COMPLETENESS OF UNBOUNDED CONVERGENCES

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Abstract. As a generalization of almost everywhere convergence to vector lattices, unbounded order convergence has garnered much attention. The concept of boundedly *uo*-complete Banach lattices was introduced by N. Gao and F. Xanthos, and has been studied in recent papers by D. Leung, V.G. Troitsky, and the aforementioned authors. We will prove that a Banach lattice is boundedly *uo*-complete iff it is monotonically complete. Afterwards, we study completeness-type properties of minimal topologies; minimal topologies are exactly the Hausdorff locally solid topologies in which *uo*-convergence implies topological convergence.

1. Introduction

In the first half of the paper, we study when norm bounded *uo*-Cauchy nets in a Banach lattice are *uo*-convergent. The section starts with a counterexample to a question posed in [LC], and culminates in a proof that a Banach lattice is (sequentially) boundedly *uo*-complete iff it is (sequentially) monotonically complete. This gives the final solution to a problem that has been investigated in [Gao14], [GX14], [GTX17], and [GLX].

The latter half of this paper focuses on the "extremal" topologies of a vector lattice X. For motivation, recall that corresponding to a dual pair $\langle E, E^*\rangle$ is a family of topologies on E "compatible" with duality. The two most important elements of this family are the weak and Mackey topologies, which are defined by their extremal nature. Analogously, given a vector lattice X, it is often possible to equip X

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with many topologies compatible (in the sense of being locally solid and Hausdorff) with the lattice structure. It is easy to see that whenever X admits some Hausdorff locally solid topology, the collection of all Riesz pseudonorms on X generates a *finest* Hausdorff locally solid topology on X. This "greatest" topology appears in many applications. Indeed, analogous to the theory of compatible locally convex topologies on a Banach space - where the norm topology is the Mackey topology - the norm topology on a Banach lattice X is the finest topology on X compatible with the lattice structure. This is [AB03, Theorem 5.20].

On the opposite end of the spectrum, a Hausdorff locally solid topology on a vector lattice X is said to be ***minimal*** if there is no coarser Hausdorff locally solid topology on X; it is ***least*** if it is coarser than every Hausdorff locally solid topology on X. Least topologies were introduced in [AB80] and studied in [AB03]; minimal topologies were studied in [Lab87], [Con05], [Tay], and [KT]. An important example of a least topology is the unbounded norm topology on an order continuous Banach lattice. The unbounded absolute weak*-topology on $L_\infty[0,1]$ is a noteworthy example of a minimal topology that is not least. In the next subsection, we briefly recall some facts about minimal and unbounded topologies; for a detailed exposition the reader is referred to [Tay] and [KT].

1.1. **Notation.** Throughout this paper, all vector lattices are assumed Archimedean. For a net (x_α) in a vector lattice X, we write $x_\alpha \xrightarrow{o} x$ if (x_α) ***converges to*** x ***in order***; that is, there is a net (y_β), possibly over a different index set, such that $y_\beta \downarrow 0$ and for every β there exists α_0 such that $|x_\alpha - x| \leq y_\beta$ whenever $\alpha \geq \alpha_0$. We write $x_\alpha \xrightarrow{uo} x$ and say that (x_α) ***uo-converges*** to $x \in X$ if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for every $u \in X_+$. For facts on uo-convergence, the reader is referred to [GTX17]. In particular, [GTX17, Theorem 3.2] will be used freely. Recall that a Banach lattice X is ***(sequentially) boundedly uo-complete*** if norm bounded uo-Cauchy nets (respectively, sequences) in X are uo-convergent in X.

