situations, V has the indicated properties. □

Remark. In this example, the subspace V is in a sense larger, relative to other ambient subspaces naturally associated to it, than is the case for corresponding examples involving lacunary series. To formulate this assertion more precisely, for each degree $D\in\mathbb{N}$ let $V_{N,D}$ be the subspace of L^2 spanned by polynomials of degrees $\le D$ in $\{r_\nu : 1\le\nu\le N\}$. Let N tend to infinity, while D remains fixed. The dimensions $\dim(V_{N,D})$ satisfy $\liminf_{N\to\infty} N^{-3}\dim(V_{N,D}) < \infty$ for any D in Example 5, while for the lacunary series example $r_\nu = 2^{1/2}\sin(2\pi 4^\nu x)$, $\dim(V_{N,D})$ has order of magnitude N^D. Thus the span of

$\{r_\nu : 1 \leq \nu \leq N\}$, for these N, is a comparatively large subspace of the associated spaces $V_{N,D}$ in Example 5.

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