Given a locally solid topology τ on a vector lattice X, one can associate a topology, $u\tau$, in the following way. If $\{U_i\}_{i\in I}$ is a base at zero for τ consisting of solid sets, for each $i \in I$ and $u \in X_+$ define

$$U_{i,u} := \{x \in X : \ |x| \wedge u \in U_i\}.$$

As was proven in [Tay, Theorem 2.3], the collection $\mathcal{N}_0 = \{U_{i,u} : i \in I, u \in X_+\}$ is a base of neighbourhoods at zero for a new locally solid topology, denoted by $u\tau$, and referred to as the ***unbounded*** ***τ-topology***. Noting that the map $\tau \mapsto u\tau$ from the set of locally solid topologies on X to itself is idempotent, a locally solid topology τ is called ***unbounded*** if there is a locally solid topology σ with $\tau = u\sigma$ or, equivalently, if $\tau = u\tau$. The following connection between minimal topologies, unbounded topologies, and *uo*-convergence was proven in [Tay, Theorem 6.4]. Recall that a locally solid topology τ is ***Lebesgue*** if order null nets are τ-null.

Theorem 1.1. *Let τ be a Hausdorff locally solid topology on a vector lattice X. TFAE:*

(i) *uo-null nets are τ-null;*
(ii) *τ is Lebesgue and unbounded;*
(iii) *τ is minimal.*

In particular, a vector lattice can admit at most one minimal topology.

Interestingly, the process of unbounding a topology can convert the greatest topology into the least topology; this happens with the norm topology on an order continuous Banach lattice.

Recall that a vector lattice X is ***universally complete*** if it is order complete and laterally complete; it is ***universally σ-complete*** if it is σ-order complete and laterally σ-complete. By [AB03, Theorem 7.21], every vector lattice has a (unique up to lattice isomorphism) universal completion, which we will denote by X^u. By [GTX17, Theorem 3.2] and order density of X in X^u, *uo*-convergence passes freely between X and its universal completion. As in [AB03, Theorem 7.37], we say that a non-empty subset A of X_+ is ***dominable*** if it is order bounded when viewed as a subset of X^u.

All other undefined terminology is consistent with [AB03]. In particular, we say that a locally solid topology τ on a vector lattice X is ***Levi*** if τ-bounded increasing nets in X_+ have supremum. Levi and

monotonically complete are synonymous; the latter terminology is that of [MN91], and is used in [GLX].

2. Boundedly uo-complete Banach lattices

Results equating the class of boundedly *uo*-complete Banach lattices to the class of monotonically complete Banach lattices have been acquired, under technical assumptions, by N. Gao, D. Leung, V.G. Troitsky, and F. Xanthos. The sharpest result is [GLX, Proposition 3.1]; it states that a Banach lattice whose order continuous dual separates points is boundedly *uo*-complete iff it is monotonically complete. In this section, we remove the restriction on the order continuous dual.

The following question was posed as Problem 2.4 in [LC]:

Question 2.1. Let (x_α) be a norm bounded positive increasing net in a Banach lattice X. Is (x_α) *uo*-Cauchy in X?

If Question 2.1 is true, it is easily deduced that a Banach lattice is boundedly *uo*-complete iff it is monotonically complete. However, the next example answer this question in the negative, even for sequences.

Example 2.2. Let S be the set of all non-empty finite sequences of natural numbers. For $s \in S$ define $\lambda(s) = \text{length}(s)$. If $s, t \in S$, define $s \leq t$ if $\lambda(s) \leq \lambda(t)$ and $s(i) = t(i)$ for $i = 1, \dots, \lambda(s)$. For $s \in S$ with $\lambda(s) = n$ and $i \in \mathbb{N}$, define $s * i = (s(1), \dots, s(n), i)$. Put

$$X = \{x \in \ell^\infty(S) : \lim_{i\to\infty} x(s * i) = \frac{1}{2}x(s) \text{ for all } s \in S\}. \tag{2.1}$$

It can be verified that X is a closed sublattice of $(\ell^\infty(S), \|\cdot\|_\infty)$ and for $t \in S$ the element $e^t : S \to \mathbb{R}$ defined by

$$e^t(s) = \begin{cases} (\frac{1}{2})^{\lambda(s)-\lambda(t)} & \text{if } t \leq s \\ 0 & \text{otherwise} \end{cases}$$

is an element of X with norm 1. Define $f_1 = e^{(1)}, f_2 = e^{(1)} \vee e^{(2)} \vee e^{(1,1)} \vee e^{(1,2)} \vee e^{(2,1)} \vee e^{(2,2)}$, and, generally,

$$f_n = \sup\{e^t : \lambda(t) \leq n \text{ and } t(k) \leq n \ \forall k \leq \lambda(t)\}.$$

The sequence (f_n) is increasing and norm bounded by 1; it was shown in [BL88, Example 1.8] that (f_n) is not order bounded in X^u. Therefore, (f_n) cannot be *uo*-Cauchy in X for if it were then it would be *uo*-Cauchy in X^u and hence order convergent in X^u by [GTX17, Theorem 3.10]. Since it is increasing, it would have supremum in X^u; this is a contradiction as (f_n) is not order bounded in X^u.

Under some mild assumptions, however, Question 2.1 has a positive solution. Recall that a Banach lattice is ***weakly Fatou*** if there exists $K \geq 1$ such that whenever $0 \leq x_\alpha \uparrow x$, we have $\|x\| \leq K \sup \|x_\alpha\|$.

Proposition 2.3. *Let X be a weakly Fatou Banach lattice. Then every positive increasing norm bounded net in X is uo-Cauchy.*

Proof. Let K be such that $0 \leq x_\alpha \uparrow x$ implies $\|x\| \leq K \sup \|x_\alpha\|$. Now assume that $0 \leq u_\alpha \uparrow$ and $\|u_\alpha\| \leq 1$. Let $u > 0$ and pick n such that $\|u\| > \frac{K}{n}$. If $0 \leq (\frac{1}{n}u_\alpha) \wedge u \uparrow_\alpha u$, then $\|u\| \leq \frac{K}{n}$. Therefore, there exists $0 < w \in X$ such that $(\frac{1}{n}u_\alpha) \wedge u \leq u - w$ for all α. But then $(nu - u_\alpha)^+ = n\left[u - (\frac{1}{n}u_\alpha) \wedge u\right] \geq nw > 0$ for all α, so that (u_α) is dominable. By [AB03, Theorem 7.37], (u_α) is order bounded in X^u, and hence $u_\alpha \uparrow \widehat{u}$ for some $\widehat{u} \in X^u$. This proves that (u_α) is *uo*-Cauchy in X^u, hence in X. $\square$

Proposition 2.4. *Let X be a weakly σ-Fatou Banach lattice. Then every positive increasing norm bounded sequence in X is uo-Cauchy.*

Proof. The proof is similar and, therefore, omitted. $\square$

Even though Question 2.1 is false, the equivalence between boundedly *uo*-complete and Levi still stands. We will show that now. First, recall that a positive sequence (x_n) in a vector lattice is said to be ***laterally increasing*** if it is increasing and $(x_m - x_n) \wedge x_n = 0$ for all $m \geq n$.

Theorem 2.5. *Let X be a Banach lattice. TFAE:*

(i) *X is σ-Levi;*
(ii) *X is sequentially boundedly uo-complete;*
(iii) *Every increasing norm bounded uo-Cauchy sequence in X_+ has a supremum.*

Proof. (i)⇒(ii): Let (x_n) be a norm bounded *uo*-Cauchy sequence in X. WLOG, (x_n) is positive; otherwise consider positive and negative parts. Define $e = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{x_n}{1+\|x_n\|}$ and consider B_e, the band generated by e. Then (x_n) is still norm bounded and *uo*-Cauchy in B_e. Also, B_e has the σ-Levi property for if $0 \leq y_n \uparrow$ is a norm bounded sequence in B_e, then $y_n \uparrow y$ for some $y \in X$ as X is σ-Levi. Since B_e is a band, $y \in B_e$ and $y_n \uparrow y$ in B_e. We next show that there exists $u \in B_e$ such that $x_n \xrightarrow{uo} u$ in B_e, and hence in X.

For each $m, n, n' \in \mathbb{N}$, since $|x_n \wedge me - x_{n'} \wedge me| \leq |x_n - x_{n'}| \wedge me$, the sequence $(x_n \wedge me)_n$ is order Cauchy, hence order converges to some u_m in B_e since the σ-Levi property implies σ-order completeness. The sequence (u_m) is increasing and

$$\|u_m\| \leq K \sup_n \|x_n \wedge me\| \leq K \sup_n \|x_n\| < \infty$$

where we use that σ-Levi implies weakly σ-Fatou. This can be proved by following the arguments in [MN91, Proposition 2.4.19]. Since B_e is σ-Levi, (u_m) increases to an element $u \in B_e$. Fix m. For any N, N' define $x_{N,N'} = \sup_{n \geq N, n' \geq N'} |x_n - x_{n'}| \wedge e$. Since (x_n) is *uo*-Cauchy, $x_{N,N'} \downarrow 0$. Now, for each m,

$$|x_n \wedge me - x_{n'} \wedge me| \wedge e \leq |x_n - x_{n'}| \wedge e \leq x_{N,N'} \ \forall n \geq N, n' \geq N'.$$

Taking order limit in n' yields:

$$|x_n \wedge me - u_m| \wedge e \leq x_{N,N'}$$

Since e is a weak unit in B_e, taking order limit in m now yields:

$$|x_n - u| \wedge e \leq x_{N,N'}, \ \forall n \geq N,$$

from which it follows that $|x_n - u| \wedge e \xrightarrow{o} 0$ in B_e. This yields $x_n \xrightarrow{uo} u$ in B_e by [GTX17, Corollary 3.5].

The implication (ii)⇒(iii) is clear. For the last implication it suffices, by [AW97, Theorem 2.4], to verify that every norm bounded laterally increasing sequence in X_+ has a supremum. Let (x_n) be a norm bounded laterally increasing sequence in X_+. By [AW97, Proposition 2.2], (x_n) has supremum in X^u, hence is *uo*-Cauchy in X^u. It follows that (x_n) is *uo*-Cauchy in X and, therefore, by assumption, *uo*-converges to some $x \in X$. It is then clear that $x_n \uparrow x$ in X. □

Theorem 2.6. *Let X be a Banach lattice. TFAE:*

(i) *X is Levi;*
(ii) *X is boundedly uo-complete;*
(iii) *Every increasing norm bounded uo-Cauchy net in X_+ has a supremum.*

Proof. If X is Levi, then X is boundedly *uo*-complete by [GLX, Proposition 3.1]. It is clear that (ii)⇒(iii) and the proof of (iii)⇒(i) is the same as in the last theorem but with [AW97, Theorem 2.4] replaced with [AW97, Theorem 2.3]. □

3. Completeness of minimal topologies

Throughout this section, X is a vector lattice and τ denotes a locally solid topology on X. We begin with a brief discussion on relations between minimal topologies and the B-property. Corollary 3.3 will be of importance as many properties of locally solid topologies are stated in terms of positive increasing nets. For minimal topologies, these properties permit a uniform and efficient treatment.

The B-property was introduced as property (B,iii) by W.A.J. Luxemburg and A.C. Zaanen in [LZ64]. It is briefly studied in [AB03] and, in particular, it is shown that the Lebesgue property does not imply the B-property. We prove, however, that if τ is unbounded then this implication does indeed hold true:

Definition 3.1. *A locally solid vector lattice (X,τ) satisfies the **B-property** if it follows from $0 \leq x_n \uparrow$ in X and (x_n) τ-bounded that (x_n) is τ-Cauchy. An equivalent definition is obtained if sequences are replaced with nets.*

Proposition 3.2. *If X is a vector lattice admitting a minimal topology τ, then τ satisfies the B-property.*

Proof. Suppose τ is minimal and (x_n) is a τ-bounded sequence satisfying $0 \leq x_n \uparrow$. By [AB03, Theorem 7.50], (x_n) is dominable. By [AB03, Theorem 7.37], (x_n) is order bounded in X^u so that $x_n \xrightarrow{uo} u$ for some $u \in X^u$. In particular, (x_n) is *uo*-Cauchy in X^u. It follows that (x_n) is *uo*-Cauchy in X. Since τ is Lebesgue, (x_n) is $u\tau$-Cauchy in X. Finally, since τ is unbounded, (x_n) is τ-Cauchy in X. □

Corollary 3.3. *Let X be a vector lattice admitting a minimal topology τ, and (x_α) an increasing net in X_+. TFAE:*

(i) *(x_α) is τ-bounded;*
(ii) *(x_α) is τ-Cauchy.*

Recall the following definition, taken from [AB03, Definition 2.43].

Definition 3.4. *A locally solid vector lattice (X,τ) is said to satisfy the **monotone completeness property (MCP)** if every increasing τ-Cauchy net of X_+ is τ-convergent in X. The σ-MCP is defined analogously with nets replaced with sequences.*

Remark 3.5. By Corollary 3.3, a minimal topology has MCP iff it is Levi.

Proposition 3.6. *Let τ be a Hausdorff locally solid topology on X. If $u\tau$ satisfies MCP then so does τ. If $u\tau$ satisfies σ-MCP then so does τ.*

Proof. Suppose $0 \leq x_\alpha \uparrow$ is a τ-Cauchy net. It is then $u\tau$-Cauchy and hence $u\tau$-converges to some $x \in X$. Therefore, $x_\alpha \uparrow x$ and $x_\alpha \xrightarrow{\tau} x$. Replacing nets with sequences yields the σ-analogue. □

Recall by [AB03, Theorem 2.46 and Exercise 2.11] that a Hausdorff locally solid vector lattice (X,τ) is (sequentially) complete iff order intervals are (sequentially) complete and τ has (σ)-MCP. Therefore, since τ-convergence agrees with $u\tau$-convergence on order intervals, $u\tau$ being (sequentially) complete implies τ is (sequentially) complete.

A locally solid topology τ on a vector lattice X is ***pre-Lebesgue*** if order bounded disjoint sequences in X are τ-null. By [AB03, Theorem 3.23], the Lebesgue property implies the pre-Lebesgue property.

Lemma 3.7. *Let (X,τ) be a Hausdorff locally solid vector lattice. If τ is unbounded then TFAE:*

(i) *τ has MCP and is pre-Lebesgue;*
(ii) *τ is Lebesgue and Levi.*

Proof. It is sufficient, by [DL98, Theorem 2.5], to prove that (X,τ) contains no lattice copy of c_0. Suppose, towards contradiction, that X

does contain a lattice copy of c_0, i.e., there is a homeomorphic Riesz isomorphism from c_0 onto a sublattice of X. This leads to a contradiction as the standard unit vector basis is not null in c_0, but the copy in X is by [Tay, Theorem 4.2]. □

Lemma 3.7 is another way to prove that a minimal topology has MCP iff it is Levi. We next present the sequential analogue:

Lemma 3.8. *Let (X, τ) be a Hausdorff locally solid vector lattice. If τ is unbounded then TFAE:*

(i) *τ has σ-MCP and is pre-Lebesgue;*
(ii) *τ is σ-Lebesgue and σ-Levi.*

Proof. (i)⇒(ii) is similar to the last lemma; apply instead [DL98, Proposition 2.1 and Theorem 2.4].

(ii)⇒(i): It suffices to show that τ is pre-Lebesgue. For this, suppose that $0 \leq x_n \uparrow \leq u$; we must show that (x_n) is τ-Cauchy. Since τ is σ-Levi and order bounded sets are τ-bounded, $x_n \uparrow x$ for some $x \in X$. Since τ is σ-Lebesgue, $x_n \xrightarrow{\tau} x$. □

Putting pieces together from other papers, we next characterize sequential completeness of *uo*-convergence.

Theorem 3.9. *Let X be a vector lattice. TFAE:*

(i) *X is sequentially uo-complete;*
(ii) *Every positive increasing uo-Cauchy sequence in X uo-converges in X;*
(iii) *X is universally σ-complete.*

In this case, uo-Cauchy sequences are order convergent.

Proof. (i)⇒(ii) is clear. (ii)⇒(iii) by careful inspection of [LC, Proposition 2.8], (iii)⇒(i) and the moreover clause follow from [GTX17, Theorem 3.10]. □

Remark 3.10. Recall that by [AB03, Theorem 7.49], every locally solid topology on a universally σ-complete vector lattice satisfies the pre-Lebesgue property. Using *uo*-convergence, we give a quick proof of this. Suppose τ is a locally solid topology on a universally σ-complete vector lattice X; we claim that *uo*-null sequences are τ-null. This follows since τ is σ-Lebesgue and *uo* and *o*-convergence agree for sequences by [GTX17, Theorem 3.9]. In particular, since disjoint sequences are *uo*-null, disjoint sequences are τ-null.

We next give the topological analogue of Theorem 3.9:

Lemma 3.11. *Let X be a vector lattice admitting a minimal topology τ. TFAE:*

(i) *τ is σ-Levi;*
(ii) *τ has σ-MCP;*
(iii) *X is universally σ-complete;*
(iv) *(X, τ) is sequentially boundedly uo-complete in the sense that τ-bounded uo-Cauchy sequences in X are uo-convergent in X.*

Proof. (i)$\Leftrightarrow$(ii) follows from Lemma 3.8. We next deduce (iii). Since τ is σ-Levi, X is σ-order complete; we prove X is laterally σ-complete. Let $\{a_n\}$ be a countable collection of mutually disjoint positive vectors in X, and define $x_n = \sum_{k=1}^{n} a_k$. Then (x_n) is a positive increasing sequence in X, and it is *uo*-Cauchy, as an argument similar to [LC, Proposition 2.8] easily shows. By Theorem 1.1, (x_n) is τ-Cauchy, hence $x_n \xrightarrow{\tau} x$ for some $x \in X$ since τ has σ-MCP. Since (x_n) is increasing and τ is Hausdorff, $x_n \uparrow x$. Clearly, $x = \sup\{a_n\}$.

(iii)$\Rightarrow$(iv) follows from Theorem 3.9; (iv)$\Rightarrow$(i) follows immediately after noticing that Proposition 2.3 is valid (similar proof) if "weakly Fatou Banach lattice" is replaced by "Hausdorff Fatou topology". $\square$

The following question(s) remain open:

Question 3.12. Let X be a vector lattice admitting a minimal topology τ. Are the following equivalent?

(i) (X, τ) is sequentially complete;
(ii) X is universally σ-complete.

Question 3.13. Let (X, τ) be Hausdorff and Lebesgue. Are the following equivalent?

(i) Order intervals of X are sequentially τ-complete;
(ii) X is σ-order complete.

Remark 3.14. Question 3.12 and Question 3.13 are equivalent. Indeed, in both cases it is known that (i)$\Rightarrow$(ii). If Question 3.13 is true then Question 3.12 is true since we have already established that minimal topologies have σ-MCP when X is universally σ-complete. Suppose Question 3.12 is true. If X is σ-order complete, then X is an ideal in its universal σ-completion, X^s. Indeed, it is easy to establish that if Y is a σ-order complete vector lattice sitting as a super order dense sublattice of a vector lattice Z, then Y is an ideal of Z; simply modify the arguments in [AB03, Theorem 1.40]. By [AB03, Theorem 4.22] we may assume, WLOG, that τ is minimal. τ then lifts to the universal completion and can be restricted to X^s.

Question 3.13 is a special case of Aliprantis and Burkinshaw's [AB78, Open Problem 4.2], which we state as well:

Question 3.15. Suppose τ is a Hausdorff σ-Fatou topology on a σ-order complete vector lattice X. Are the order intervals of X sequentially τ-complete?

The case of complete order intervals is much easier than the sequentially complete case. The next result is undoubtedly known, but fits in nicely; we provide a simple proof that utilizes minimal topologies.

Proposition 3.16. *Suppose τ is a Hausdorff Lebesgue topology on X. Order intervals of X are complete iff X is order complete.*

Proof. If X is order complete then order intervals are complete by [AB03, Theorem 4.28].

By [AB03, Theorem 4.22] we may assume, WLOG, that τ is minimal. If order intervals are complete then X is an ideal of $\widehat{X} = X^u$ by [AB03, Theorem 2.42] and [Tay, Theorem 5.2]. Since X^u is order complete, so is X. $\square$

Remark 3.17. If X is an order complete and laterally σ-complete vector lattice admitting a minimal topology τ, then τ is sequentially complete. Although these conditions are strong, they do not force X to be universally complete. This can be seen by equipping the vector lattice of [AB03, Example 7.41] with the minimal topology given by restriction of pointwise convergence from the universal completion.

The key step in the proof of Theorem 2.5 is [AW97, Theorem 2.4] which states that a Banach lattice is σ-Levi if and only if it is *laterally* σ-Levi. We say that a locally solid vector lattice (X,τ) has the ***lateral σ-Levi property*** if $\sup x_n$ exists whenever (x_n) is laterally increasing and τ-bounded. For minimal topologies, the σ-Levi and lateral σ-Levi properties do not agree, as we now show:

Proposition 3.18. *Let X be a vector lattice admitting a minimal topology τ. TFAE:*

(i) *X is laterally σ-complete;*
(ii) *τ has the lateral σ-Levi property;*
(iii) *Every disjoint positive sequence, for which the set of all possible finite sums is τ-bounded, must have a supremum.*

Proof. (i)$\Rightarrow$(iii) is clear, as is (ii)$\Leftrightarrow$(iii); we prove (ii)$\Rightarrow$(i). Assume (ii) and let (x_n) be a disjoint sequence in X_+. Since (x_n) is disjoint, (x_n) has a supremum in X^u. Define $y_n = x_1 \vee \cdots \vee x_n$. The sequence (y_n) is laterally increasing and order bounded in X^u. By [AB03, Theorem 7.37], (y_n) forms a dominable set in X_+. By [Tay, Theorem 5.2(iv)], (y_n) is τ-bounded, and hence has supremum in X by assumption. This implies that (x_n) has a supremum in X and, therefore, X is laterally σ-complete. $\square$

In [Lab84] and [Lab85], many completeness-type properties of locally solid topologies were introduced. For entirety, we classify the remaining properties, which he refers to as "BOB" and "POB".

Definition 3.19. *A Hausdorff locally solid vector lattice (X,τ) is said to be* ***boundedly order-bounded (BOB)*** *if increasing τ-bounded nets in X_+ are order bounded in X. (X,τ) satisfies the* ***pseudo-order boundedness property (POB)*** *if increasing τ-Cauchy nets in X_+ are order bounded in X.*

Remark 3.20. It is clear that a Hausdorff locally solid vector lattice is Levi iff it is order complete and boundedly order-bounded. It is also clear that BOB and POB coincide for minimal topologies.

Proposition 3.21. *Let X be a vector lattice admitting a minimal topology τ. TFAE:*

(i) (X,τ) *satisfies BOB;*
(ii) X *is majorizing in* X^u.

Proof. (i)$\Rightarrow$(ii): Let $0 \leq u \in X^u$. Since X is order dense in X^u, there exists a net (x_α) in X such that $0 \leq x_\alpha \uparrow u$. In particular, (x_α) is order bounded in X^u, hence dominable in X by [AB03, Theorem 7.37]. By [Tay, Theorem 5.2], (x_α) is τ-bounded. By assumption, (x_α) is order bounded in X, hence, $(x_\alpha) \subseteq [0,x]$ for some $x \in X_+$. It follows that $u \leq x$, so that X majorizes X^u.

(ii)$\Rightarrow$(i): Suppose (x_α) is an increasing τ-bounded net in X_+. It follows from [AB03, Theorem 7.50] that (x_α) is dominable, hence order bounded in X^u. Since X majorizes X^u, (x_α) is order bounded in X. □

Remark 3.22. By [AB03, Theorem 7.15], laterally complete vector lattices majorize their universal completions.

Remark 3.23. If τ is a Hausdorff Fatou topology on X, it is easy to see that (X,τ) satisfies BOB iff every increasing τ-bounded net in X_+ is order Cauchy in X. Compare with Question 2.1.

We next state the σ-analogue of Proposition 3.21.

Proposition 3.24. *Let X be an almost σ-order complete vector lattice admitting a minimal topology τ. TFAE:*

(i) (X,τ) *satisfies σ-BOB;*
(ii) X *is majorizing in the universal σ-completion X^s of X.*

Proof. (i)$\Rightarrow$(ii) is similar to Proposition 3.21.

(ii)$\Rightarrow$(i): Suppose (x_n) is an increasing τ-bounded sequence in X_+. It is then dominable in X, hence in X^s by [AB03, Lemma 7.11]. It follows by [AB03, Theorem 7.38] that (x_n) is order bounded in X^s. Since X is majorizing in X^s, (x_n) is order bounded in X. □

The next definition is standard in the theory of topological vector spaces:

Definition 3.25. *Let (E,σ) be a Hausdorff topological vector space. E is* ***quasi-complete*** *if every σ-bounded σ-Cauchy net is σ-convergent.*

Remark 3.26. Since Cauchy sequences are bounded, there is no sequential analogue of quasi-completeness.

We finish with the full characterization of completeness of minimal topologies:

Theorem 3.27. *Let X be a vector lattice admitting a minimal topology τ. TFAE:*

(i) *X is universally complete;*
(ii) *τ is complete;*
(iii) *τ satisfies MCP;*
(iv) *τ is Levi;*
(v) *τ is quasi-complete;*
(vi) *(X, τ) is boundedly uo-complete in the sense that τ-bounded uo-Cauchy nets in X are uo-convergent in X.*

Proof. (i)⇔(ii) by [Tay, Corollary 5.3] combined with [Tay, Theorem 6.4]. Clearly, (ii)⇒(iii)⇔(iv). (iii)⇒(ii) since if τ satisfies MCP then τ is topologically complete by [AB03, Corollary 4.39]. We have thus established that (i)⇔(ii)⇔(iii)⇔(iv). It is clear that (ii)⇒(v), and (v)⇒(iii) by Corollary 3.3.

(ii)⇒(vi): Let (x_α) be a *uo*-Cauchy net in X; (x_α) is then τ-Cauchy and hence τ-convergent. The claim then follows from [Tay, Remark 2.26].

(vi)⇒(iv): Suppose $0 \leq x_\alpha \uparrow$ is τ-bounded. (x_α) is then *uo*-Cauchy, hence *uo*-convergent to some $x \in X$. Clearly, $x = \sup x_\alpha$. □

Remark 3.28. This is in good agreement with Proposition 3.6. If the minimal topology satisfies MCP then Proposition 3.6 states that every Hausdorff Lebesgue topology satisfies MCP. Universally complete spaces, however, admit at most one Hausdorff Lebesgue topology by [AB03, Theorem 7.53].

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